SECRET STUDENT NUMBER: 58

FUN FACTS TO KNOW AND TELL

$$
\begin{aligned}
I_n &\equiv \int_0^\infty dx\, \frac{x^{n-1}}{e^x-1} = \Gamma(n)\zeta(n), \qquad \int_0^\infty dx\, \frac{x^{n-1}}{e^x+1} = \Gamma(n)\zeta(n)\left[1-(1/2)^{n-1}\right], \\
\zeta(n) &\equiv \sum_{m=1}^{\infty} m^{-n}, \quad \Gamma(n) \equiv (n-1)!, \\
\zeta(3/2) &= 2.612375..., \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) = 1.20205..., \quad \zeta(4) = \frac{\pi^4}{90}, \\
&\int_{-\infty}^{\infty} dx\, e^{-x^2/2} = \sqrt{2\pi}, \qquad \int_0^\infty dx\, x^n e^{-x} = n!
\end{aligned}
$$

LONG ANSWER SECTION

1. (10 pts) Beginning with the fundamental thermodynamic relation,

$$TdS = dE + PdV - \mu dN,$$

derive the Maxwell relation

$$\left.\frac{\partial S}{\partial P}\right|_{T,\mu} = -\left.\frac{\partial V}{\partial T}\right|_{\mu,P}.$$

2. (10 pts) Consider a non-relativistic particle in a one-dimensional potential

$$V(x) = \frac{A}{6}x^6.$$

Using some combination of the equipartition, generalized equipartition and/or virial theorems, find the thermal average $\langle x^6 \rangle$ in terms of T and A.

3. A one-dimensional gas of non-relativisitic spin-1/2 fermions of mass m confined within a length L is thermalized according to a chemical potential μ and temperature T, i.e., the phase space occupancy is

$$f(p) = \frac{e^{-(E_p-\mu)/T}}{1+e^{-(E_p-\mu)/T}}.$$

Originally, the temperature is $T = 0$ and the chemical potential is μ.

(a) (5 pts) In terms of the m, L and μ find the average number of particles N when $T = 0$.

(b) (5 pts) In terms of m and L, find the single-particle density of states D (number of states per energy) as a function of the single-particle energy ϵ.

(c) (10 pts) Assuming μ is held constant while the temperature is slighty raised, find the change in the average number of particles to second order in the temperature. Express your answer in the form,

$$N = N_0 + AT + BT^2,$$

solving for A and B in terms of D and $dD/d\epsilon$ evaluated at the Fermi surface.

SHORT ANSWER SECTION

4. (2 pts each) Consider three single particle levels, $-\epsilon$, 0, and ϵ, which are populated by two indistinguishable spin-zero bosons. Let the system be thermalized at temperature T.

   (a) What is the average total energy when $T = 0$? __________

   (b) What is the total entropy when $T = 0$? __________

   (c) What is the average total energy when $T = \infty$? __________

   (d) What is the total entropy when $T = \infty$? __________

5. (3 pts each) Consider a two-dimensional square lattice of coupled three-dimensional oscillators that supports both longitudinal and transverse modes with the same speed of sound c_s. Assume the lattice is one atom thick in the z direction and infinitely long in the x and y direction. Let C/N refer to the specific heat per oscillator.

   (a) As $T \to 0$, the specific heat from phonons behaves as $C \sim T^n$. What is n? __________

   (b) What is C/N as $T \to \infty$? __________

6. (2 pts each) Consider a one-dimensional Ising model at temperature $T > 0$. Label each of the following as true or false.

   (a) In the exact solution there is no phase transition. __________

   (b) In the mean-field solution there is no phase transition. __________

   (c) In the mean-field solution, the critical exponents are the same as they would be for a two-dimensional model. __________

7. (1 pt each) For the following choose between *maximize or minimze* and between S=*entropy, E=energy, F =Helmoltz free energy, P =pressure, or G = Gibbs free energy.* Circle your choices.

   (a) For a thermalized system at fixed energy, volume and particle number, the system would adjust any order parameter to minimize / maximize the thermodynamic quantity S, E, F, P, G.

   (b) For a thermalized system at fixed volume, temperature, and particle number, the system would adjust any order parameter to minimize / maximize the thermodynamic quantity S, E, F, P, G.

   (c) For a thermalized system at fixed temperature, particle number and pressure, the system would adjust any order parameter to minimize / maximize the thermodynamic quantity S, E, F, P, G.

   (d) For a thermalized system at fixed volume, temperature, and chemical potential, the system would adjust any order parameter to minimize / maximize the thermodynamic quantity S, E, F, P, G.

8. (2 pts each) Pick the appropriate number of dimensions d for each of the following:

   (a) For Bose condensation of a non-relativistic gas to occur, d must be greater than __________.

   (b) According to the Ginzburg criteria, mean field theories will be valid near T_c for d greater than __________.

9. (1 pt each) Assume the free energy density has the form,

$$f(x,T) = \frac{A}{2}m(x)^2 + \frac{B}{4}m(x)^4 + \frac{\kappa}{2}(\nabla m)^2,$$

where $m(x)$ is the magnetization density, and A, B and κ are functions of the temperature T. For each of the following quantities, choose among *ZERO*, *INFINITE* and *FINITE* for what values these quantities approach as $T \to T_c$ in a standard Ginzburg-Landau picture of a phase transition. Circle your answers.

(a) A [*ZERO*, *INFINITE*, *FINITE*].
(b) B [*ZERO*, *INFINITE*, *FINITE*].
(c) κ [*ZERO*, *INFINITE*, *FINITE*].
(d) ξ (the correlation length) [*ZERO*, *INFINITE*, *FINITE*].
(e) $\Delta F/A$ (the surface free energy for the interface between positive and negative m domains below T_c) [*ZERO*, *INFINITE*, *FINITE*].
(f) $\langle F/V \rangle$ (thermal averaged free energy density) [*ZERO*, *INFINITE*, *FINITE*].
(g) C_V (the specific heat) [*ZERO*, *INFINITE*, *FINITE*].

10. (1 pt each) Graph several isotherms on a P vs. V graph illustrating the characteristics of a liquid gas phase transition. The graph should include:

(a) An isotherm with $T > T_c$.
(b) An isotherm with $T = T_c$.
(c) An isotherm with $T < T_c$.
(d) Label the critical point.
(e) For the isotherm with $T < T_c$, label the coexistence points.