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Bose-enhancement and Pauli-blocking effects in transport models

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Abstract

The ability of semi-classical transport models to correctly simulate Pauli blocking and Bose enhancement is discussed. In the context of simple quantum mechanical systems, it is shown that using $(1 \pm f)$ enhancements is inadequate to describe systems far from equilibrium. The relaxation of $(1+f)$ descriptions toward equilibrium is studied both in the context of simple models, including a hot pion gas confined to a box. We give simple estimates for the characteristic number of collisions for a system to relax toward equilibrium.

Semi-classical simulations form the backbone of the phenomenology for high-energy and intermediate-energy heavy-ion physics. The semi-classical nature of such descriptions is justified if Compton wave lengths are smaller than the size of the reaction volume, and if phase space filling factors, f, are much less than unity. The first condition is generally fulfilled whenever incident beam energies exceed $\approx 40 \cdot A$ MeV. The second condition is violated for lower energy collisions by nucleons, as excitation energies are less than the Fermi energy until beam energies exceed $\approx 100 \cdot A$ MeV. For relativistic heavy-ion collisions, $E/A > 10$ GeV, phase-space-filling factors for pions also approach unity. For these two regimes, quantum statistics must be incorporated into simulations.

Nordheim [1] first pointed out that modifying the Boltzmann transport equation by incorporating phase-space occupation factors $(1 \pm f)$ into the collision integral, showing the phase space distribution functions have the Bose-Einstein or Fermi-Dirac distributions as their respective equilibrium solutions. The usual way to accomplish this is by modifying the cross sections from their in-vacuum values $\sigma^{\rm vac}_{ab\to cd}$ to the in-medium value [2–5]:

$$\sigma^{\rm med}_{ab\to cd} = \sigma^{\rm vac}_{ab\to cd} \cdot (1 \pm f_c)(1 \pm f_d). \tag{1}$$

Given a system with measured occupations, a scattering will choose its final state according to the weights described above. This problem was originally cast in terms of the Einstein A_{nm} and B_{nm} coefficients [6], which is the correct treatment given measurements are made after each emission. However, when measurements are not made between individual scatterings; correct treatment of Bose and Fermi statistics requires the incorporation of many-body effects. Our principal objective here is to compare models where full n-body symmetrization has been accounted for with the $(1 \pm f)$ method.

We start by examining the Einstein problem: If in free space the rate at which quanta are scattered into a cell k is α_k and the rate at which quanta are scattered out of the cell is $\beta_k f_k$, the rate equations for the phase-space occupation f_k in and out of the medium are

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given by

$$\frac{df_k^{\rm vac}}{dt} = \alpha_k - f_k^{\rm vac}\beta_k, \quad (2)$$

$$\frac{df_k^{\rm med}}{dt} = \alpha_k(1 \pm f_k^{\rm med}) - f_k^{\rm med}\beta_k. \quad (3)$$

This results in equilibrium values of f_k,

$$f_k^{\rm vac} = \frac{\alpha_k}{\beta_k}, \quad (4)$$

$$f_k^{\rm med} = \frac{\alpha_k/\beta_k}{1 \mp \alpha_k/\beta_k}. \quad (5)$$

If the free space values of α_k and β_k result in a Boltzmann factor, $\alpha_k/\beta_k = \exp(-(E_k - \mu)/T)$, the in-medium result is a Bose-Einstein or Fermi-Dirac form. Thus one can be confident that a simulation that incorporates $(1 \pm f)$ corrections will ultimately obtain correct equilibrium behavior given sufficient time such that gain and loss terms balance.

In a relativistic heavy-ion collision ($E/A \approx 200$ GeV) the reaction can be viewed as having two stages. During the creation stage pion production takes place, while during the reinteraction stage expansion and cooling occur, during which the pion number stays roughly constant. We divide our model into two stages correspondingly. For the first stage we consider only production. Given the large energies of the colliding nucleons, reformation of the energetic initial states is negligible and one can consider only formation processes. In the absence of Bose effects, we assume the probability of having created n pions is Poissonian:

$$P_k^0(n,t) = \exp(-\eta_k(t))\frac{\eta_k(t)^n}{n!}. \quad (6)$$

Fully including symmetrization, the probability of creating n pions is enhanced by a factor of $n!$ assuming the time t is after emission has stopped and that no measurements were made at intermediate times. This enhancement can be considered as either the contribution of interference terms or as the amplitude of a matrix element with n identical Boson creation operators, $\langle 0|a^n a^{\dagger n}|0\rangle$,

$$P_k^{\rm full}(n,t) = \frac{1}{Z_k}P_k^0(n,t)n! = (1-\eta_k(t))\eta_k(t)^n \quad (7)$$

where Z_k is a normalization.

If the state is measured at each intermediate time t during the emission, $(1 \pm f)$ methods are justified. The probability of seeing n particles at time t, $P_k^{(1+f)}(n,t)$ is then determined by the equation

$$\frac{d}{dt}P_k^{(1+f)}(n,t) = \frac{d\eta_k(t)}{dt}\left\{n \cdot P_k^{(1+f)}(n,t) - (1+n)\cdot P_k^{(1+f)}(n+1,t)\right\}, \quad (8)$$

with the solution

$$P_k^{(1+f)}(n,t) = (1-\gamma_k(t))\gamma_k(t)^n,$$
$$\gamma_k(t) = 1 - \exp(-\eta_k(t)). \quad (9)$$

Simulating systems populated according to $(1+f)$ weights results in dendritic growth of phase-space occupation. The first few particles can bias the behavior of all the following particles, leading to large fluctuations in the populations. Whereas, when no measurement is made until the end of the emission, all of the emissions are affected equally by the symmetrization. The average occupation is lower for the $(1+f)$ case, Eq. (8), than for the case where full n-body symmetrization weighting is applied at the end of the emission as in Eq. (7), but certainly higher than for the case where symmetrization is neglected, $\langle n_k^0\rangle$.

$$\langle n_k^0\rangle = \eta_k(t=\infty),$$
$$\langle n_k^{(1+f)}\rangle = \frac{\gamma_k(t=\infty)}{1-\gamma_k(t=\infty)},$$
$$\langle n_k^{\rm full}\rangle = \frac{\eta_k(t=\infty)}{1-\eta_k(t=\infty)}. \quad (10)$$

If the function η_k has a thermal form, $\eta_k = \exp(-(E_k - \mu)/T)$, the full n-body result corresponds to the thermal Bose-Einstein form. The additional created particles account for the difference of the Bose-Einstein form as compared to the Boltzmann form for the spectra. For the $(1+f)$ case, the population of the different levels of energy E_k can not be described by one temperature and chemical potential. Fig. 1(a) illustrates the three solutions for the mean occupancy. There is more than just a quantitative difference. The $(1+f)$ result never diverges as compared to the full n-body case which results in an infinite occupancy when η_k exceeds unity. The $(1+f)$ result even differs to the second power of η_k.

One can solve the same models for fermions. In this case $(1+f)$ terms are replaced by $(1-f)$ terms

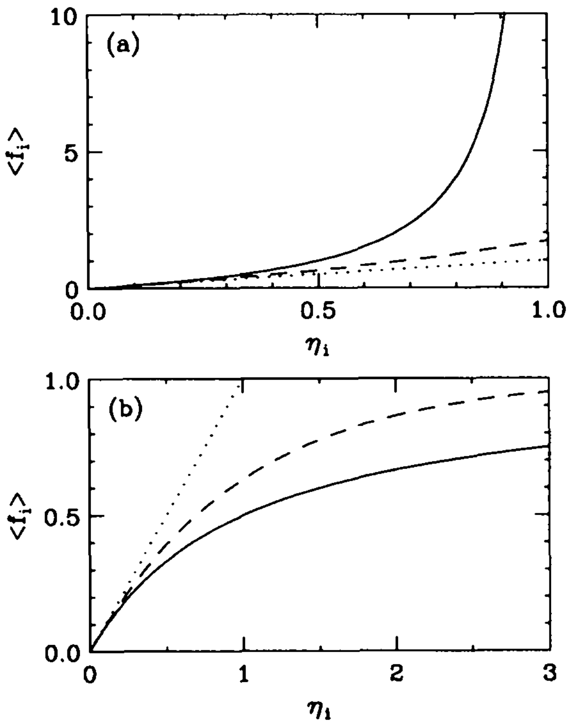

Fig. 1. a: Average population of a Bosonic state as a function of the Poissonian parameter η for the cases of neglected symmetrization (dotted line), $(1+f)$-enhancement (dashed line), and full n-body enhancement (solid line). b: Same comparison for a Fermi gas; $(1-f)$ suppression (dashed line), full n-body suppression (solid line).

in Eq. (8) and the $n!$ in Eq. (7) is replace by zero for $n > 1$. The results look very similar for the mean occupations:

$$\langle n_k^{(1-f)} \rangle = \gamma_k(t=\infty),$$
$$\langle n_k^{\mathrm{full}} \rangle = \frac{\eta_k(t=\infty)}{1+\eta_k(t=\infty)}. \tag{11}$$

The difference is illustrated in Fig. 1(b) which shows that $(1-f)$ models can modestly underestimate Pauli-blocking effects.

Even if $(1+f)$ models grossly underestimate initial populations of low-energy states, rescattering effects can lead to equilibrium. If sufficient scattering occurs to produce equilibrium, the only fault of the $(1+f)$ models will lie in the prediction of the overall multiplicity. For this reason we wish to investigate the amount of scattering required to produce equilibrium.

We consider a closed system with N_l levels labeled by k where each level has a degeneracy d_k and base weights η_k. For initial conditions we fill the levels with N particles according to the $(1+f)$ prescription, neglecting any loss terms. Collisions are modeled by randomly removing one of the N particles and placing it into a level according to the weight $d_k(1+n_k)\eta_k$. After sufficient collisions populations approach equilibrium.

The equilibrium population for fixed N systems can be calculated by using diagrammatic methods [8]. Although these method have only been used to study multi-particle symmetrization of outgoing states, they can also be used to calculate equilibrium populations. Ref. [8] describes how to calculate the probability of emitting N particles in states $k_1 \cdots k_N$ given the source function $S(p,x)$, where $S(p,x)$ is the probability of emitting a particle from space-time point x into final state p neglecting symmetrization. Under the assumption that the momentum dependence in S is not correlated to the space-time dependence, $S(p,x) = S(x)\eta(p)$ the expression in Ref. [8] becomes

$$P(k_1,\cdots k_N) = \frac{1}{Z}\int dx_1 \cdots dx_N \eta_{k_1} \cdots \eta_{k_N} \times S(x_1)\cdots S(x_N)|U(x_1,\cdots x_n;k_1,\cdots k_N)|^2, \tag{12}$$

where U is the evolution matrix from the creation of pions at x to their measurement in states k. If we assume that the system is finite with discreet eigenstates, and if we assume that there is no x dependence aside from confining the region of emission, the expression above becomes

$$P(k_1,\cdots k_N) = \frac{1}{Z}\int dx_1 \cdots dx_N \eta_{k_1} \cdots \eta_{k_N} \times |\langle x_1,\cdots x_n|k_1,\cdots k_N\rangle|^2. \tag{13}$$

The squared quantity is the squared n-particle wave function with $N!$ terms. The same equation applies for an open (non-bounded) system if the emission occurs all at one time, except that the labels refer to continuum states rather than discreet eigenstates. By making the substitution $\eta_i = \exp(-E_i/T)$ one sees that Eq. (13) is equivalent to a thermal trace, hence it is the equilibrium answer.

Methods for accounting for all $N!$ terms in the outgoing wave function were devised in Ref. [8]. Translating the method of Ref. [8] to the case of discreet states results in the prescription below for calculating $\langle n_k \rangle$ given η_k and N. First define quantities $G_n(k)$, C_n and $\omega(n)$:

$$G_n(k) \equiv (\pm 1)^{n-1} d_k \eta_k^n,$$

$$C_n(k) \equiv \frac{1}{n} \sum_k G_n(k),$$

$$\omega(n) \equiv \sum_{i_1 \cdots i_n} \prod \frac{C_1^{i_1}}{i_1!} \frac{C_2^{i_2}}{i_2!} \cdots \frac{C_n^{i_n}}{i_n!}. \tag{14}$$

The sum over $i_1 \cdots i_N$ is constrained such that the overall order is N, $i_1 + 2i_2 + 3i_3 + \cdots = N$. An algorithm for calculating the sum in the expression of $\omega(n)$ is also given in Ref. [8]. The expression for the equilibrium occupations is

$$\langle n_k \rangle_{\text{equil}} = [G_1(k)\omega(n-1) + G_2(k)\omega(n-2) + \cdots + G_{n-1}(k)\omega(1) + G_n(k)][\omega(N)]^{-1}. \tag{15}$$

Now we return to the evolution of $\langle n_k \rangle$ averaged over many samplings. Fig. 2(a) shows the occupation as a function of the number of collisions for the case where there are 8 levels with filling weights $\eta_i = \exp(-k \cdot \Delta E/T)$, with $\Delta E = T/8$, and 40 particles. Given the large number of particles in a limited phase space one can easily see that the initial conditions underpopulate the ground state, but that a sufficient number of collisions restores equilibrium. The dashed lines in Fig. 2(a) represent the equilibrium occupancies calculated above. In order to make a detailed study of the approach to equilibrium Fig. 2(b) shows the discrepancies from equilibrium as a function of the number of collisions on a logarithmic scale. This allows the extraction of characteristic scales for the exponential approach to equilibrium.

One striking aspect of Fig. 2(b) is that as one approaches equilibrium the lines become parallel. The inverse slope of this line corresponds to the characteristic number of collisions for approaching equilibrium τ_d. In order to understand how the parameters (level density, temperature, ground-state population) determine τ_d, we linearize the evolution equation and treat f_k as a continuous variable. This allows us to quickly find τ_d for a variety of systems, which will lend insight into what characteristics determine τ_d. Ignoring the discreet nature of f_k,

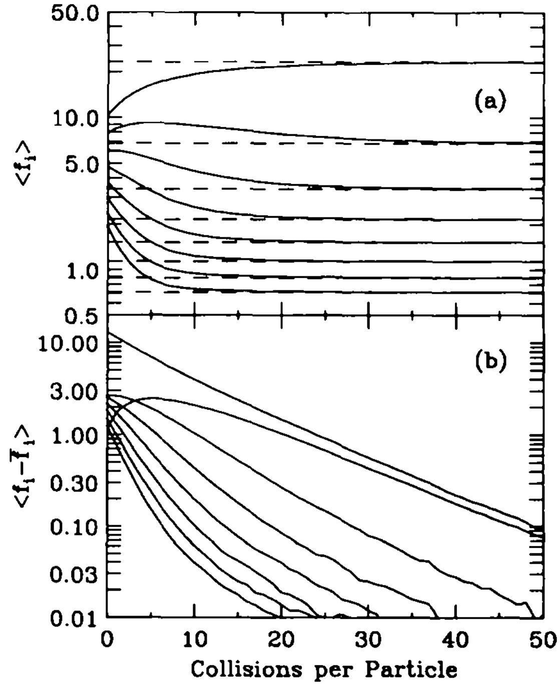

Fig. 2. Eight level system with 40 particles and level spacing $\Delta E = T/8$. a: Average populations of the eight levels as a function of the number of collisions per particle. Dashed lines: exact equilibrium values computed from Eq. 15. Solid lines: $(1+f)$-type simulations, averaged over 3×10^6 events. b: Deviation of the $(1+f)$-type simulations from equilibrium populations. After many collisions the lines become parallel. Statistical noise causes the ragged behavior for small deviations.

$$\frac{df_k}{d\tau} = \frac{(1+f_k)\eta_k}{Z} - \frac{f_k}{N}, \tag{16}$$

$$Z = N \sum (1+f_k)\eta_k. \tag{17}$$

The normalization factor Z which is a function of the distribution f_k, ensures that particle number is conserved. For small deviations from equilibrium, one can expand Z keeping the lowest order terms in $\delta f_k = f_k - f_k^{\text{eq}}$ to obtain

$$\frac{d\delta f_k}{d\tau} = \frac{-\delta f_k}{1+f_k^{\text{eq}}} + \frac{f_k^{\text{eq}}}{N} \sum_j \frac{\delta f_i}{1+f_j^{\text{eq}}}. \tag{18}$$

One can solve the N coupled differential equations with matrix methods. There are $N-1$ solutions given the constraint $\sum_k \delta f_k = 0$. Each solution has the form

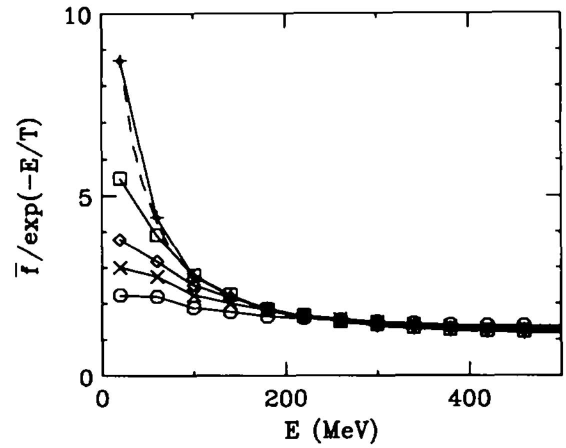

Fig. 3. Populations for states, divided by $\exp(-E_i/T)$, as a function of the energy for a system of 200 relativistic particles confined to a 10 fm cube at a temperature of 175 MeV, obtained from $(1+f)$-type simulations and averaged over 10^5 events. Initial populations (circles), after one (crosses), two (diamonds), five (squares), and fifty (upright crosses) collisions per particle. After many collisions populations have approached equilibrium values (dashed line).

$$\delta f_k \propto \exp\left\{-\frac{N_{\rm coll}}{N\tau_d}\right\}. \tag{19}$$

Each eigenvector has a different value of τ_d. We find that the slowest decaying eigenvector is the one where the ground state is furthest from equilibrium. For thermal systems we also find one can estimate $\tau_d \approx T/(E_2 - E_1)$ where E_1 and E_2 refer to the energies of the ground state and the first excited state. This approximation works to better than 20% accuracy whenever the first state has an occupation much greater than unity. This is true even when the second state is degenerate. Thus in the case of a nearly degenerate ground state, equilibration is very slow.

The simulation of the eight-level model described above can be easily modified to handle more sophisticated systems. To that end we consider the eigenstates of a cube of length $L = 10$ fm on a side with a base filling factor, $\eta_i = \exp(-E_i/T)$, with $T = 175$ MeV. Assuming the eigenstates are determined by a relativistic dispersion,

$$E^2_{k_x,k_y,k_z} = m^2 + \frac{\pi^2}{L^2}\left(k_x^2 + k_y^2 + k_z^2\right), \tag{20}$$

we simulate the behavior of $\langle f_i \rangle$ for 200 particles. Fig. 3 shows symmetrization enhancements to spectra. The symmetrization enhancement is defined by the average phase-space filling for states of that energy divided by $\exp(-E_i/T)$. The lowest curve demonstrates the symmetrization enhancement after states have been filled according to the $(1+f)$ prescription with no collisions. The figure shows that after five collisions per particle symmetrization enhancements for small energies are half way to equilibrium values. The estimate of τ_d mentioned above for this system is six collisions per particle.

In heavy-ion collisions the number of collisions pions feel is on the order of a half dozen for sulphur on lead collisions at 200 GeV incident laboratory energy. For the upcoming lead beam that number could roughly double. Thus we see that for larger systems, $(1+f)$ theory should be able to recover from its shortcomings and predict a spectra in line with what would be expected had all n-body effects been included. However, the multiplicity would still be underestimated. The amount by which the multiplicity might be underestimated would be similar to the amount of emission due to stimulated emission. Fig. 1 shows that the extra multiplicity could possibly be very large if values of η_k become larger than unity. Although a real system can not emit an infinite number of particles which is possible with the Poissonian model used above, the possibility of a significant fraction of pions being due to enhanced emission is possible. Phase-space occupations should be even more overpopulated for conditions created by the relativistic heavy ion collider (RHIC), scheduled to come on line in 1999.

In proton-induced collisions, conclusions are simpler. For these systems reinteraction of hadrons is not so important. Here $(1+f)$ models are completely inappropriate and full n-particle symmetrization effects must be incorporated. This issue has been addressed in the context of CENTAURO phenomena [7–10].

The possibility of creating super-radiant pion sources is one of the most exciting prospects of relativistic-heavy-ion physics, cosmic-ray physics and particle physics. Symmetrization phenomena should manifest themselves in a variety of observables: multiplicity distributions, isospin distributions and spectra. Making confident predictions of collective symmetrization phenomena is inherently difficult due to the unstable nature of enhanced emission. We have shown here that $(1+f)$ based simulations can grossly underpredict such phenomena. Should super-radiant conditions be attained in relativistic heavy-ion collisions, one of the most daunting theoretical challenges will be finding a way to incorporate many-body symmetrization into descriptions where emission and collisions are treated realistically.

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