March 18

Induction and Inductance
Chapter 31
Review - Self Inductance

> Self-induce emf, $\mathcal{E}_L$ appears in any coil in which the current is changing

$$\mathcal{E}_L = -L \frac{di}{dt}$$

> Direction of $\mathcal{E}_L$ follows Lenz’s law and opposes the change in current
What is induced emf in coil 1 from a changing current in coil 2?

\[ \mathcal{E}_1 = -M \frac{di_2}{dt} \]

where

\[ M = \frac{N_1 \Phi_{12}}{i_2} = \frac{N_2 \Phi_{21}}{i_1} \]
Review: RC circuit

> **RC circuit** is a resistor and capacitor in series

- Charging up a capacitor (switch at a)
- Kirchhoff rule for loop:

\[
\frac{dq}{dt} + \frac{q}{C} - \mathcal{E} = 0
\]

(=differential equation)

- Solution

\[
q = C \mathcal{E} (1 - e^{-t/\tau_c})
\]

where \( \tau_c = RC \)

- Discharging capacitor (switch at b)

\[
q = q_0 e^{-t/\tau_c}
\]
Inductance

- **RL circuit** is a resistor and inductor in series
- Close switch to point a
  - **Initially** $i$ is increasing through inductor so $\mathcal{E}_L$ opposes rise and $i$ through $R$ will be
    \[
    i < \frac{\mathcal{E}}{R}
    \]
  - **Long time later**, $i$ is constant so $\mathcal{E}_L = 0$ and $i$ in circuit is
    \[
    i = \frac{\mathcal{E}}{R}
    \]
RL Circuit - Differential Equation

> Initially an inductor acts to oppose changes in current through it
> Long time later inductor acts like ordinary conducting wire
> Apply Kirchhoff loop rule right after switch has been closed at \( a \)

\[
\mathcal{E} - iR - L\frac{di}{dt} = 0
\]
RL Circuit Solution

> Differential equation ....... similar to RC circuit:

\[
L \frac{di}{dt} + iR - E = 0
\]

\[
\frac{dq}{dt} R + \frac{q}{C} - E = 0
\]

> Solution is

\[
i = \frac{E}{R} \left(1 - e^{-t/\tau_L}\right)
\]

> Inductive time constant is

\[
\tau_L = \frac{L}{R}
\]

> Satisfies conditions:
  - At \( t=0 \), \( i = 0 \)
  - At \( t=\infty \), \( i = \frac{E}{R} \)
“Discharging” = Stop Current

> Now move switch to position b so battery is out of system
> Current will decrease with time and loop rule gives

\[ iR + L \frac{di}{dt} = 0 \]

> Solution is

\[ i = \frac{\mathcal{E}}{R} e^{-t/\tau_L} = i_0 e^{-t/\tau_L} \]

> Satisfies conditions

- At \( t=0 \), \( i = i_0 = \frac{\mathcal{E}}{R} \)
- At \( t=\infty \), \( i = 0 \)
RL circuits Summary

- Circuit is closed (switch to “a”)
  \[ i = \frac{E}{R} \left( 1 - e^{-t/\tau_L} \right) \]

- Circuit is opened (switch to “b”)
  \[ i = \frac{E}{R} e^{-t/\tau_L} = i_0 e^{-t/\tau_L} \]

- Time constant is
  \[ \tau_L = \frac{L}{R} \]

Switch at a, current through inductor is:

- Initially \( i = 0 \) (acts like broken wire)
- Long time later \( i = \frac{E}{R} \) (acts like simple wire)
Problem 31-5

> Have a circuit with resistors and inductors
> What is the current through the battery just after closing the switch?

> Inductor oppose change in current through it
> Right after switch is closed, current through inductor is 0
> Inductor acts like broken wire
Problem 31-5, continued

> Apply loop rule

\[ E - iR = 0 \]

> Immediately after switch closed, current through the battery is

\[ i = \frac{E}{R} \]
Problem 31-5, continued

> What is the current through the battery a long time after the switch has been closed?

> Currents in circuit have reached equilibrium so inductor acts like simple wire

> Circuit is 3 resistors in parallel

\[
i = \frac{E}{R_{eq}} \quad R_{eq} = \frac{R}{3}
\]

Remember the “water slides”
Energy

> How much energy is store in a $B$ field?

> Conservation of energy expressed in loop rule

\[ \mathcal{E} = L \frac{di}{dt} + iR \]

> Multiply each side by $i$

\[ \mathcal{E}i = Li \frac{di}{dt} + i^2 R \]

> $P = i \mathcal{E}$ is the rate at which the battery delivers energy to rest of circuit

> $P = i^2 R$ is the rate at which energy appears as thermal energy in resistor
Energy (2)

> Middle term is rate at which energy $\frac{dU_B}{dt}$ is stored in the $B$ field

$$\frac{dU_B}{dt} = L_i \frac{di}{dt}$$

> Energy stored in magnetic field

$$U_B = \frac{1}{2} L_i^2$$

> Similar to energy stored in electric field

$$U_E = \frac{1}{2} \frac{q^2}{C}$$
Energy Density

- What is the energy density of a $B$ field?

- Energy density, $u_B$ is energy per unit volume

\[ u_B = \frac{U_B}{Al} \]

- Magnetic energy density

\[ u_B = \frac{1}{2} \frac{B^2}{\mu_0} \]

- Similar to electric energy density

\[ u_E = \frac{1}{2} \varepsilon_0 E^2 \]
Where does this come from?

\[ B = \mu_0 ni \]

\[ L = l\mu_0 n^2 A \Rightarrow \]

\[ U_B = \frac{1}{2} i^2 L = \frac{1}{2} l\mu_0 n^2 i^2 A = \frac{1}{2} Al(n^2 i^2 \mu_0^2) / \mu_0 \]

\[ U_B / Al = \frac{1}{2} B^2 / \mu_0 \]