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Problem Set 12

1. The Schrödinger equation reads
$$-\frac{\hbar^2}{2m}\nabla^2\psi(\bar r) = \left(E - U(\bar r)\right)\psi(\bar r), \text{ or}$$
$$\nabla^2\psi(\bar r) + \frac{2m}{\hbar^2}E\psi(\bar r) = \frac{2m}{\hbar^2}U(\bar r)\psi(\bar r)$$
Far from the region where the potential is localized we have a free particle. We seek the solution there as $\psi^{(0)}(\bar r) = e^{i\bar k\cdot\bar r}$, the energy is $\frac{\hbar^2k^2}{2m}$ (the potential there is absent). Then we account for the perturbation:
$$\psi \simeq \psi^{(0)}(\bar r) + \psi^{(1)}(\bar r), \quad E = \frac{\hbar^2k^2}{2m}.$$
Substitute this into the Schrödinger equation and keep terms of the first order in $U(\bar r)$, i.e., disregard $U(\bar r)\psi^{(1)}(\bar r)$.
This gives
$$\nabla^2\psi^{(1)}(\bar r) + k^2\psi^{(1)}(\bar r) = \frac{2m}{\hbar^2}U(\bar r)\psi^{(0)}(\bar r)$$

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2. Let x_i be the roots of $f(x)$, i.e. $f(x_i)=0$. We assume that the roots are nondegenerate, $f'(x_i) \equiv \left.\frac{df}{dx}\right|_{x=x_i} \neq 0$. Function $\delta(f(x))$ should be a sum of δ-functions $\delta(x-x_i)$ with a weight. To find the weight consider

$$\int_{x_i-\varepsilon}^{x_i+\varepsilon} \delta\big(f(x)\big)\,dx = \int_{x_i-\varepsilon}^{x_i+\varepsilon} \delta\big(f'(x_i)(x-x_i)\big)\,dx =$$

$$= \int_{x_i-\varepsilon}^{x_i+\varepsilon} \frac{d\left[f'(x_i)x\right]}{f'(x_i)}\,\delta\big(f'(x_i)(x-x_i)\big) = \frac{1}{|f'(x_i)|}$$

(if $f'(x_i)<0$, we are integrating in the negative direction of $f'(x_i)dx$ for $dx>0$, hence we have to change the sign)

Then $\delta\big(f(x)\big) = \sum_i \frac{1}{|f'(x_i)|}\,\delta(x-x_i)$

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3. Seek the solution of the Schrödinger equation

$$i\hbar \frac{\partial\psi}{\partial t} = \left[H_0 + \frac{1}{2}\mathcal{V}e^{i\omega t} + \frac{1}{2}\mathcal{V}e^{-i\omega t}\right]\psi$$

in the form

$$\psi = c_1(t)\, e^{-iE_1 t/\hbar}|\psi_1\rangle + c_2(t)e^{-iE_2 t/\hbar}|\psi_2\rangle$$

The full equations for $c_{1,2}$ read

$$\dot c_1 = \frac{1}{2i\hbar} V^* c_2 e^{i\Delta t}\left(1+e^{-2i\omega t}\right), \quad \dot c_2 = \frac{1}{2i\hbar} V c_1 e^{-i\Delta t}\left(1+e^{2i\omega t}\right)$$

where $\Delta = \omega - \omega_{21}$, $V = \langle\psi_2|\mathcal{V}|\psi_1\rangle$, and we have set $\langle\psi_1|\mathcal{V}|\psi_1\rangle = \langle\psi_2|\mathcal{V}|\psi_2\rangle = 0$, as we did in class

We set $c_1 = c_1^{(0)} + c_1^{(1)}$, $c_2 = c_2^{(0)} + c_2^{(1)}$

$$\dot c_1^{(0)} = \frac{1}{2i\hbar} V^* c_2^{(0)} e^{i\Delta t}, \quad \dot c_2^{(0)} = \frac{1}{2i\hbar} V c_1^{(0)} e^{-i\Delta t}$$

$$\dot c_1^{(1)} = \frac{1}{2i\hbar} V^* c_2^{(0)} e^{i\Delta t - 2i\omega t}, \quad \dot c_2^{(1)} = \frac{1}{2i\hbar} V c_1^{(0)} e^{-i\Delta t + 2i\omega t}$$

$$\Rightarrow \quad c_1^{(1)} \simeq \frac{V^*}{2\hbar(2\omega-\Delta)} c_2^{(0)} e^{i\Delta t - 2i\omega t} \simeq \frac{V^*}{4\hbar\omega} c_2^{(0)} e^{i\Delta t - 2i\omega t}$$

$$c_2^{(1)} \simeq -\frac{V}{2\hbar(2\omega-\Delta)} c_1^{(0)} e^{-i\Delta t + 2i\omega t} \simeq -\frac{V}{4\hbar\omega} c_1^{(0)} e^{-i\Delta t + 2i\omega t}$$

The solution of the equations for $c_1^{(0)}$ and $c_2^{(0)}$ was given in class,

$$c_1^{(0)}(t) = A_1 e^{i(\Delta+\Omega)t/2} + B_1 e^{i(\Delta-\Omega)t/2}, \quad \Omega = \sqrt{\Delta^2 + \frac{|V|^2}{\hbar^2}}$$

$$c_2^{(0)}(t) = -\frac{\hbar(\Delta+\Omega)}{V^*} A_1 e^{-i(\Delta-\Omega)t/2} - \frac{\hbar(\Delta-\Omega)}{V^*} B_1 e^{-i(\Delta+\Omega)t/2}$$

with the initial conditions $A_1 + B_1 = c_1(0)$, $(\Delta+\Omega)A_1 + (\Delta-\Omega)B_1 = -\frac{V^*}{\hbar}c_2(0)$

The corrections $c_1^{(1)}$ and $c_2^{(1)}$ are $\sim |V|/\omega\hbar \ll 1 \Rightarrow$ this inequality is the condition of the perturbation $\mathcal{V}\cos\omega t$ being small.

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4. We found in class that the probability to make a transition over time dt is

$$dP = W|c_1|^2 dt$$

where $|c_1|^2$ is the occupation of the bound state. As a result of a transition, this occupation decreases by dP. Therefore we have

$$\frac{d|c_1|^2}{dt} = -\frac{dP}{dt} = -W|c_1|^2, \qquad |c_1(t)|^2 = |c_1(0)|^2 e^{-Wt}$$

This exponential decrease is familiar from the radioactive decay, with W being the decay rate and $1/W$ being the lifetime of a radioactive isotope.