PHY 472 - 2020

Problem Set 9

1. We consider the perturbation $H_z = \frac{\mu_B}{\hbar}(2\vec{S}+\vec{L})\cdot\vec{B}$ where we have set $g=2$. We found in class that, upon averaging for a given $\vec{J}$, we have

$$\vec{L}+2\vec{S} \Rightarrow \vec{J} + \frac{\langle\vec{S}\cdot\vec{J}\rangle}{J^2}\vec{J} = \left[1 + \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}\right]\vec{J}$$

$$\equiv g_J(l,s)\vec{J}$$

Then the energy levels are

$$E_{m_j} = \mu_B B\, m_j\, g_J(l,s), \text{ with } m_j = -j, -j+1, \dots, j.$$

- total of $2j+1 = 4$ levels

There are 2 ways to have a single-electron state with $j=\frac{3}{2}$

(a) $l=1$, $j=l+s$; (b) $l=2$, $j=l-s$.

For (a) $g_J(l,s) = \frac{4}{3}$, for (b) $g_J(l,s) = \frac{4}{5}$

PHY 472 - 2020

Problem Set 9

2. We have $H^{(0)} = \hbar\omega\left(a^{+}a + \frac{1}{2}\right)$, $E_n^{(0)} = \hbar\omega\left(n + \frac{1}{2}\right)$, $n = 0, 1, 2, \ldots$

$$H^{(1)} = \beta\left(a^{+}a^{2} + a^{+2}a\right).$$

Since $\langle m|a|n\rangle = \sqrt{n}\,\delta_{m,n-1}$, $\langle m|a^{+}|n\rangle = \sqrt{n+1}\,\delta_{m,n+1}$

we have $\langle n|H^{(1)}|n\rangle = 0$ and the lowest-order corrections are of the second order in $H^{(1)}$.

$$\Delta E_n^{(2)} = \sum_m \frac{|\langle m|H^{(1)}|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \beta^2\left[\frac{\left(\sqrt{n}\,(n-1)\right)^2}{E_n^{(0)} - E_{n-1}^{(0)}} + \frac{\left(n\sqrt{n+1}\right)^2}{E_n^{(0)} - E_{n+1}^{(0)}}\right]$$

from $m = n-1$ from $m = n+1$

$$= \frac{\beta^2}{\hbar\omega}\left[+\,n(n-1)^2 - n^2(n+1)\right] = \frac{\beta^2}{\hbar\omega}\,n(1-3n)$$

Then $\Delta E_0^{(2)} = 0$, $\Delta E_1^{(2)} = -\dfrac{2\beta^2}{\hbar\omega}$

PHY 472 - 2020

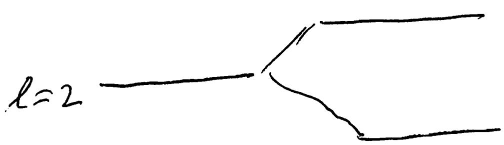
Problem Set 9

3. The energy of a state with given J^2, L^2, S^2 is

$$E_{jls} = E_{so}\left[j(j+1) - l(l+1) - s(s+1) \right],$$

where E_{so} depends on the principal quantum number and, generally, on the value of l.

For $l=2$ we have $j=\frac{5}{2}$ and $j=\frac{3}{2}$

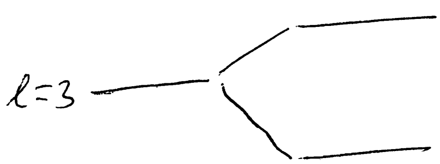


$l=2$

$j=\frac{5}{2}$, $E_{jls} = E_{so} \cdot 2$

$j=\frac{3}{2}$, $E_{jls} = -3E_{so}$

For $l=3$ we have $j=\frac{7}{2}$ and $j=\frac{5}{2}$



$l=3$

$j=\frac{7}{2}$, $E_{jls} = 3E_{so}$

$j=\frac{5}{2}$, $E_{jls} = -4E_{so}$

PHY 472 - 2020

Problem Set 9

4. The wave functions are

$$\psi_{210} = R_{2p}(r)\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$$

$$\psi_{21\pm1} = \mp R_{21}(r)\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{\pm i\varphi}, \qquad R_{21} = \frac{1}{\sqrt{24}}\frac{r}{a^{5/2}}e^{-r/2a}$$

and $\psi_{200} = R_{20}(r)\cdot\left(\frac{1}{4\pi}\right)^{1/2}$, $\quad R_{20} = \frac{1}{\sqrt{2a^3}}\left(1-\frac{r}{2a}\right)e^{-r/2a}$

The perturbation is $H^{(1)} = +eEr\cos\theta$

Clearly, $\langle\psi_{210}|H^{(1)}|\psi_{21\pm1}\rangle = 0$, because $H^{(1)}$ is independent of φ. Similarly, $\langle\psi_{21m}|H^{(1)}|\psi_{21m'}\rangle = 0$ for all m, m' because of the parity. $|\psi_{21m}\rangle$ change sign on inversion, as does also $H^{(1)}$. Therefore the matrix elements must be equal to zero. On the other hand, $\langle\psi_{200}|H^{(1)}|\psi_{21m}\rangle$ is nonzero by parity, but since $\psi_{21\pm1}$ has the factor $\exp(\pm i\varphi)$, the only matrix element that survives is

$$W = \langle\psi_{200}|H^{(1)}|\psi_{210}\rangle = eE\int_0^\infty r^2dr\, r\,\frac{1}{\sqrt{24}}\frac{r}{a^{5/2}}\frac{1}{\sqrt{2a^3}}\left(1-\frac{r}{2a}\right)e^{-r/a}$$

$$\times\int_{-1}^{1}\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta\,\frac{1}{\sqrt{4\pi}}\cos\theta\, d\varphi\, d\cos\theta$$

$$= -3eEa \qquad \left[\int dr = -\frac{9}{\sqrt{3}}a, \quad \int d\Omega = \frac{1}{\sqrt{3}}\right]$$

We see that states $|21\pm1\rangle$ are not changed. The states $|200\rangle$ and $|210\rangle$ are mixed and split into a doublet with the levels shifted by $\pm W$