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Problem Set 1

1. We have $\langle\alpha|(A^\dagger)^\dagger|\beta\rangle = \left(\langle\beta|A^\dagger|\alpha\rangle\right)^* = \left(\langle\alpha|A|\beta\rangle^*\right)^* = \langle\alpha|A|\beta\rangle$

Since $|\alpha\rangle, |\beta\rangle$ are arbitrary, it means $(A^\dagger)^\dagger = A$

Using the definition of the Hermitian conjugate again:

$$\langle\alpha|(AB)^\dagger|\beta\rangle = \left(\langle\beta|AB|\alpha\rangle\right)^* = \underset{\substack{\uparrow \\ \text{any complete set}}}{\sum_\gamma} \left(\langle\beta|A|\gamma\rangle\langle\gamma|B|\alpha\rangle\right)^*$$

$$= \sum_\gamma \langle\alpha|B^\dagger|\gamma\rangle\langle\gamma|A^\dagger|\alpha\rangle = \langle\alpha|B^\dagger A^\dagger|\beta\rangle$$

$\Rightarrow$ thus $(AB)^\dagger = B^\dagger A^\dagger$

We have

$$[A,BC] = ABC - BCA = \underbrace{ABC - BAC}_{[A,B]C} + \underbrace{BAC - BCA}_{B[A,C]}$$

$$= [A,B]C + B[A,C]$$

Similarly, $[AB,C] = ABC - CAB = \underbrace{ABC - ACB}_{A[B,C]} + \underbrace{ACB - CAB}_{[A,C]B}$

$$= A[B,C] + [A,C]B$$

$$[AB,C]^\dagger = (ABC)^\dagger - (CAB)^\dagger = C^\dagger B^\dagger A^\dagger - B^\dagger A^\dagger C^\dagger = [C^\dagger, B^\dagger A^\dagger]$$

$$= B^\dagger[C^\dagger, A^\dagger] + [C^\dagger, B^\dagger]A^\dagger$$

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2. We have $[p_m, r_n] = -i\hbar\delta_{mn}$, where $m,n = 1,2,3$ and

$p_1 = p_x$, $p_2 = p_y$, $p_3 = p_z$ ; $r_1 = x$, $r_2 = y$, $r_3 = z$

$L_x = (\bar{r}\times\bar{p})_x = y p_z - z p_y$, $L_y = (\bar{r}\times\bar{p})_y = z p_x - x p_z$

Then $[p_x, L_y] = [p_x, z p_x - x p_z] = \underbrace{[p_x, z p_x]}_{=0} - [p_x, x p_z]$

$= -[p_x, x] p_z - x \underbrace{[p_x, p_z]}_{=0} = i\hbar p_z$

We have

$[L_x, x] = [y p_z - z p_y, x] = 0 \Rightarrow$ all operators in $L_x$ commute with $x$

$[L_x, y] = [y p_z - z p_y, y] = \underbrace{[y p_z, y]}_{=0} - [z p_y, y]$

$= -z[p_y, y] - \underbrace{[z, y]}_{=0} p_y = i\hbar z$

$[L_x, z] = [y p_z - z p_y, z] = [y p_z, z] - \underbrace{[z p_y, z]}_{=0}$

$= y[p_z, z] + \underbrace{[y, z]}_{=0} p_z = -i\hbar y$

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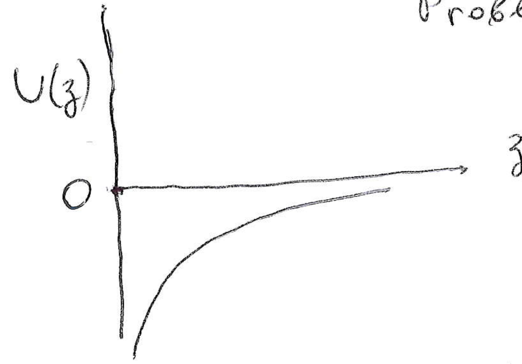
3.



Bound states exist in the range where the full energy $E$ is less than the potential energy $U(z)$ for $z \to \pm\infty$.

This is the range $E < 0$

The Schrödinger equation reads

$$H\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2} + U(z)\psi = E\psi; \quad \psi(0) = 0, \text{ since } U(z) \to \infty,\ z \to 0,\ z < 0$$

For $E < 0$ and $z \to 0$ but $z > 0$ we have

$$\psi(z) \propto z^\alpha; \quad -\frac{\hbar^2}{2m}\alpha(\alpha-1)z^{\alpha-2} - \frac{\Lambda}{z}z^\alpha = E z^\alpha$$

Therefore $\alpha = 1 \Rightarrow$ this is the only way to "kill" the term $\propto z^{\alpha-2}$, which would be much larger than other terms for small $z$. Then $\psi(z) \propto z$ for $z \to 0$.

For $E < 0$ and $z \to \infty$ seek the solution as $\psi(z) = e^{-\gamma z}$.

Then $$-\frac{\hbar^2}{2m}\gamma^2 e^{-\gamma z} - \underbrace{\frac{\Lambda}{z}e^{-\gamma z}}_{\text{small for large } z} = E e^{-\gamma z}$$

Then $\psi(z) \propto \exp\left[-\frac{1}{\hbar}\sqrt{-2mE}\, z\right]$ for $z \to \infty$, $E < 0$

The eigenfunctions are defined for $0 \le z < \infty$. The set of the eigenfunctions of $H$ is incomplete for $-\infty < z < \infty$

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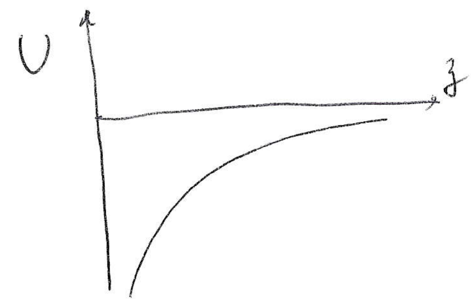
4. 

The Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dz^2} - \frac{\Lambda}{z}\psi(z) = E\,\psi(z)$$

$$\psi(0) = 0$$

Look for eigenfunctions $\psi(z)$ for $E<0$ in the form

$$\psi(z) = \sum_{n\geq 1} b_n z^n \exp\left[-\frac{z}{\hbar}\sqrt{-2mE}\right].$$ Then

$$\psi'(z) = \sum_n \left(n b_n z^{n-1} - b_n z^n \frac{\sqrt{-2mE}}{\hbar}\right) e^{-z\sqrt{-2mE}/\hbar}$$

$$\psi''(z) = \sum_n \left(n(n-1) b_n z^{n-2} - \frac{2n b_n}{\hbar} z^{n-1}\sqrt{-2mE} - \frac{2mE}{\hbar^2} b_n z^n\right) e^{-z\sqrt{-2mE}/\hbar}$$

Plug into the Schrödinger equation, use $\sum_{n=1} n(n-1) b_n z^{n-2} e^{-\gamma z}$
$= \sum_{n=0} (n+1)(n+2) b_{n+2} z^n e^{-\gamma z}$ and similarly for other terms.

This gives, with $\gamma = \sqrt{-2mE}/\hbar$

$$-\frac{\hbar^2}{2m}\sum_{n=0}\left[(n+1)(n+2)b_{n+2} - \frac{2(n+1)b_{n+1}}{\hbar}\sqrt{-2mE}\right] z^n e^{-\gamma z} - \Lambda \sum_n b_{n+1} z^n e^{-\gamma z} = 0$$

Then $$b_{n+2} = b_{n+1}\left[\frac{2}{\hbar(n+2)}\sqrt{-2mE} - \frac{2m\Lambda}{\hbar^2}\frac{1}{(n+1)(n+2)}\right]$$

For $\psi(z)$ being a polynomial times $e^{-\gamma z}$, we have to have the coefficient at $b_{n+1}$ becoming $0$ for some $n$. This gives the energy value in the corresponding eigenstate, $\sqrt{-2mE_{n+1}} = \frac{m\Lambda}{\hbar(n+1)}$

or $E_n = -\frac{R}{n^2}$, $n = 1, 2, \ldots$, $R = \frac{m\Lambda^2}{2\hbar^2}$