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# Problem Set 8

1. Change variables: $Q_1 = \frac{1}{\sqrt{2}}(q_1+q_2)$, $Q_2 = \frac{1}{\sqrt{2}}(q_1-q_2)$

$$\partial_{q_1} = \frac{1}{\sqrt{2}}\left(\partial_{Q_1} + \partial_{Q_2}\right), \quad \partial_{q_2} = \frac{1}{\sqrt{2}}\left(\partial_{Q_1} - \partial_{Q_2}\right)$$

or $P_1 = -i\hbar\,\partial_{Q_1} = -i\hbar\frac{1}{\sqrt{2}}(\partial_{q_1}+\partial_{q_2}) = \frac{1}{\sqrt{2}}(p_1+p_2)$

$$P_2 = -i\hbar\,\partial_{Q_2} = -i\hbar\frac{1}{\sqrt{2}}(\partial_{q_1}-\partial_{q_2}) = \frac{1}{\sqrt{2}}(p_1-p_2)$$

Therefore $[P_1, Q_1] = \frac{1}{2}[p_1+p_2, q_1+q_2] = -i\hbar$

$$[P_2, Q_2] = \frac{1}{2}[p_1-p_2, q_1-q_2] = -i\hbar$$

$$[P_1, Q_2] = \frac{1}{2}[p_1+p_2, q_1-q_2] = 0 = [P_2, Q_1]$$

Thus $(Q_1, P_1)$, $(Q_2, P_2)$ are pairs of conjugate variables.

We have $q_1^2+q_2^2 = Q_1^2+Q_2^2$, $p_1^2+p_2^2 = P_1^2+P_2^2$ - easy to check.

Also, $q_1 q_2 = \frac{1}{4}(q_1+q_2)^2 - \frac{1}{4}(q_1-q_2)^2 = \frac{1}{2}(Q_1^2-Q_2^2)$

Therefore in the variables $Q_{1,2}$, $P_{1,2}$ the hamiltonian reads

$$H = \frac{1}{2m}(P_1^2+P_2^2) + \frac{1}{2}m(\omega^2+V)Q_1^2 + \frac{1}{2}m(\omega^2-V)Q_2^2$$

This is the Hamiltonian of two harmonic oscillators with frequencies $\omega_1 = \sqrt{\omega^2+V}$ and $\omega_2 = \sqrt{\omega^2-V}$.

The energy levels are

$$E(n_1,n_2) = \hbar\omega_1\left(n_1+\frac{1}{2}\right) + \hbar\omega_2\left(n_2+\frac{1}{2}\right)$$

where $n_{1,2} = 0, 1, 2, \ldots$

For small $|V|$ we have $\omega_1 \simeq \omega + \frac{V}{2\omega}$, $\omega_2 \simeq \omega - \frac{V}{2\omega}$

A similar analysis can be done by writing the Hamiltonian in terms of the ladder operators, $H \approx \hbar\omega(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) + \frac{\hbar V}{2\omega}(a_1^\dagger a_2 + a_2^\dagger a_1)$

and writing $a_1 = \frac{1}{\sqrt{2}}(b_1+b_2)$, $a_2 = \frac{1}{\sqrt{2}}(b_1-b_2)$, $[b_i, b_j^\dagger] = \delta_{ij}$

$$H = \hbar\left(\omega+\frac{V}{2\omega}\right)b_1^\dagger b_1 + \hbar\left(\omega-\frac{V}{2\omega}\right)b_2^\dagger b_2$$

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2. The energy of the electron is $H = g\mu_B \vec{s}\cdot\vec{B}$

The Schrödinger equation

$$H\psi = E\psi \;, \qquad \psi = \alpha\chi_+ + \beta\chi_-$$

$$s_z\chi_\pm = \pm\frac{1}{2}\chi_\pm \;, \quad s_x\chi_+ = \frac{1}{2}\chi_- \;, \quad s_x\chi_- = \frac{1}{2}\chi_+$$

Then

$$H\psi = \frac{1}{2}g\mu_B\left[B_x(\alpha\chi_- + \beta\chi_+) + B_z(\alpha\chi_+ - \beta\chi_-)\right] =$$

$$= \frac{1}{2}g\mu_B\left[\chi_+(\alpha B_z + \beta B_x) + \chi_-(-\beta B_z + \alpha B_x)\right]$$

$$= E(\alpha\chi_+ + \beta\chi_-)$$

Multiply by $\chi_+$ or by $\chi_-$, or, using that the functions $\chi_+$ and $\chi_-$ are orthogonal, we obtain

$$\left(\frac{1}{2}g\mu_B B_z - E\right)\alpha + \frac{1}{2}g\mu_B B_x\beta = 0$$

$$\frac{1}{2}g\mu_B B_x\alpha + \left(-\frac{1}{2}g\mu_B B_z - E\right)\beta = 0$$

Then $E^2 - \left(\frac{1}{2}g\mu_B B_z\right)^2 - \left(\frac{1}{2}g\mu_B B_x\right)^2 = 0$, $E_\pm = \pm\frac{1}{2}g\mu_B B$, $B = \sqrt{B_x^2 + B_z^2}$

$\Rightarrow$ as expected

Respectively $\dfrac{\alpha_\pm}{\beta_\pm} = \dfrac{B_x}{\pm B - B_z}$ , $|\alpha_\pm|^2 + |\beta_\pm|^2 = 1$

Orthogonality:

$$\langle \alpha_+\chi_+ + \beta_+\chi_- | \alpha_-\chi_+ + \beta_-\chi_- \rangle = \alpha_+\alpha_- + \beta_+\beta_-$$

$$= \beta_+\beta_-\left[\frac{B_x}{B - B_z}\,\frac{B_x}{-B - B_z} + 1\right] = 0$$

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3. Assume that, for a given $\lambda$, we have solved the Schrödinger equation $H(\lambda)|\psi_n\rangle = E_n(\lambda)|\psi_n\rangle$, with $|\psi_n\rangle \equiv |\psi_n(\lambda)\rangle$.

Consider now $H(\lambda+\delta\lambda) = H(\lambda) + H^{(1)}$, $H^{(1)} = \frac{\partial H}{\partial \lambda}\delta\lambda$.

The first-order correction to the energy is

$$E_n^{(1)} = \langle\psi_n|\frac{\partial H}{\partial\lambda}\delta\lambda|\psi_n\rangle = \delta\lambda\langle\psi_n|\frac{\partial H}{\partial\lambda}|\psi_n\rangle$$

Then the energy is $E_n(\lambda+\delta\lambda) = E_n(\lambda) + E_n^{(1)} = E_n(\lambda) + \frac{\partial E_n}{\partial\lambda}\delta\lambda$

and we have $$\frac{\partial E_n}{\partial\lambda} = \langle\psi_n|\frac{\partial H}{\partial\lambda}|\psi_n\rangle$$

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4. We have $v = \dfrac{pc}{\sqrt{p^2 + m^2c^2}}$. Then $v^2 = \dfrac{p^2c^2}{p^2 + m^2c^2}$

and $p^2 = \dfrac{m^2v^2c^2}{c^2 - v^2} = \dfrac{m^2v^2}{1 - \frac{v^2}{c^2}}$

Then $p^2 + m^2c^2 = \dfrac{m^2c^2}{1 - \frac{v^2}{c^2}}$ and $T = c\sqrt{p^2 + m^2c^2} - mc^2 = \dfrac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2$