PHYSICS 831 (Statistical Mechanics)
Final (Subject) Examination
December 15, 2005
Time: 3 hours (3:00-6:00 pm)

Student Number:

There are five problems. Please use different sheets for each part of each problem. Do not write your name. Write your student number on every page. Please show your work neatly so that partial credits can be given.

$$I_n = \int_{-\infty}^{+\infty} \frac{t^n e^t}{(e^t+1)^2} dt = 0 \text{ for odd } n;\ I_0 = 1, I_2 = \pi^2/3$$

$$I_n = \int_0^{\infty} \frac{x^{n-1}}{e^x - 1} dx = \Gamma(n)\varsigma(n),$$

where $\Gamma(n)$ is the Gamma function and $\varsigma(n)$ is the Riemann zeta function, some of whose values are given below.

$$\varsigma(2) \approx 1.645, \varsigma(3) \approx 1.202, \varsigma(4) \approx 1.082$$
$$\varsigma(5) \approx 1.037, \varsigma(6) \approx 1.017$$

$\Gamma(n) = (n-1)!$; $\int_0^{\infty} e^{-bx^2} dx = \sqrt{\pi}/2b$ $\quad \int_0^{\infty} x^n e^{-\mu x} dx = n!/\mu^{n+1}$

Volume of a n-dimensional sphere of radius R: $V_n(R) = \frac{\pi^{n/2}}{(n/2)!} R^n$

For small y : $Sinh y \approx y + y^3/6 + ...$; $Cosh y \approx 1 + y^2/2 + ...$

For values of the fundamental constants please use the attached sheet.

Table of Values

Quantity	Symbol	Value	CGS	SI
Velocity of light	c	2.997925	10^{10} cm s^{-1}	10^{8} m s^{-1}
Proton charge	e	1.60219	—	10^{-19} C
		4.80325	10^{-10} esu	—
Planck's constant	h	6.62620	10^{-27} erg s	10^{-34} J s
	$\hbar = h/2\pi$	1.05459	10^{-27} erg s	10^{-34} J s
Avogadro's number	N	6.02217×10^{23} mol^{-1}	—	—
Atomic mass unit	amu	1.66053	10^{-24} g	10^{-27} kg
Electron rest mass	m	9.10956	10^{-28} g	10^{-31} kg
Proton rest mass	M_p	1.67261	10^{-24} g	10^{-27} kg
Proton mass/electron mass	M_p/m	1836.1	—	—
Reciprocal fine structure constant $\hbar c/e^2$	$1/\alpha$	137.036	—	—
Electron radius e^2/mc^2	r_e	2.81794	10^{-13} cm	10^{-15} m
Electron Compton wavelength $\hbar/mc$	$\lambda\!\!\!^{-}_e$	3.86159	10^{-11} cm	10^{-13} m
Bohr radius $\hbar^2/me^2$	r_0	5.29177	10^{-9} cm	10^{-11} m
Bohr magneton $e\hbar/2mc$	μ_B	9.27410	10^{-21} erg G^{-1}	10^{-24} J T^{-1}
Rydberg constant $me^4/2\hbar^2$	R_∞ or Ry	2.17991	10^{-11} erg	10^{-18} J
		13.6058 eV		
1 electron volt	eV	1.60219	10^{-12} erg	10^{-19} J
	eV/h	2.41797×10^{14} Hz	—	—
	eV/hc	8.06546	10^{3} cm^{-1}	10^{5} m^{-1}
	eV/k_B	1.16048×10^{4} K	—	—
Boltzmann constant	k_B	1.38062	10^{-16} erg K^{-1}	10^{-23} J K^{-1}
Permittivity of free space	ϵ_0	—	1	$10^{7}/4\pi c^2$
Permeability of free space	μ_0	—	1	$4\pi \times 10^{-7}$

Source: B. N. Taylor, W. H. Parker, and D. N. Langenberg, Rev. Mod. Phys. 41, 375 (1969). See also E. R. Cohen and B. N. Taylor, Journal of Physical and Chemical Reference Data 2(4), 663 (1973).

Problem # 1

(a) The van der Waals gas is described by the equation of state

$$P = nRT/(V-b) - a/V^2.$$

Justify this formula qualitatively in terms of interactions between molecules, and describe what the coefficients a and b correspond to (*exactly*) in terms of molecular properties. Derive formulae for the isothermal compressibility κ_T and coefficient of thermal expansion α in terms of volume V and temperature T and sketch their dependence on V for a given T. (12 points)

(b) Derive a formula for the Joule-Thompson coefficient $(\partial U/\partial V)_T$ in terms of T, P and $(\partial P/\partial T)_V$. Find the Joule-Thompson coefficient for a van der Waals gas. What happens to this coefficient for an ideal gas? (8 points)

Problem # 2

(a) Consider a system with two single-particle orbitals (states) with energies $+\varepsilon_0$ and $-\varepsilon_0$. The system is populated by two non-interacting identical bosons of the same spin S. Find the partition function and average energy/particle in terms of ε_0, S and temperature T. (10 points)
(b) Derive an expression for the Bose Einstein condensation temperature for an ideal gas of non-relativistic ^{4}He atoms (mass M) in terms of the density N/V and M. (10 points)

Problem # 3

Consider an ideal ultra-relativistic two-dimensional electron gas of density $n=N/A=$ $3.0\text{x}10^{11}$ cm^{-2}; the energy of each particle is given by $\varepsilon = pc$, where p is the electron momentum and c is the speed of light.
(a) Find the numerical values of Fermi energy and spreading pressure (force/length) at zero temperature. (10 points)
(b) Show that at low temperature ($k_B T \ll \varepsilon_F$) the chemical potential for fixed density is given by

$$\mu(T) = \varepsilon_F\left(1 + aT^r\right)$$

Find the power r and the coefficient a in terms of the density n and other fundamental constants. State what terms you neglect to get the above power law. (10 points)

Problem # 4

Consider a system of N non-interacting localized quantum spins $\vec{S}_i$ (I = 1, N) in the presence of an external field of strength B in the z-direction and temperature T. The magnetic moment of each spin is given by $\vec{M} = g\mu_B \vec{S}$. Show that the zero-field magnetic susceptibility shows Curie law behaviour

$$\chi = C/T$$

and express the Curie constant C in terms of N, S, $g\mu_B$ (15 points)

Problem # 5

You aspire to make an electron beam that delivers a perfectly constant current. The electrons are emitted from a cathode and accelerated towards the detector that records the current. The electrons are filtered to render the beam absolutely monochromatic (no velocity dispersion). The electrons leave the cathode and travel to the detector independently of one another. You do everything possible to reduce the noise in the detector and beam and yet you notice fluctuations in the current. The fluctuation (noise) increase with increasing mean current but more slowly, so that the relative noise decreases. Interestingly, the noise does *not* depend on temperature.

(a) Identify the physical origin of this noise. (4points)

(b) Give a formula for the magnitude of this fluctuation in terms of the mean current I and the charge of the electron e. (9 points)

(c) Can you measure the charge of the electron using this approach? (2 points)