FUN FACTS TO KNOW AND TELL

\[ I_n \equiv \int_0^\infty dx \frac{x^{n-1}}{e^x - 1} = \Gamma(n)\zeta(n), \quad \int_0^\infty dx \frac{x^{n-1}}{e^x + 1} = \Gamma(n)\zeta(n) \left[ 1 - (1/2)^{n-1} \right], \]

\[ \zeta(n) \equiv \sum_{m=1}^\infty m^{-n}, \quad \Gamma(n) \equiv (n-1)!, \]

\[ \zeta(3/2) = 2.612375... , \quad \zeta(2) = \frac{\pi^2}{6} , \quad \zeta(3) = 1.20205... , \quad \zeta(4) = \frac{\pi^4}{90} , \]

\[ \int_{-\infty}^\infty dx \ e^{-x^2/2} = \sqrt{2\pi} , \quad \int_0^\infty dx \ x^n e^{-x} = n! \]

LONG ANSWER SECTION

1. (10 pts) Beginning with the fundamental thermodynamic relation,

\[ TdS = dE + PdV - \mu dN , \]

derive the Maxwell relation

\[ \left. \frac{\partial S}{\partial P} \right|_{T,\mu} = - \left. \frac{\partial V}{\partial T} \right|_{\mu,P} . \]
2. (10 pts) Consider a non-relativistic particle in a one-dimensional potential

\[ V(x) = \frac{A}{6} x^6. \]

Using some combination of the equipartition, generalized equipartition and/or virial theorems, find the thermal average \( \langle x^6 \rangle \) in terms of \( T \) and \( A \).
3. A one-dimensional gas of non-relativistic spin-1/2 fermions of mass $m$ confined within a length $L$ is thermalized according to a chemical potential $\mu$ and temperature $T$, i.e., the phase space occupancy is

$$f(p) = \frac{e^{-(E_p-\mu)/T}}{1 + e^{-(E_p-\mu)/T}}.$$  

Originally, the temperature is $T = 0$ and the chemical potential is $\mu$.

(a) (5 pts) In terms of the $m$, $L$ and $\mu$ find the average number of particles $N$ when $T = 0$.

(b) (5 pts) In terms of $m$ and $L$, find the single-particle density of states $D$ (number of states per energy) as a function of the single-particle energy $\epsilon$.

(c) (10 pts) Assuming $\mu$ is held constant while the temperature is slightly raised, find the change in the average number of particles to second order in the temperature. Express your answer in the form,

$$N = N_0 + AT + BT^2,$$

solving for $A$ and $B$ in terms of $D$ and $dD/d\epsilon$ evaluated at the Fermi surface.
SHORT ANSWER SECTION

4. (2 pts each) Consider three single particle levels, $-\epsilon$, 0, and $\epsilon$, which are populated by two indistinguishable spin-zero bosons. Let the system be thermalized at temperature $T$.

(a) What is the average total energy when $T = 0$? 

(b) What is the total entropy when $T = 0$? 

(c) What is the average total energy when $T = \infty$? 

(d) What is the total entropy when $T = \infty$? 

5. (3 pts each) Consider a two-dimensional square lattice of coupled three-dimensional oscillators that supports both longitudinal and transverse modes with the same speed of sound $c_s$. Assume the lattice is one atom thick in the $z$ direction and infinitely long in the $x$ and $y$ direction. Let $C/N$ refer to the specific heat per oscillator.

(a) As $T \to 0$, the specific heat from phonons behaves as $C \sim T^n$. What is $n$? 

(b) What is $C/N$ as $T \to \infty$? 

6. (2 pts each) Consider a one-dimensional Ising model at temperature $T > 0$. Label each of the following as true or false.

(a) In the exact solution there is no phase transition. 

(b) In the mean-field solution there is no phase transition. 

(c) In the mean-field solution, the critical exponents are the same as they would be for a two-dimensional model. 

7. (1 pt each) For the following choose between maximize or minimize and between $S=entropy$, $E=energy$, $F=Helmholtz$ free energy, $P=pressure$, or $G=Gibbs$ free energy. Circle your choices.

(a) For a thermalized system at fixed energy, volume and particle number, the system would adjust any order parameter to **minimize** / **maximize** the thermodynamic quantity $S$, $E$, $F$, $P$, $G$. 

(b) For a thermalized system at fixed volume, temperature, and particle number, the system would adjust any order parameter to **minimize** / **maximize** the thermodynamic quantity $S$, $E$, $F$, $P$, $G$. 

(c) For a thermalized system at fixed temperature, particle number and pressure, the system would adjust any order parameter to **minimize** / **maximize** the thermodynamic quantity $S$, $E$, $F$, $P$, $G$. 

(d) For a thermalized system at fixed volume, temperature, and chemical potential, the system would adjust any order parameter to **minimize** / **maximize** the thermodynamic quantity $S$, $E$, $F$, $P$, $G$. 

8. (2 pts each) Pick the appropriate number of dimensions $d$ for each of the following:

(a) For Bose condensation of a non-relativistic gas to occur, $d$ must be greater than __________.

(b) According to the Ginzburg criteria, mean field theories will be valid near $T_c$ for $d$ greater than __________.
9. (1 pt each) Assume the free energy density has the form,

\[ f(x, T) = \frac{A}{2} m(x)^2 + \frac{B}{4} m(x)^4 + \frac{\kappa}{2} (\nabla m)^2, \]

where \( m(x) \) is the magnetization density, and \( A, B \) and \( \kappa \) are functions of the temperature \( T \). For each of the following quantities, choose among \text{ZERO}, \text{INFINITE} and \text{FINITE} for what values these quantities approach as \( T \to T_c \) in a standard Ginzburg-Landau picture of a phase transition. Circle your answers.

(a) \( A \) [ \text{ZERO}, \text{INFINITE}, \text{FINITE} ].

(b) \( B \) [ \text{ZERO}, \text{INFINITE}, \text{FINITE} ].

(c) \( \kappa \) [ \text{ZERO}, \text{INFINITE}, \text{FINITE} ].

(d) \( \xi \) (the correlation length) [ \text{ZERO}, \text{INFINITE}, \text{FINITE} ].

(e) \( \Delta F/A \) (the surface free energy for the interface between positive and negative \( m \) domains below \( T_c \)) [ \text{ZERO}, \text{INFINITE}, \text{FINITE} ].

(f) \( \langle F/\mathcal{V} \rangle \) (thermal averaged free energy density) [ \text{ZERO}, \text{INFINITE}, \text{FINITE} ].

(g) \( C_V \) (the specific heat) [ \text{ZERO}, \text{INFINITE}, \text{FINITE} ].

10. (1 pt each) Graph several isotherms on a \( P \) vs. \( V \) graph illustrating the characteristics of a liquid gas phase transition. The graph should include:

(a) An isotherm with \( T > T_c \).

(b) An isotherm with \( T = T_c \).

(c) An isotherm with \( T < T_c \).

(d) Label the critical point.

(e) For the isotherm with \( T < T_c \), label the coexistence points.