# Precise Representation The Enlightenment: Newton

the apple, or not.

# remember, at home during plague (and fire that flattened London)

his “anni mirabiles”

first ideas about gravitation, optics, light, calculus

# Principia

**Definitions, Axioms (Laws), Propositions, Lemmas (assumptions), Corollaries and Scholia (notes)**

**First, relevant definitions:**

**mass:** “The quantity of matter is the measure of the same, arising from its density and bulk conjointly…”

not very satisfying… More notions of “mass” will follow…

**Think of it as the amount of ‘stuff’ in an object**

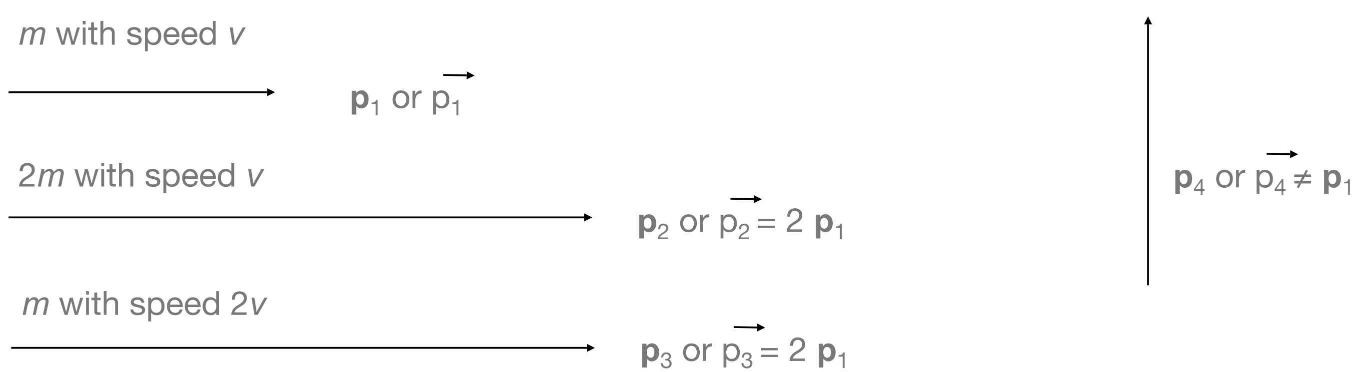
But boy, is mass a tricky concept.

# Principia

**"quantity of motion"**: "The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly…"

Or…finally…in modern terms:

**momentum**: $\mathbf{p} = m\,\mathbf{v}$…which we recognize now as a **vector.**



# ta -da!

**Axiom I** "Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it".

(Descartes, Huygens, and Galileo knew this…so-called "Principle of Inertia")

***we say:*** **Every body remains at rest or continues uniform motion in a straight line unless a net force acts on it.**

**Axiom II** "The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force acts."

(Galileo and Kepler almost knew this…)

***we say:*** **The change of momentum of a body is directly proportional to an external force applied to it. We now say, that the change of momentum with respect to time is equal to an external force applied to it.**

**Axiom III** "To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts."

***we say:*** **To every action there is an equal and oppositely directed reaction.**

(Bingo. Brand New…#3 makes mechanics possible)

# PHILOSOPHIÆ NATURALIS Principia MATHEMATICA.

## Definitiones.

### Def. I.

*Quantitas Materiæ est mensura ejusdem orta ex illius Densitate & Magnitudine conjunctim.*

AEr duplo densior in duplo spatio quadruplus est. Idem intellige de Nive et Pulveribus per compressionem vel liquefactionem condensatis. Et par est ratio corporum omnium, quæ per causas quascunq; diversimode condensantur. Medii interea, si quod fuerit, interstitia partium libere pervadentis, hic nullam rationem habeo. Hanc autem quantitatem sub nomine corporis vel Massæ in sequentibus passim intelligo. Innotescit ea per corporis cujusq; pondus. Nam ponderi proportionalem esse reperi per experimenta pendulorum accuratissime instituta, uti posthac docebitur.





### Def. II.

*Quantitas motus est mensura ejusdem orta ex Velocitate et quantitate Materiæ conjunctim.*

Motus totius est summa motuum in partibus singulis, adeoq; in corpore duplo majore æquali cum Velocitate duplus est, et dupla cum Velocitate quadruplus.

### Def. III.

*Materiæ vis insita est potentia resistendi, qua corpus unumquodq;, quantum in se est, perseverat in statu suo vel quiescendi vel movendi uniformiter in directum.*

Hæc semper proportionalis est suo corpori, neq; differt quicquam ab inertia Massæ, nisi in modo concipiendi. Per inertiam materiæ fit ut corpus omne de statu suo vel quiescendi vel movendi difficulter deturbetur. Unde etiam vis insita nomine significantissimo vis inertiæ dici possit. Exercet vero corpus hanc vim solummodo in mutatione status sui per vim aliam in se impressam facta, estq; exercitium ejus sub diverso respectu et Resistentia et Impetus: Resistentia quatenus corpus ad conservandum statum suum reluctatur vi impressæ; Impetus quatenus corpus idem, vi resistentis obstaculi difficulter cedendo, conatur statum ejus mutare. Vulgus Resistentiam quiescentibus et Impetum moventibus tribuit; sed motus et quies, uti vulgo concipiuntur, respectu solo distinguuntur ab invicem, neq; semper vere quiescunt quæ vulgo tanquam quiescentia spectantur.

### Def. IV.

*Vis impressa est actio in corpus exercita, ad mutandum ejus statum vel quiescendi vel movendi uniformiter in directum.*

Consistit hæc vis in actione sola, neq; post actionem permanet in corpore. Perseverat enim corpus in statu omni novo per solam vim

# Newton's Second Law

**that's the one with the mathematical content**

**In general terms it is not:**

**F** = m**a**

**It says that**

**"The change of motion is proportional to the motive force impressed."**

$$\Delta\vec{p} \propto \vec{F}$$

Strictly speaking, what he stated is

$$\Delta\vec{p} \propto \vec{F}$$

which we today represent as

$$\Delta\vec{p} = \vec{F}\Delta t$$

There is some physics in this. It says that if you apply a force to a body for a time, $\Delta t$, that body will change its momentum by $\Delta\vec{p}$. Today, this quantity on the right is called the Impulse.

You make use of the concept unconsciously. Suppose you jump off a chair. Just before you hit the ground, your velocity is $\mathbf{v}_0 = V(-\mathbf{j})$, pointing DOWN. Just after you stop, your velocity is $\mathbf{v}_f = 0$. The CHANGE in your velocity is

$$\begin{aligned} \Delta\mathbf{v} &= \mathbf{v}_f - \mathbf{v}_0 = 0\mathbf{j} - V(-\mathbf{j}) \\ & \quad \text{using just magnitudes in the } y \text{ direction...} \\ \Delta v &= 0 - (-V) = V \end{aligned}$$

That's a given. So, in magnitude, $Ft = mV$.

What's important to you is the stress on your knees…the force, $\mathbf{F}$ that results. You can keep that as small as possible by making $t$ as long as possible: you bend your knees.

On the other hand, if you want to impart a large change of momentum, you apply the largest force you can for the longest time that you can. Think baseball.

So, the formal way of viewing this notion of the Second Law, is to note that we can write

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t}$$

If we use the calculus, and then take the limit as $\Delta t \to 0$, then we get the formal definition of the force

$$\vec{F} = \frac{d\vec{p}}{dt}$$

The other way that the second law is used... was never written down by Newton, rather was first used by Leonhard Euler in 1752 (one of the many brilliant physicists who came after and cleaned up and formalized Newton's ideas).

Notice that $\Delta p$ means calculate the CHANGE of momentum...which consists of $mv$. So, if the mass is constant, then $\Delta p = m\Delta v$. So, we get:

$$\vec{F} = m\frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_0}{\Delta t}$$

Since, $\Delta v/\Delta t$ is the acceleration, we have the famous statement which has bedeviled engineering students for 250 years:

$$\begin{aligned} \vec{F} &= m\vec{a} \quad \ldots\text{actually, this means:} \\ \sum_{i=1}^{n}\vec{F}_i &= m\vec{a}. \end{aligned}$$

This says that the total sum of all forces (vector sum) acting on a body will result in an acceleration $\vec{a}$ imparted to that body. That's it...that's all of engineering and much of the space program.

“If you substitute the word “force” [vis] for the word “soul” [anima], you have the very principle on which the celestial physics in Astronomia Nova is based. For I formerly believed completely that the cause moving the planets is a soul...But when I recognized that this motive cause grows weaker as the distance from the sun increases, just as the light of the sun is attenuated, I concluded that this force must be as it were corporeal.

Kepler

# speaking of Kepler

Back to the moon, Alice...

# notice: limits figure into his arguments

even though it’s strictly geometrical

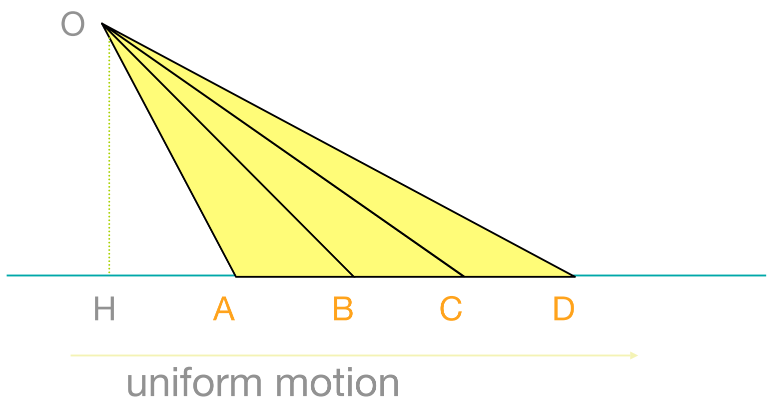
he’s working on enhancing the earlier approach

# Kepler's 2nd Law

As an example of his (non-calculus, but geometrical) reasoning in *Principia*:

Suppose we have a body executing inertial motion, marking out AB, BC, etc in time intervals T.

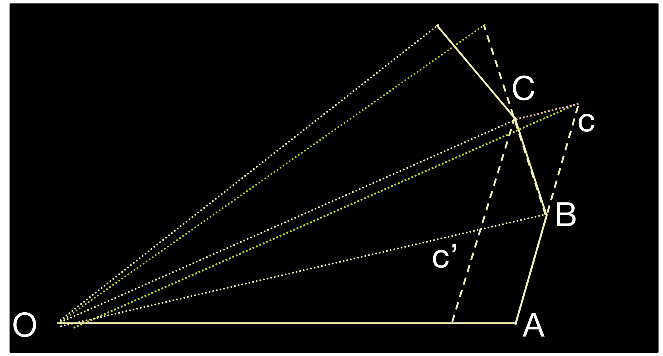
Then with respect to some point in space:




from the constant velocity condition and the rule for triangle area, the areas of all of these triangles are equal:

$$1/2AB\times OH = 1/2BC\times OH = 1/2CD\times OH = \ldots$$

Now, suppose that at B, the object is given a kick toward O



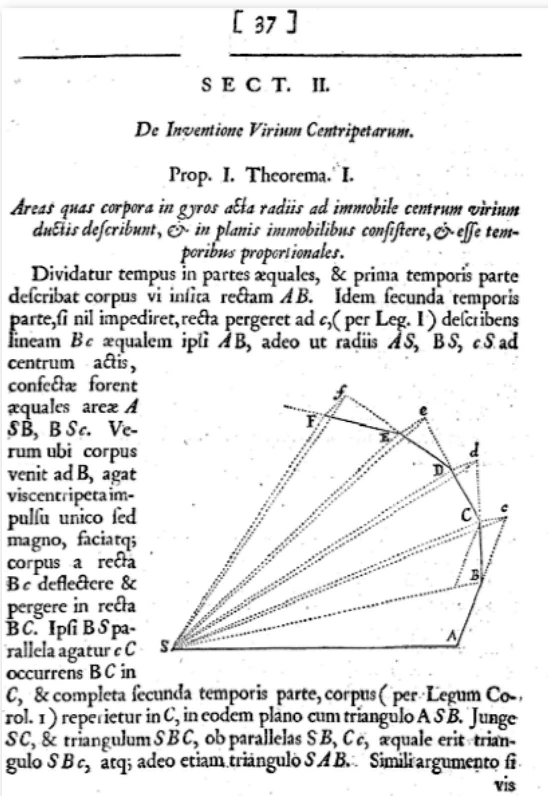

He showed that if the body would have gone to c during T, in absence of the kick, that it actually traveled distance equivalent to Bc' during T. He then showed that the area of OCB is equal to the area OBc…(if cC and OB are parallel)

So the equal area rule holds when the circumstance is a impulsive force toward O.

He then presumes the distances small, so the number of triangles is infinite. Then, the object travels on a circle. It was a short step to Kepler's 3rd law from here.

This is Kepler's Area Law…

# it falls into place



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SECT. II.

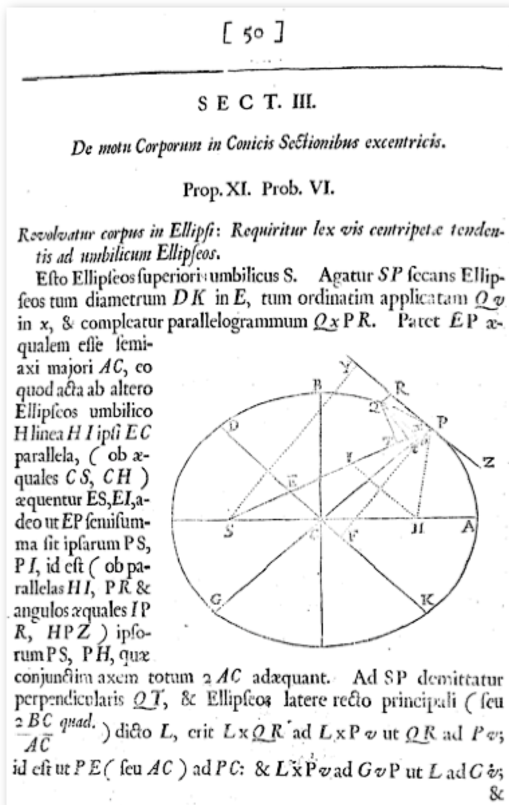
*De Inventione Virium Centripetarum.*

Prop. I. Theorema. I.

*Areas quas corpora in gyros acta radiis ad immobile centrum virium ductis deſcribunt, & in planis immobilibus conſiſtere, & eſſe temporibus proportionales.*

Dividatur tempus in partes æquales, & prima temporis parte deſcribat corpus vi inſita rectam AB. Idem ſecunda temporis parte, ſi nil impediret, recta pergeret ad c, (per Leg. I) deſcribens lineam Bc æqualem ipſi AB, adeo ut radiis AS, BS, cS ad centrum actis, confectæ forent æquales areæ ASB, BSc. Verum ubi corpus venit ad B, agat vis centripeta impulſu unico ſed magno, faciatq; corpus a recta Bc deflectere & pergere in recta BC. Ipſi BS parallela agatur cC occurrens BC in C, & completa ſecunda temporis parte, corpus (per Legum Corol. 1) reperietur in C, in eodem plano cum triangulo ASB. Junge SC, & triangulum SBC, ob parallelas SB, Cc, æquale erit triangulo SBc, atq; adeo etiam triangulo SAB. Simili argumento ſi

vis



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SECT. III.

*De motu Corporum in Conicis Sectionibus excentricis.*

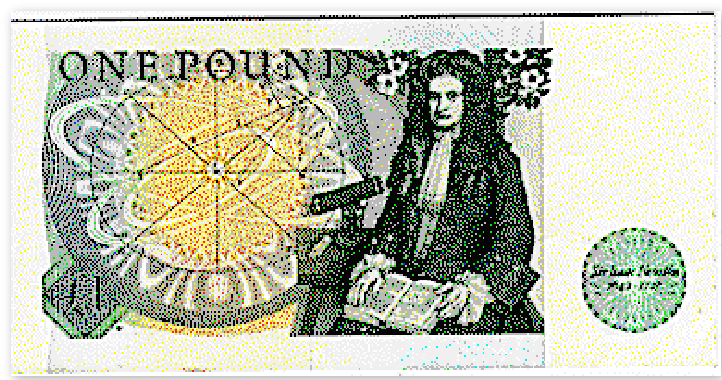
Prop. XI. Prob. VI.

*Revolvatur corpus in Ellipſi: Requiritur lex vis centripetæ tendentis ad umbilicum Ellipſeos.*

Eſto Ellipſeos ſuperioris umbilicus S. Agatur SP ſecans Ellipſeos tum diametrum DK in E, tum ordinatim applicatam Qv in x, & compleatur parallelogrammum QxPR. Patet EP æqualem eſſe ſemiaxi majori AC, eo quod acta ab altero Ellipſeos umbilico H linea HI ipſi EC parallela, (ob æquales CS, CH) æquentur ES, EI, adeo ut EP ſemiſumma ſit ipſarum PS, PI, id eſt (ob parallelas HI, PR & angulos æquales IPR, HPZ) ipſorum PS, PH, quæ conjunctim axem totum 2AC adæquant. Ad SP demittatur perpendicularis QT, & Ellipſeos latere recto principali (ſeu $\frac{2BC\ quad.}{AC}$) dicto L, erit L×QR ad L×Pv ut QR ad Pv; id eſt ut PE (ſeu AC) ad PC: & L×Pv ad GvP ut L ad Gv;

&

an embarrassment for …the Mint:



# $1/r^2$ force laws were "in the air"

**maybe because that's how the surface area of a sphere increases with radius?**

$$A(sphere) = 4\pi r^2$$

# remember Kepler's 3rd law:

$$\frac{R^3}{T^2} = \text{constant}$$

**R: mean radius**

**T: period**

Kepler determined it for planets going around the sun

# the moon, redux

There is a geometrical theorem that says: $MA^2 = AM' \cdot AD$...let the distance AM' = $s$

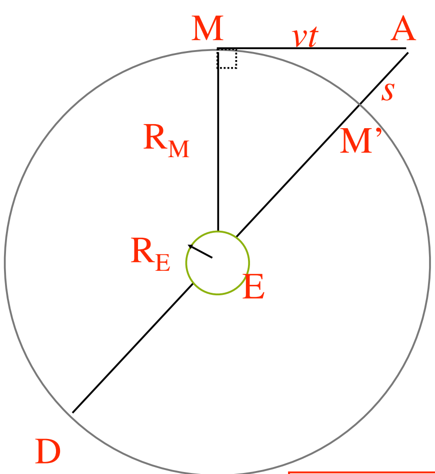
Moon has speed, v...tangent to its path.

In some time, $t$, absent a pull from earth, the moon would travel the distance MA

so, $vt$ = MA and from the geometrical theorem: $(vt)^2 = s\,(s + 2\,R_M)$

The actual trajectory is presumed by him to be one of **infinitesimal tugs**

So, MA is very short, and hence $s$ is tiny, certainly relative to $R_M$




Then, $(vt)^2 = 2\,s\,R_M$

From Galileo's relation for constant acceleration: $s = 1/2\ a\ t^2$ where $a$ is a constant and pointing to the earth.

Solve for $s\,/\,t^2$ and substitute and the result is: $a = \dfrac{v^2}{R_M}$

Write this differently. Let T be the period of the moon's orbit, so $vT = 2\pi R_M$

$$a = \frac{v^2}{R_M} = \frac{4\pi^2 R_M^2}{R_M T^2} = \left(\frac{4\pi^2}{R_M^2}\right)\left(\frac{R_M^3}{T^2}\right)$$

So, the acceleration is: $a = \dfrac{4\pi^2 K_E}{R_M^2}$

and hence, the force is $\dfrac{1}{r^2}$

From Kepler's 3rd Law = constant, $K_E$ for the Earth-Moon system
(a **different** K for orbits around another center...like planets-sun

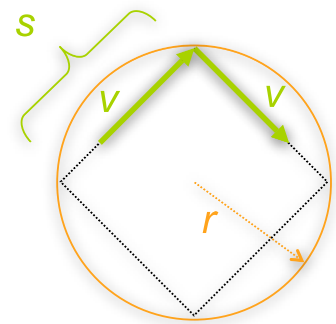
# circular motion was a toughy

"centripetal" acceleration…it points in

\'sen-'trip-et-'l\

adj [NL centripetus, fr. centr-+L petere to go to, seek] suppose we have an object approaching the inside of a rim of radius *r* with speed *v*, traveling on a square of side *s* inscribed in the circle

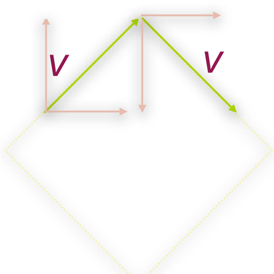
it perfectly rebounds - reflects - at an angle equal to its original incidence

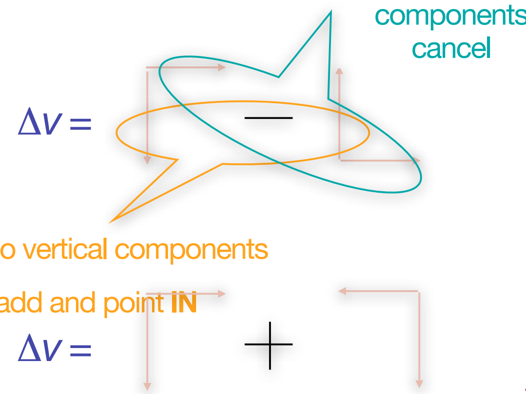



*look at components of the velocity before and after the rebound and calculate the vector difference:*

*change in:* **v** = **v**(final) – **v**(initial)

**What is that change in velocity? It's due ONLY to the change in direction: the final velocity minus the initial velocity:**




Δv = twice the vertical component, <u>down</u>

= 2 v / √2, since it's a square

**Since there is a change in velocity, there is an acceleration:**

$$
\begin{aligned}
\Delta v &= 2\left(v/\sqrt{2}\right) && \text{we just did this} \\
\Delta t &= s/v && \text{the time between collisions} \\
r &= s/\sqrt{2} && \text{from the geometry} \\
a &= \frac{\Delta v}{\Delta t} && \text{definition of acceleration} \\
&= 2\left(\frac{v}{\sqrt{2}}\right)\frac{v}{r\sqrt{2}} && \text{just plugging in...}
\end{aligned}
$$

$$a_c = \frac{v^2}{r} \quad \text{the centripetal acceleration} \tag{30}$$

...an acceleration which points toward the center of the circle… called "centripetal" - **ALL NON-STRAIGHT MOTION IS ACCELERATED MOTION**

In the spirit of extrapolating to limiting cases…any polygon, up to an infinitely sided one would give same result. That's circular motion, in that limit…1665 or so.

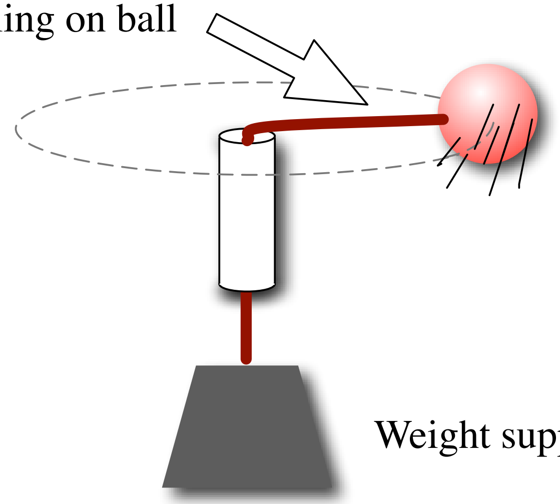
# centripetal force

**So, for something (moon) to move in a circle requires lots of little tugs towards the center**

- This overall force toward the center is the Centripetal Force
- It has the same direction as the centripetal acceleration

centripetal force of rope pulling on ball

$$F_{cent} = m a_{cent} = m\frac{v^2}{r}$$



$v$, the speed tangent to the path

and remember: $$a = \frac{4\pi^2 K_E}{R_M^2}$$

Weight supplies a force to the rope

# put on your seat belt

this is cool:

Galileo and Newton knew that the period of a pendulum was "powered" by the same acceleration source that things on inclined planes were..or freely falling bodies

call that acceleration of gravity on earth: $g$

Further, they knew that the period ($\tau$) depended only on the length of the string, *L*. Newton calculated it and then measured them carefully.

$$\tau = 2\pi\sqrt{\frac{L}{g}}$$

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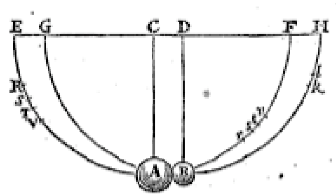
vallis deſcribantur ſemicirculi *EAF*, *GBH* radijs *CA*, *DB* biſecti. Trahatur corpus *A* ad arcus *EAF* punctum quodvis *R*, & (ſubducto corpore *B*) demittatur inde, redeatq; poſt unam oſcillationem ad punctum *V*. Eſt *RV* retardatio ex reſiſtentia aeris. Hujus *RV* fiat *ST* pars quarta ſita in medio, & hæc exhibebit retardationem in deſcenſu ab *S* ad *A* quam proxime. Reſtituatur corpus *B* in locum ſuum. Cadat corpus *A* de puncto *S*, & velocitas ejus in loco reflexionis *A*,




He did experiments on 10' long pendula and measured many things… among them, he found that the **acceleration due to gravity, *g***, was approximately

**g = 32 ft per second per second**

# the apple moment:

**the centripetal force**

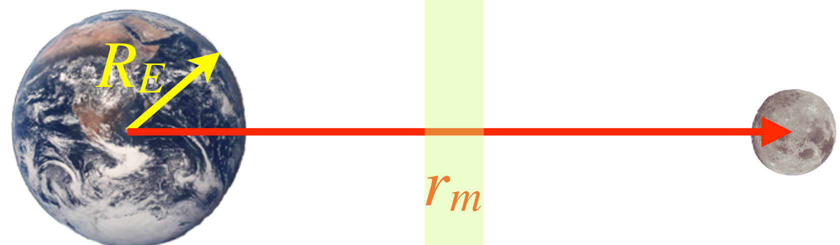
**of an object orbiting at the earth's surface**


$$a_{cent}(\text{near earth}) = \frac{4\pi^2 K_E}{R_E^2}$$

$$F = ma_{cent}$$

**How about the moon?**

**same form**




$$a_{cent}(\text{moon}) = \frac{4\pi^2 K_E}{r_m^2}$$

So, use the moon to calculate the Kepler constant:

$$K_E = \frac{r_m^3}{\tau_m^2}$$

$$a_{cent}(\text{near earth}) = \frac{4\pi^2}{R_E^2}\left(\frac{r_m^3}{\tau_m^2}\right)$$

# bingo.

$$a_{cent} = \frac{4\pi^2}{R_E^2}\left(\frac{r_m^3}{\tau_m^2}\right)$$

He knows: $r_m = 60.1 \bullet R_E$; $R_E = 4000$ mi; $\tau_m = 27.3$ d

$$\begin{aligned} g &= \frac{4 \times \pi^2 \times (60.1 R_E)^3}{R_E^2 \tau_m^2} \\ &= \frac{4 \times \pi^2 (60.1)^3 \times (4000 mi) \times (5280 ft/mi)}{[(27.3d)(3600 \times 24 s/d)]^2} \\ g &= 32.5 ft/s^2 \end{aligned}$$

He did experiments on 10' long pendula and measured many things among them, he found that the **acceleration due to gravity, *g***, wa[s] approximately

**g = 32 ft per second per second**

**Lookit!** The same value for the acceleration due to gravity that he measured with pendula.

...calculated using numbers **for the Moon**.

# so...

# the Moon is held in orbit by the same force

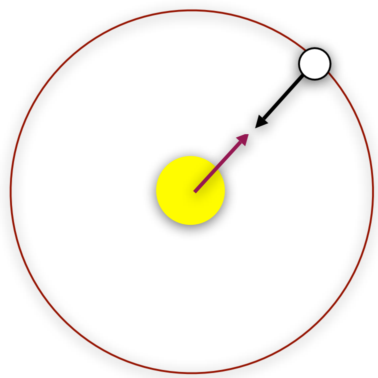
that holds things on earth

Planets the same idea?

Different “*K*” for things around the Sun from the Moon (and us) around the Earth!

**Using: $M_S$ as the mass of the sun & $m$ as the mass of a planet at radius $r$ and**

$F_{cent} = m\, a_c$:



$$F_c = 4\pi^2 K \frac{m}{R^2}$$

multiply by $1 = M_S/M_S$

$$F = \left(\frac{4\pi^2 K}{M_S}\right)\frac{M_S m}{r^2}$$

$$= G\frac{M_S m}{r^2}$$

Now, he makes a breathtaking leap: Suppose *m* and *m'* are the masses of any two objects that are separated by distance *r*…He postulates that there is a gravitational force acting between them of

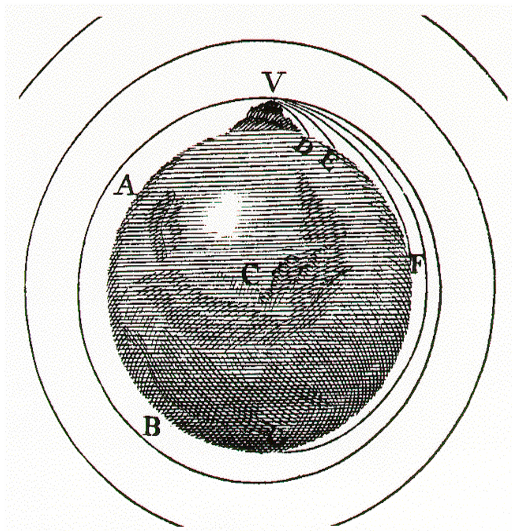
$$F = G\frac{mm'}{r^2}$$

G is a universal constant, for all bodies. Notice that his 3rd law is working here as each body attracts the other - symmetric in m and m'.

**Note: this is an example of "Action-Reaction": planet attracts the sun and the sun attracts the planet - *equal and opposite***

**his link between the orbit of the moon and acceleration of gravity on earth led him to a new interpretation of planetary motion**

- it's all falling




recall our discussion of centripetal acceleration derived by little impulses of force

that's the same as saying that objects go a bit, fall a bit, go a bit, fall a bit...

in the limit that "a bit" is infinitesimal - this description is of continuous, **orbital motion**

He showed that in the *Principia* by the above legendary picture of a cannon being shot at increasingly large velocities.

at some point*, the cannon ball falls and misses the earth...and just continues to fall (orbit)

* the escape velocity: $v_{esc} = \sqrt{\frac{2GM_E}{R_E}}$ ...good for any object, of any mass.

# “Gee”-Whiz.

G, the gravitational constant, is very hard to measure precisely

was first done only about a century later by the totally strange Henry Cavendish

Today, $G = 6.673(10) \times 10^{-11}\ m^3/kgs^2$

# note

G is still hard to measure, but needs to be done better

# the "2" in $1/r^2$ is important

**is directly related to there being 3 space dimensions...**

**the degree to which "2" has been determined is surprisingly not so great**

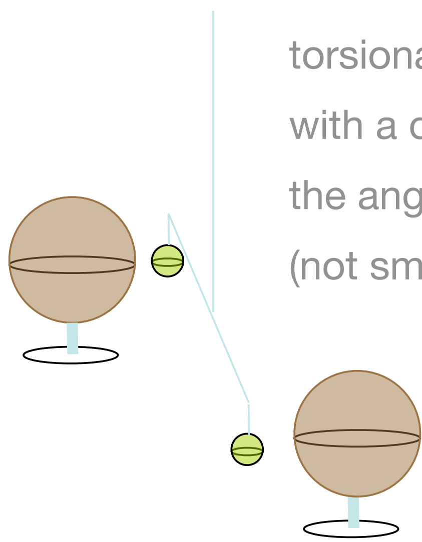
Current elementary particle theory is inching toward interest in space+time dimensions which are more than 3+1

it seems that we cannot rule out the possibility of more than 3 space dimensions from any experiment

indeed, we will be searching for the effects of this in upcoming experiments in Illinois and Switzerland

# Cavendish Experiment

G was finally measured in 1798, but not known to have been measured (!) until nearly a century later. Henry was a little secretive.



torsional balance

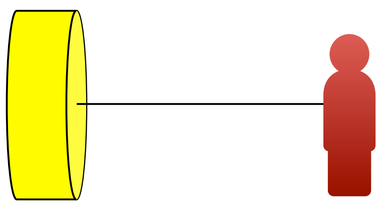
with a quartz fiber...carefully measure

the angle of deflection using light and mirrors

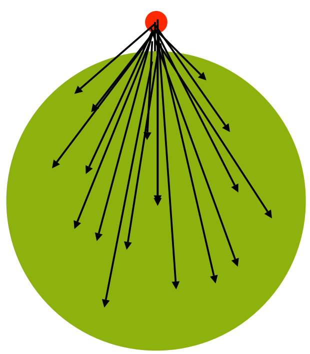
(not smoke and mirrors...)

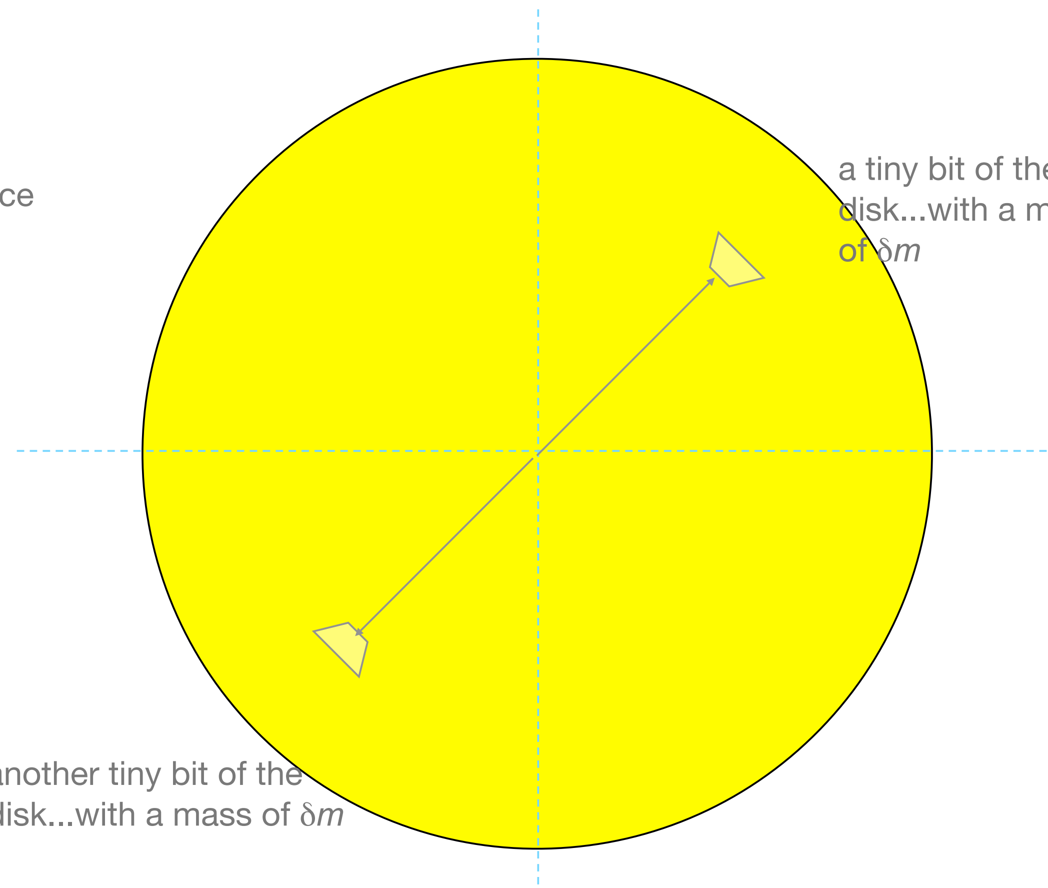
Cavendish got within 1% of the modern value...among other very precise, extraordinary experiments...he was 50 years ahead of his time in optics, chemistry, electromagnetism...

# calculus by waving your arms




the presumption about gravitational bodies was that they were point-sized… what about extended objects?





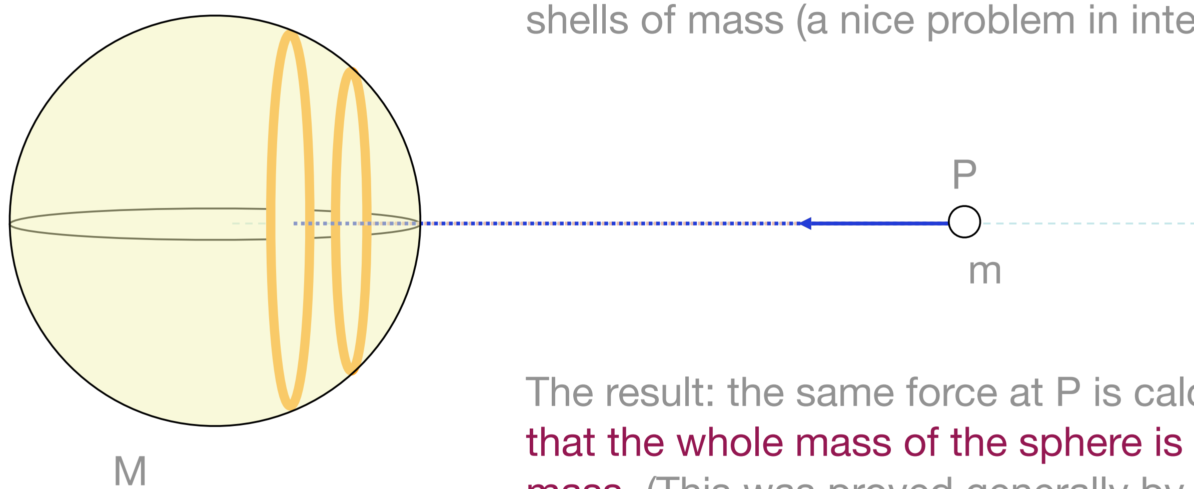

***what's*** **the direction of the force between you and each tiny bit?**

***what's*** **the direction of the NET attractive force between you and the two bits together?**

**add up enough bits to cover the entire disk...*****where*** **does the attractive force between you and the whole disk point?**

## There is magic in the inverse-square law…

outside of  spherical volume of mass, the force on a mass m at a point outside can be calculated by adding up contributions from shells of mass (a nice problem in integral calculus).



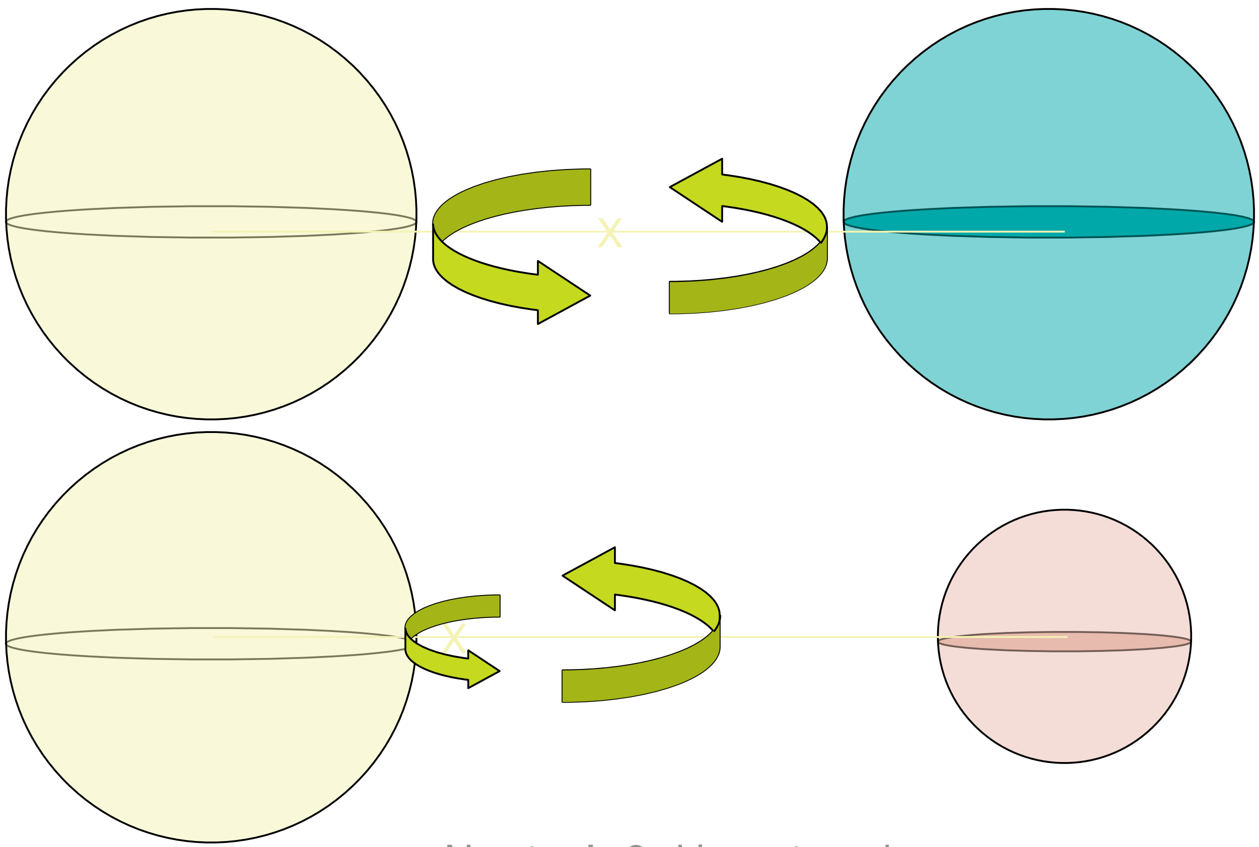

The result: the same force at P is calculated as if one just assumes that the whole mass of the sphere is concentrated at its center of mass. (This was proved generally by Gauss a century later, which is applicable in electromagnetic configurations.)

we need to be careful about our terminology: does the moon really orbit the earth? Does the earth really orbit the sun? no.

some LARGE gravitational mass

another LARGE, identical gravitational mass

who orbits whom?



Newton’s 3rd law at work...

# But weight, there's more.

If acceleration varies like the distance from a mass

what about Galileo's conclusion that free-fall was a constant acceleration phenomenon?

Consider a mass m at a distance y above the surface of the earth, where y is, say, less than an airliner's typical altitude of 5 miles.

Remember, that $R_E$ = 4000 mi, so fractionally, we're 5/4000 = 0.00125 further away than the surface…


$$F_y = G\frac{M_E m}{(R_E + y)^2}$$

$$= G\frac{M_E m}{R_E^2(1+\frac{y}{R_E})^2}$$

$$= G\frac{M_E m}{R_E^2}(1+\frac{y}{R_E})^{-2}$$

$$= G\frac{M_E m}{R_E^2}\left[1 - 2\left(\frac{y}{R_E}\right) + 3\left(\frac{y}{R_E}\right)^2 - 4\left(\frac{y}{R_E}\right)^3 + \ldots\right]$$

remember the Binomial expansion

Clearly, the first term is sufficient, so that for this tiny r:

$$F_y = G\frac{M_E m}{R_E^2}$$

…which is a constant, so we're justified in defining and using

$$g = G\frac{M_E}{R_E^2}$$

so that:

$$F_y = mg$$

This is the **Weight**…the force of the Earth's gravitational force.

# mass

heavy idea, man.

Remember, Newton's definition of mass was less than satisfying.

The modern, intuitive notion works: mass is the *quantity of matter* which constitutes an object.

However, this masks a deeper conceptual issue: mass has been used in two ways in the previous discussion:

> inertial mass: We can keep track of an "inertial mass" $m_i$, which is in the Second Law, $F = m_i a$. This is the resistance that the body has to being accelerated under the application of any force…the inertia.
>
> gravitational mass: The use of mass in the gravitational law, $m_g$, is different: here the force is $F = GMm_g / r^2$ and m is just a measure of the response that a body feels under the gravitational force of attraction.

Until the 20th century, these were dealt with as equal with experiment as the excuse.

**Newton knew:** **The quantities of matter in pendulous bodies, whose centers are equally distant from the center of suspension are in the ratio compounded of the ratio of the weights and the squared ratio of the times of the oscillation in a vacuum.**

this is his olde-timey way of saying that the inertial mass is proportional to the product of the period of a pendulum times the weight…

The period (back and forth) of a pendulum of length L is,

$$\tau = 2\pi\sqrt{\frac{m_i L}{W}}$$

substituting for the weight,

$$\tau = 2\pi\sqrt{\frac{L}{g}\frac{m_i}{m_g}}$$

By carefully measuring and comparing periods of pendulum bobs of different materials, but same weights, he concluded that the two masses were the same. Also, if you look at the use of Kepler's 3rd law in the gravitation discussion, you'll see that the Moon argument would not work

**just to be sure we're on the same page:**

$$W = mg$$ *changes, depending on planet*

$$W/g = m$$ *same, anywhere in the universe*

$$W/m = g$$ *same for all objects on the surface of the earth*

**Units**

- English System: mass: slugs; Force: pounds, lb: 1lb = 1 slug ft/s$^2$
- MKS System: mass: kilogram, kg; Force: Newtons, N: 1N = 1 kg m/s$^2$