DO NOT WRITE YOUR NAME OR STUDENT NUMBER ON ANY SHEET!

**FUN FACTS TO KNOW AND TELL**

\[
\int_0^\infty dx \frac{x^{n-1}}{e^x - 1} = \Gamma(n)\zeta(n) \quad \int_0^\infty dx \frac{x^{n-1}}{e^x + 1} = \Gamma(n)\zeta(n) \left[ 1 - (1 - 2)^{n-1} \right]
\]

\[\zeta(n) = \sum_{m=1}^\infty m^{-n} \quad \Gamma(n) \equiv (n - 1)!
\]

\[\zeta(2) = 2.612375 \quad (\zeta(2) = \frac{\pi^2}{6}) \quad \zeta(3) = 1.20205 \quad \zeta(4) = \frac{\pi^4}{90}
\]

\[
\int_{-\infty}^{\infty} dx \ e^{-x^2} = \sqrt{2\pi} \quad \int_0^{\infty} dx \ x^n e^{-x} = n!
\]
1. (10 pts) Consider two single-particle energy levels, 0 and $\epsilon$. Spin-1 bosons ($m = -1 \ 0 \ 1$) are allowed to populate the levels and equilibrate with a heat and particle bath defined by a temperature $T$ and chemical potential $\mu < 0$. The bosons are indistinguishable aside from their spin. What is the average number of bosons in each level?
2. (10 pts) Assume that the free energy in a two-dimensional system obeys the following form,

\[ F = \int d^2r \left\{ \frac{A}{2} \phi^2 + \frac{C}{2} \phi^4 \right\} \]

Assuming that near \( T_c \), \( A \sim \alpha t \), and the critical exponent in mean field theory \( \beta \) where,

\[ \langle \phi \rangle \sim t^\beta \]

below \( T_c \).
3. $N$ ink molecules are placed in a liquid at a time $t = 0$ and diffuse according to a diffusion constant $D$, i.e., the density of molecules satisfies the diffusion equation,

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}$$

For example, if the $N$ molecules are initially positioned at $x = 0$ in a translationally-invariant medium, the density evolves as,

$$\rho(x, t) = \frac{N}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}$$

(a) (10 pts) Now, add an absorptive boundary at $x = 0$, and place the drop at a small distance $a$ from the boundary. By small we will only consider times such that $2Dt >> a^2$. Solve for the density $\rho(x, t)$. You should include only the lowest order in $a$.

(b) (5 pts) What fraction of molecules survive to time $t$? Again assume $2Dt >> a^2$. 
4. Suppose the average energy $E$ and the average number of particles $N$ in a one-dimensional system of extent $L$ are given as a function of $T$, $L$ and $\alpha \equiv -\mu \ T$. Further assume that $L$ is much larger than any microscopic scale or correlation length of the system.

(a) (10 pts) Derive an expression for the specific heat per unit length,

$$C \equiv \frac{1}{L} \frac{\partial E}{\partial T} \bigg|_N$$

in terms of $T \ L \ E \ N \ \partial_T E \ \partial_\alpha E \ T$ and $\partial_\alpha N \ T$.

(b) (10 pts) Assume the correlations in the system are sufficiently local they can be expressed in terms of delta functions,

$$\langle \Delta \rho(0) \Delta \rho(x) \rangle_{\alpha T} = A \delta(x)$$
$$\langle \Delta \epsilon(0) \Delta \epsilon(x) \rangle_{\alpha T} = B \delta(x)$$
$$\langle \Delta \epsilon(0) \Delta \rho(x) \rangle_{\alpha T} = D \delta(x)$$

where $\epsilon$ and $\rho$ are the energy density and number density respectively. Express $C$ in terms of $T$, $\alpha$, $A$, $B$ and $D$. 
SHORT ANSWER SECTION

5. (1 pt each) Graph several isotherms on a $P$ vs. $V$ graph illustrating the characteristics of a liquid gas phase transition. The graph should include:

(a) An isotherm with $T > T_c$.
(b) An isotherm with $T = T_c$.
(c) An isotherm with $T < T_c$.
(d) Label the critical point.
(e) For the isotherm with $T < T_c$, label the coexistence points.

6. (2 pts each) Consider a one-dimensional Ising model. Label each of the following statements as true or false.

(a) In the exact solution there is no phase transition. 
   __________

(b) In the mean-eld solution there is no phase transition. 
   __________

(c) In the mean-eld solution, the critical exponents are the same for the one-dimensional and two-dimensional solutions. 
   __________
9. (3 pts) One might expect a Goldstone boson from a phase transition with: (circle one)

- spontaneous breaking of a continuous symmetry
- spontaneous breaking of a discrete symmetry
- explicit breaking of a continuous symmetry
- explicit breaking of a discrete symmetry

10. (2 pts) For a system of massless bosons, $E = cp$, what dimensionality, $D$, is required for Bose condensation?
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\[ \int_0^\infty dx \, e^{-x^2} = \frac{\sqrt{\pi}}{2} \quad \int_0^\infty dx \, x^n e^{-x} = n! \]