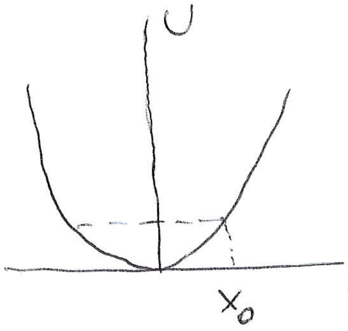
PHY 472 - 2020
Problem Set 2

1. Consider a particle with the Hamiltonian



$$H = \frac{p^2}{2m} + U(x), \quad U(x) = \frac{1}{2} m\omega^2 x^2$$

If the particle is "smeared" over the range $\sim x_0$, its potential energy is $U \sim \frac{1}{2} m\omega^2 x_0^2$. The uncertainty of the momentum is $\Delta p \sim \frac{\hbar}{x_0}$, and thus the kinetic energy is $\frac{p^2}{2m} \sim \frac{\hbar^2}{2m x_0^2}$. Then the total energy is $E \sim \frac{\hbar^2}{2m x_0^2} + \frac{1}{2} m\omega^2 x_0^2$; Minimize over x_0 (or x_0^2 - easier): $-\frac{\hbar^2}{2m (x_0^2)^2} + \frac{1}{2} m\omega^2 = 0$, $x_0^2 = \frac{\hbar}{m\omega}$

and then $E \sim \hbar\omega$. The correct answer is, as you know, $E_{min} = \frac{\hbar\omega}{2}$, but we have obtained a reasonable estimate

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2. We have from $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$:

$$\frac{\partial}{\partial\theta} = z\cos\varphi\frac{\partial}{\partial x} + z\sin\varphi\frac{\partial}{\partial y} - z\tan\theta\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial\varphi} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Use $y\cot\theta = z\sin\varphi$

$x\cot\theta = z\cos\varphi$

Then $L_\pm$ = (from textbook) $\pm\hbar e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\varphi}\right)$

$$= \pm\hbar e^{\pm i\varphi}\Big[z\cos\varphi\frac{\partial}{\partial x} + z\sin\varphi\frac{\partial}{\partial y} - z\tan\theta\frac{\partial}{\partial z}$$

$$\pm i\cot\theta\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\Big]$$

$$= \pm\hbar e^{\pm i\varphi}\Big[\frac{\partial}{\partial x}\underbrace{(z\cos\varphi \mp iz\sin\varphi)}_{z e^{\mp i\varphi}} + \frac{\partial}{\partial y}\underbrace{(z\sin\varphi \pm iz\cos\varphi)}_{= \pm iz e^{\mp i\varphi}} - z\tan\theta\frac{\partial}{\partial z}\Big]$$

On the other hand,

$$L_\pm = L_x \pm iL_y = yp_z - zp_y \pm i(zp_x - xp_z) = -i\hbar\left[\pm iz\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + (y \mp ix)\frac{\partial}{\partial z}\right]$$

Noting that $y \mp ix = z\tan\theta(\sin\varphi \mp i\cos\varphi) = \mp ize^{\pm i\varphi}\tan\theta$, we see that the expressions for $L_\pm$ coincide.

We have $L_-(\sin\theta)^l e^{-il\varphi} = -\hbar e^{-i\varphi}\left[\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right](\sin\theta)^l e^{-il\varphi}$

$$= -\hbar e^{-i\varphi}\left[l(\sin\theta)^{l-1}\cos\theta\, e^{-il\varphi} - i\cot\theta(\sin\theta)^l(-il)e^{-il\varphi}\right]$$

$$= -\hbar e^{-i\varphi} l e^{-il\varphi}\left[(\sin\theta)^{l-1}\cos\theta - \frac{\cos\theta}{\sin\theta}(\sin\theta)^l\right] = 0$$

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3. We have $L^2 = L_+ L_- + L_z^2 - \hbar L_z$ | Using the previous problem, $L_z = -i\hbar\, \partial/\partial\varphi$

$$= -\hbar^2 e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right] e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right) - \hbar^2\frac{\partial^2}{\partial\varphi^2} + i\hbar^2\frac{\partial}{\partial\varphi}$$

$$= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \frac{i}{\sin^2\theta}\frac{\partial}{\partial\varphi} - i\cot\theta\frac{\partial^2}{\partial\theta\partial\varphi} + i\cot\theta\frac{\partial^2}{\partial\varphi\partial\theta} + \cot\theta\frac{\partial}{\partial\theta}\right.$$

$$\left. + \cot^2\theta\frac{\partial^2}{\partial\varphi^2} - i\cot^2\theta\frac{\partial}{\partial\varphi}\right] - \hbar^2\frac{\partial^2}{\partial\varphi^2} + i\hbar^2\frac{\partial}{\partial\varphi}$$

$$= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right] - \hbar^2(\cot^2\theta + 1)\frac{\partial^2}{\partial\varphi^2}$$

$$= -\hbar^2\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{\hbar^2}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

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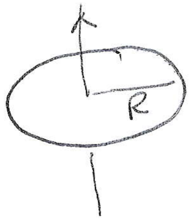
4. The magnetic moment is $\mu = I \cdot A$

I: current; A: area of the loop



$A = \pi R^2$. Current $= \dfrac{\text{charge through crosssection}}{\text{time}}$

Make a cross section of the donut, assume that it spins with angular velocity ω. Period is $\frac{2\pi}{\omega}$. Current is

$$I = \frac{\text{charge}}{\text{time}} = \frac{Q}{2\pi/\omega} = \frac{\omega}{2\pi} Q. \quad \text{Then } \mu = \frac{\omega}{2} Q R^2$$

Angular momentum is $L = M \omega R^2$.

The gyromagnetic ratio is $\frac{\mu}{L} = \frac{Q}{2M}$