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Problem Set 13

1. For $t>0$ one can think of the system as of a spin-$\frac{1}{2}$ particle in a "magnetic field" with componente $\vec{B} = C(\hbar\omega, V, 0)$ where C is a scaling parameter, you can think of the reciprocal gyromagnetic ratio. Therefore we immediately know the energies

$E_\pm = \pm \frac{1}{2}\sqrt{\hbar^2\omega^2 + V^2}$ for $t>0 \Rightarrow$ will check

Let us solve the Schrodinger equation

$$\left(\frac{1}{2}\hbar\omega\sigma_z + \frac{1}{2}V\sigma_y\right)\Psi_\pm = E_\pm \Psi_\pm$$

$$|\uparrow\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix}, \quad |\downarrow\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}$$

$$\Psi_\pm = a_\pm|\uparrow\rangle + b_\pm|\downarrow\rangle \ ;$$

$$\frac{1}{2}\hbar\omega\left(a_\pm|\uparrow\rangle - b_\pm|\downarrow\rangle\right) + \frac{1}{2}Vi\left(a_\pm|\downarrow\rangle - b_\pm|\uparrow\rangle\right) = E_\pm\left(a_\pm|\uparrow\rangle + b_\pm|\downarrow\rangle\right)$$

This gives two equations by equating the coefficients at $|\uparrow\rangle$ and at $|\downarrow\rangle$, as these functions are orthogonal

$$a_\pm\left(\frac{1}{2}\hbar\omega - E_\pm\right) - \frac{1}{2}iVb_\pm = 0$$

$$a_\pm\cdot\frac{1}{2}iV + b_\pm\left(-\frac{1}{2}\hbar\omega - E_\pm\right) = 0$$

Determinant $\begin{vmatrix} \frac{1}{2}\hbar\omega - E_\pm & -\frac{1}{2}iV \\ \frac{1}{2}iV & -\frac{1}{2}\hbar\omega - E_\pm \end{vmatrix} = 0 = E_\pm^2 - \left(\frac{1}{2}\hbar\omega\right)^2 - \left(\frac{1}{2}V\right)^2 = 0$

$$\Rightarrow E_\pm = \pm\frac{1}{2}\sqrt{\hbar^2\omega^2 + V^2}$$

$$b_\pm = a_\pm \frac{iV}{\hbar\omega + 2E_\pm} \ ; \quad |a_\pm|^2 + |b_\pm|^2 = 1 \Rightarrow |a_\pm|^2\left(1 + \frac{V^2}{(\hbar\omega + 2E_\pm)^2}\right) = 1$$

$$a_\pm = e^{i\alpha_\pm}\frac{\hbar\omega + 2E_\pm}{\sqrt{V^2 + (\hbar\omega + 2E_\pm)^2}}, \quad b_\pm = ie^{i\alpha_\pm}\frac{V}{\sqrt{V^2 + (\hbar\omega + 2E_\pm)^2}}$$

with an arbitrary $\alpha_\pm$ at this time. We can set $\alpha_\pm = 0$

$$\Psi(t) = A_+\left(a_+|\uparrow\rangle + b_+|\downarrow\rangle\right)e^{-iE_+t/\hbar} + A_-\left(a_-|\uparrow\rangle + b_-|\downarrow\rangle\right)e^{-iE_-t/\hbar}, \ t>0$$

At $t=0$ we have $\Psi(0) = |\uparrow\rangle \Rightarrow A_+a_+ + A_-a_- = 1, \ A_+b_+ + A_-b_- = 0$

$$A_- = -A_+ b_+/b_- = -A_+\frac{\sqrt{V^2 + (\hbar\omega + 2E_-)^2}}{\sqrt{V^2 + (\hbar\omega + 2E_+)^2}} \ ;$$

Notation : $G_\pm = \sqrt{V^2 + (\hbar\omega + 2E_\pm)^2}$

continued

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1 - Continued

With the notation $G_\pm$ we have $a_\pm = \dfrac{\hbar\omega + 2E_\pm}{G_\pm}$,

$$b_\pm = \frac{iV}{G_\pm}, \qquad A_- = -A_+ \frac{G_-}{G_+}$$

$$A_+ a_+ + A_- a_- = A_+ \left(\frac{\hbar\omega + 2E_+}{G_+} - \frac{G_-}{G_+} \frac{\hbar\omega + 2E_-}{G_-} \right) = A_+ \frac{2(E_+ - E_-)}{G_+} = 1$$

$$A_+ = \frac{G_+}{2(E_+ - E_-)} \equiv \frac{G_+}{4E_+}, \qquad A_- = -\frac{G_-}{4E_+}$$

We have $\langle\downarrow|\Psi(t)\rangle = A_+ b_+ e^{-iE_+t/\hbar} + A_- b_- e^{-iE_-t/\hbar}$

$$= \frac{iV}{4E_+} e^{-iE_+t/\hbar} - \frac{iV}{4E_+} e^{-iE_-t/\hbar} = -\frac{V}{2E_+} \sin E_+ t, \quad \text{since } E_- = -E_+$$

Then the population of the state $|\downarrow\rangle$ is

$$P_\downarrow = \frac{V^2}{4E_+^2} \sin^2 E_+ t$$

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2. The Schrödinger equation in free space in spherical coordinates reads:

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2}\frac{\partial^2\psi}{\partial\varphi^2}\right] = E\psi$$

We have

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[(ikr-1)e^{ikr}\right] = \frac{1}{r^2}\left[(-k^2r+ik-ik)e^{ikr}\right]f(\theta)$$

$$= -k^2\frac{e^{ikr}}{r}f(\theta)$$

In contrast, $\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) = \frac{e^{ikr}}{r^3\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{df}{d\theta}$

$$= \frac{e^{ikr}}{r^3\sin\theta}\left[\cos\theta\frac{df}{d\theta} + \sin\theta\frac{d^2f}{d\theta^2}\right] \propto \frac{1}{r^3}$$

Since we consider the limit of large r, the last term can be disregarded compared with the first term, which is $\propto\frac{1}{r}$. Also, obviously, $\frac{\partial\psi}{\partial\varphi}=0$. Then

$$H\frac{e^{ikr}}{r}f(\theta) \simeq \frac{\hbar^2k^2}{2m}\frac{e^{ikr}}{r}f(\theta), \quad E = \frac{\hbar^2k^2}{2m}$$

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3. Following the lecture notes, in the Born approximation

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar}\frac{m^2}{(2\pi\hbar)^3}\left|U_{\vec p\vec p'}\right|^2, \quad U_{\vec p\vec p'} = \int d\vec r\, U(r)\, e^{i(\vec p-\vec p')\vec r/\hbar}$$

For the screened Coulomb potential

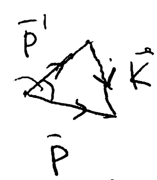
$$U_{\vec p\vec p'} \equiv F(K) = \int d\vec r\, e^{iKr\cos\theta}\, U(r) =$$

$\vec K = (\vec p - \vec p')/\hbar$, choose the z-axis along $\vec K$

$$= \frac{4\pi}{K}\int_0^\infty r\,dr\,\sin Kr\, U(r) = -\frac{4\pi}{K}\frac{Ze^2}{4\pi\varepsilon_0}\int_0^\infty dr\,\sin Kr\, e^{-br}$$

$$= -\frac{2\pi}{Ki}\frac{Ze^2}{4\pi\varepsilon_0}\int_0^\infty dr\, e^{-br}\left(e^{iKr} - e^{-iKr}\right) = -\frac{4\pi}{b^2+K^2}\frac{Ze^2}{4\pi\varepsilon_0}$$

In terms of the scattering angle θ



, $K = \frac{2p}{\hbar}\sin\frac{\theta}{2}$

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar}\frac{m^2}{(2\pi\hbar)^3}\frac{16\pi^2}{\left(b^2 + \frac{4p^2}{\hbar^2}\sin^2\frac{\theta}{2}\right)^2}\cdot\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)^2$$

$$= \left(\frac{mZe^2}{2\pi\varepsilon_0}\right)^2\left(\frac{1}{\hbar^2 b^2 + 4p^2\sin^2\frac{\theta}{2}}\right)^2 \Rightarrow \text{explicitly depends on } \hbar$$

No divergence for $\theta \to 0$, in contrast to the Coulomb scattering

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4. The total scattering cross-section is

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \left(\frac{m Z e^2}{2\pi \varepsilon_0}\right)^2 \cdot 2\pi \int_0^{\pi} \frac{\sin\theta \, d\theta}{\left(\hbar^2 b^2 + 4p^2 \sin^2\frac{\theta}{2}\right)^2}$$

Use $\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta)$, change $x = \cos\theta$

$$\sigma = 2\pi \left(\frac{m Z e^2}{2\pi\varepsilon_0}\right)^2 \int_{-1}^{1} \frac{dx}{\left(\hbar^2 b^2 + 2p^2 - 2p^2 x\right)^2}$$

$$= 2\pi \left(\frac{m Z e^2}{2\pi\varepsilon_0}\right)^2 \frac{1}{2p^2} \left.\frac{1}{\hbar^2 b^2 + 2p^2(1-x)}\right|_{-1}^{1} =$$

$$= 4\pi \left(\frac{m Z e^2}{2\pi\varepsilon_0}\right)^2 \frac{1}{\hbar^2 b^2 (\hbar^2 b^2 + 4p^2)}$$

σ diverges for $b \to 0$

Note that b^{-1} is the typical radius of the potential well. If the particle wavelength $2\pi\hbar/p$ is small compared to b^{-1}, i.e., $p^2 \gg \hbar^2 b^2$, we have

$$\sigma \cong \frac{\pi}{2} \frac{1}{E} \left(\frac{Z e^2}{2\pi\varepsilon_0}\right)^2 \frac{m}{\hbar^2 b^2} \propto \frac{1}{b^2},$$

which makes sense: the cross-section is proportional to the "area" in which the potential is localized

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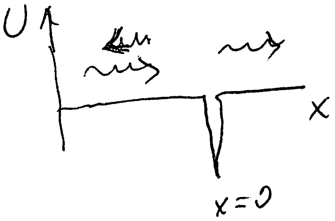
5. The Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi - \alpha\,\delta(x)\,\psi = E\psi$$



Away from $x=0$ the solution is $C_+ e^{ikx}$ or $C_- e^{-ikx}$, $E = \frac{\hbar^2 k^2}{2m}$.

Seek the solution in the form

$$\psi(x) = C\left[\underset{\uparrow\ \text{incident wave}}{e^{ikx}} + \underset{\uparrow\ \text{reflected wave}}{r e^{-ikx}}\right] \quad x<0, \qquad \psi(x) = \underset{\uparrow\ \text{transmitted wave}}{C t e^{ikx}}$$

Continuity: $\psi(x\to -0) = \psi(x\to +0)$, $\underline{1 + r = t}$,

Integrate over a small region centered at $x=0$:

$$-\frac{\hbar^2}{2m}\left[\left(\frac{\partial\psi}{\partial x}\right)_{x\to+0} - \left(\frac{\partial\psi}{\partial x}\right)_{x\to-0}\right] - \alpha\,\psi(0) = 0$$

$$-\frac{\hbar^2}{2m}\left[t\,ik - (ik - rik)\right] \cancel{-t\alpha} = 0 \Rightarrow \quad t + r - 1 = \frac{2mi\alpha}{\hbar^2 k}\,t$$

This gives $t = \frac{1}{1-i\mu}$, $r = \frac{i\mu}{1-i\mu}$, with $\mu = \frac{m\alpha}{\hbar^2 k}$.

The reflection coefficient is $|r|^2 = \frac{\mu^2}{1+\mu^2}$

The transmission coefficient is $|t|^2 = \frac{1}{1+\mu^2}$

Please note that this is a scattering problem! In one dimension you can scatter forward or back ward