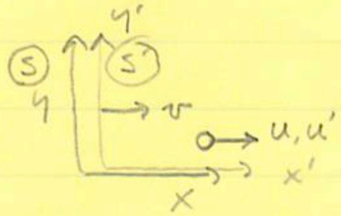
Relativity.

"invariance" is the key concept.

a simple invariance: 2 comoving inertial frames.



Galilean Transformations:

$$x = x' + vt'$$

$$y = y'$$

$$t = t'$$

so, $$\frac{dx}{dt} = \frac{dx'}{dt} + v\frac{dt'}{dt}$$

$$= \frac{dx'}{dt'}\frac{dt'}{dt} + v$$

$$u = u' + v$$

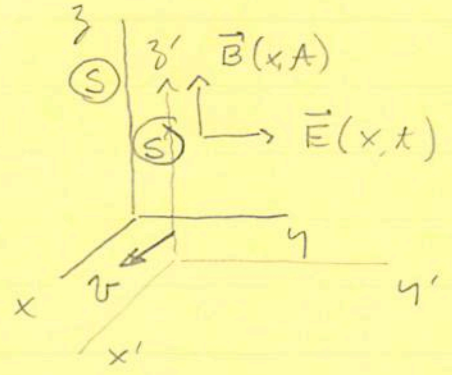
$$\frac{du}{dt} = \frac{du'}{dt'}\frac{dt'}{dt} + \frac{dv}{dt}$$

$$a = a' \quad \Rightarrow \quad f = ma$$

$$= f' = ma'$$

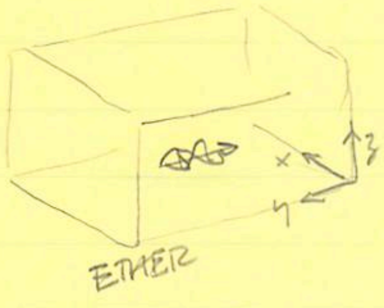
presuming $m = m'$

and everyone knows this. Newtonian mechanics is invariant wrt inertial frame observations now:



What about an electromagnetic wave? What happens to the equations of electromagnetism?

Lorentz tackled this and developed the Lorentz force to deal with his atomistic theory



E&M waves in ether can be produced by oscillating ions.

↑
● → ⟿ • ↙ other ions. feel force
↓

Ions feel $\vec{F}_N = m\vec{a}$ and $\vec{F}_L = q\vec{E} + q\vec{v} \times \vec{B}$

Know $F_N = F_N'$ what about $\vec{F}_L$?

Presume $F = qE - quB = F' = qE' - qu'B'$

"Know" $u' = u - v$

$$qE' - qu'B' = qE' - quB' + qvB'$$

$$E' - u'B' = E' + vB' - uB'$$

$$E' - uB' \text{ want } = E - uB$$

so, assign $E = E' + vB'$

$B = B'$

Okay..

How about M.E. ?

Look at $\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$ $\vec{E} = E(x,t)\,\hat{j}$

$\vec{B} = B(x,t)\,\hat{k}$

$\vec{\nabla} \times \vec{B} = -\frac{\partial B}{\partial x}\,\hat{j}$

$\frac{\partial \vec{E}}{\partial t} = \frac{\partial E}{\partial t}\,\hat{j}$

So, this M.E. is $\frac{\partial B}{\partial x} = -\frac{1}{c^2}\frac{\partial E}{\partial t}$ ☆

Transform to other frame

⋮

$$\frac{\partial B'}{\partial x'} = -\frac{1}{c^2}\frac{\partial E'}{\partial t'} + \frac{1}{c^2}\left[v\frac{\partial E'}{\partial x'} - v\frac{\partial B'}{\partial t'} + v^2\frac{\partial B'}{\partial x'} \right]$$

which is not the same form as ☆

This is the beginning of Special Relativity

Lorentz calculated how to make this transformation work for E&M

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

in 1899. Called the Lorentz Transformations

For Lorentz:
- Ether exists
- speed of light is c in ether
- ME hold in ether

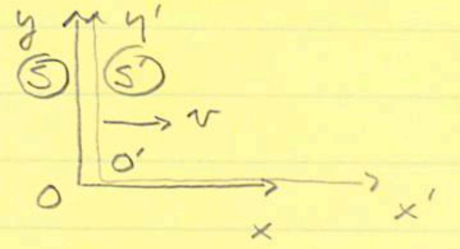
(also figured out by Voigt in 1887 and Larmor in 1898)

You know the rest of the story. Einstein was bothered by these things but took a completely different point of view.

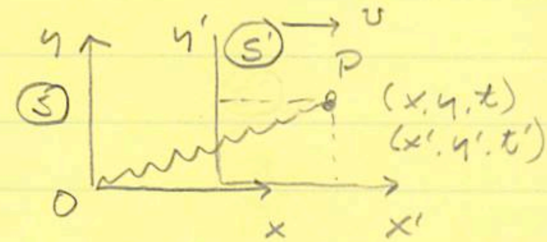
2 POSTULATES OF RELATIVITY

1. No experiment can be performed which can detect an absolute velocity through space..
   - ⇒ the laws of physics are the same in all inertial rest frames.
   - ⇒ can't detect ether? ⇒ ether can't exist.
2. The speed of light is constant in all inertial rest frames

One of his derivations of the consequences:



when O and O' coincide, flash a light bulb.



Observer at rest in O: beam traveled $\overline{OP}$ such that $\frac{\overline{OP}}{t} = c$

Observer in O': beam traveled $\frac{\overline{O'P}}{t'} = c$

So,

$$\frac{\overline{OP}}{t} = \frac{\sqrt{x^2+y^2+z^2}}{t} = c$$

$$x^2+y^2+z^2 = c^2t^2$$

$$\frac{\overline{O'P}}{t'} = c \Rightarrow \quad x'^2+y'^2+z'^2 = c^2t'^2$$

Obviously, $y = y'$; $z = z'$ so,

$$x'^2 - c^2t'^2 = x^2 - c^2t^2 = 0$$

A G.T. : $(x - vt)^2 - c^2t^2 = x^2 + v^2t^2 - 2xvt - c^2t^2$

$\neq 0$

Presume linear transformation

$$x' = \alpha x + \eta t$$
$$t' = \epsilon x + \gamma t$$

I.C. : @ S' origin $x' = 0$ $x = vt$

so,

$$0 = \alpha v t + \eta t \quad \Rightarrow \quad \eta = -\alpha v$$

generally, then

$$x' = \alpha (x - vt)$$

Substitute into $x'^2 - c^2 t'^2 = 0$

$$(\alpha^2 - c^2\epsilon^2)x^2 - (\alpha^2 v^2 + c^2\gamma^2)t^2$$
$$- (2\alpha^2 v + 2c^2\epsilon\gamma)xt = 0$$

which we need to be equal to $x^2 - c^2t^2 = 0$

So,

$$\alpha^2 - c^2\epsilon^2 = 1$$
$$c^2\gamma^2 + \alpha^2 v^2 = c^2$$
$$2\alpha^2 v + 2c^2\epsilon\gamma = 0$$

Solve:

$$\alpha = \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\epsilon = -\frac{\gamma v}{c^2} = -\frac{v/c^2}{\sqrt{1 - v^2/c^2}}$$

Which are the L.T. $\quad x' = \gamma(1 - vt)$

$$t' = \gamma\left(1 - \frac{\beta x}{c}\right)^{*} \qquad \beta = v/c$$

Of course the reverse transformations: $v \to -v$

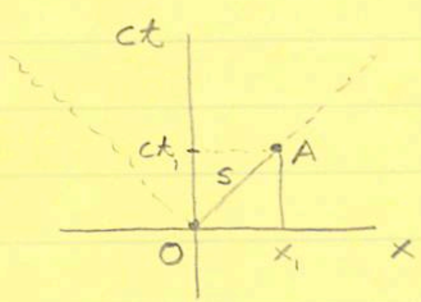
In 3d, Euclidean space, the quantity which is invariant wrt all coordinate systems is the length

$$L^2 = x^2 + y^2 + z^2$$

In relativity, the corresponding "length" is the spacetime "interval"

$$s^2 = c^2t^2 - (x^2 + y^2 + z^2) = s'^2$$

1 space dimension:



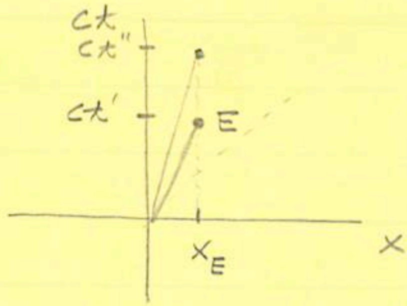
at 45°

$$\text{slope} = \frac{\Delta(ct)}{\Delta x} = 1$$

$$= c\,\frac{\Delta t}{\Delta x} \Rightarrow \frac{\Delta x}{\Delta t} = c$$

the "light cone"

$$* \quad \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$



object sitting still in Ⓢ

In rest frame of Ⓢ , $\Delta x = 0$

$$(\Delta s)^2 = (c \cdot \Delta t)^2 - (\Delta x)^2$$

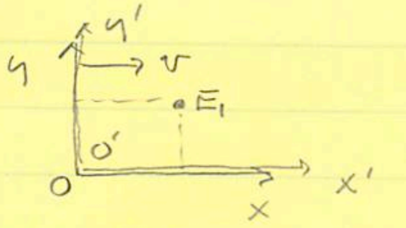
$$\sqrt{s^2} = c\Delta t$$

the Δt is a clock carried by the Ⓢ object → PROPER TIME

In general, rearranging $(c\Delta t)^2 = (\Delta s)^2 + (\Delta x)^2$

proper time is the shortest time

→ all clocks hear proper time

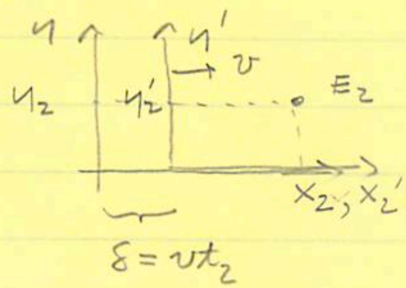


@ $t = t_1 = t_1' \equiv 0$

E_1 happens & then E_2 at same place in (S')

at $x_1 y_1$ (S)

$x_1' y_1'$ (S')

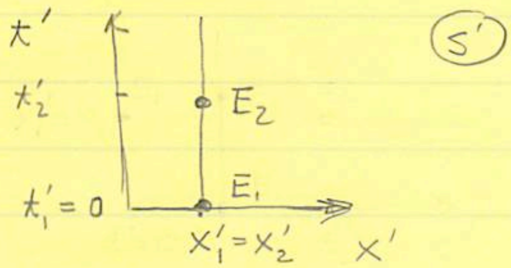


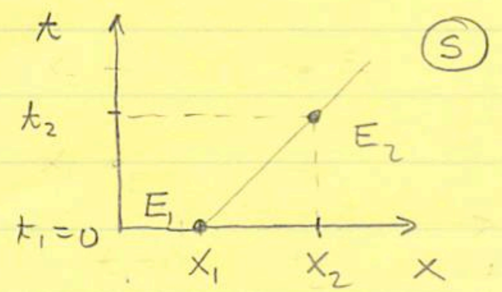
G.T. ⇒ $x_2 = x_2' + \delta$

$x_2 = x_2' + vt_2$

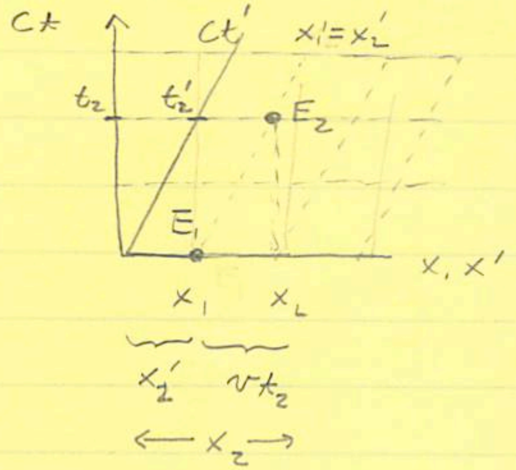
$t_2 = t_2'$

On a world-diagram



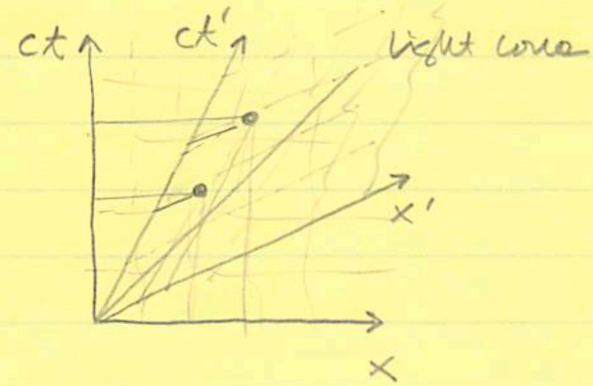


Can put this on a single diagram



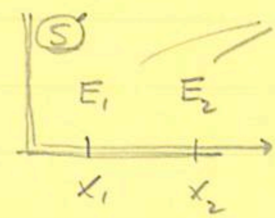
$$x_2 = x_2' + vt_2$$
$$x_2' = x - vt_2$$
$$t_2 = t_2'$$

The same can be done for L.T. → the distinguishing feature? A light signal must be the same slope in each frame.



Immediate consequences.

1. Simultaneity



same time in S
$t_1 = t_2 = t$.

What about S'?

L.T. : $t' = \gamma\left(t - \beta \frac{x}{c}\right)$

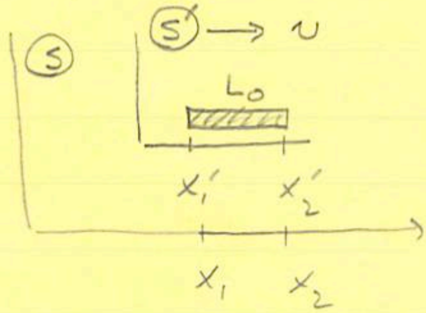
$$t_2' - t_2' = \gamma\left(\underbrace{t_2 - t_1}_{=0} - \beta\frac{x_2}{c} + \beta\frac{x_1}{c}\right)$$

$$= \frac{\gamma\beta}{c}\left(x_1 - x_2\right) \neq 0$$

not simultaneous in S'

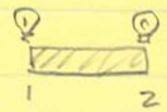
MEANING: a measurement cannot be done to <u>show</u> simultaneity ⇒ doesn't exist.

2. Length Contraction



definition of length:
distance between two points measured at the same time.

For S to measure length, must also make simultaneous position measurement.

suppose  lights which (S') can time

(S') causes flashes at x_2', t_2' and x_1', t_1' which (S) observes at $x_2\ t_2$ and $x_1\ t_1$

$$x_2' - x_1' = (x_2 - vt_2)\gamma - (x_1 - vt_1)\gamma$$

$$= (x_2 - x_1)\gamma - (t_2 - t_1)v\gamma$$

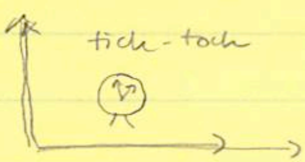
(S') times the flashes until (S) sees them as simultaneous.
THEN $x_2 - x = L$

$$x_2' - x_1' = L_0 = L\gamma$$

or what (S) measures is $L = \dfrac{L_0}{\gamma}$

$$L < L_0$$

(3) Time dilation



T = time for one tick-tock interval.

$$t_2 - t_1 = \gamma\left(t_2' - \frac{\beta x_2'}{c} - t_1' + \frac{\beta}{c}x_1'\right)$$

$$T = \gamma\underbrace{(t_2' - t_1')}_{T_0 \text{ the natural frequency}} - \underbrace{(x_2' - x_1')}_{\text{at rest in (S')} = 0}\frac{\beta}{c}$$

$T = \gamma T_0$

$T > T_0$ time dilation.

Standard proof is μ decay in cosmic rays.

$p\, N_2 \rightarrow$ stuff + π's

$\hookrightarrow \mu\, \nu_\mu$

From $N(t) = N_0 e^{-t/t_0}$ t_0 = mean lifetime $= 2.2\,\mu s$ for μ.

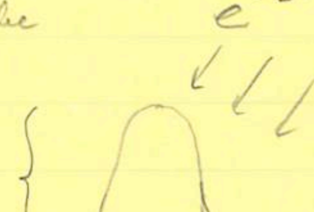
Through the atmosphere $v_\mu \hat{=} 0.992c$

$$\gamma = \frac{1}{\sqrt{1-(0.992)^2}} \cong 8$$

Average distance $d \sim ct_0 = (3\times10^8)(2.2\times10^{-6})$ $\sim 660\,m$.

Atmosphere is $\sim 30 \times d$, or the attenuation of μ's should be e^{-30} .. too many are seen.

First experiment: 6255'



$\leftarrow 563 \pm 10\ \mu/h$

$\leftarrow 408 \pm 9\ \mu/h$

Mt Washington

$N(t) = N_0 e^{-t/t_0}$

$408 = 563\, e^{-t/2.2\times10^{-6}}$

$t = 7.1\times10^{-7}$ sec in μ frame

In earth frame $\quad v_\mu = \dfrac{6255}{T}$

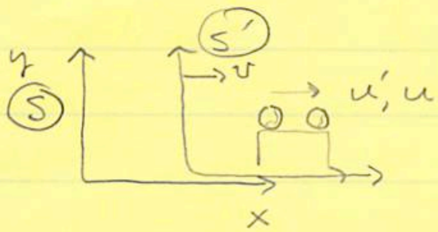
$$T \cong 6.4 \times 10^{-6}\ \text{s.}$$

So, experiment says

$$T = x\, t$$

$$x = \frac{6.4 \times 10^{-6}}{7.1 \times 10^{-7}} = 9$$

4 velocity transformation



velocity of object in S' is $\dfrac{dx'}{dt'} = u'$

$$x' = \gamma(x - \beta c t) \qquad\qquad t' = \gamma\left(t - \frac{\beta}{c} x\right)$$

$$u' = \frac{dx'}{dt'} = \gamma\left(\frac{dx}{dt'} - \beta c \frac{dt}{dt'}\right)$$

$$= \gamma\left(\frac{dx}{dt}\frac{dt}{dt'} - \beta c \frac{dt}{dt'}\right) = \gamma(u - \beta c)\frac{dt}{dt'}$$

$$\frac{dt'}{dt} = \gamma\left(1 - \frac{\beta}{c}\frac{dx}{dt}\right) = \gamma\left(1 - \frac{\beta}{c} u\right)$$

$$u' = \gamma(u - \beta c)\,\frac{1}{\gamma\left(1 - \frac{\beta}{c} u\right)} = \frac{u - \beta c}{1 - \frac{\beta}{c} u}$$

$$u'_x = \frac{u_x - \beta c}{1 - \frac{\beta}{c} u}$$

if $u' = c$

$$c = \frac{u - \beta c}{1 - \frac{\beta}{c} u}$$

$$c - \beta u = u - \beta c$$

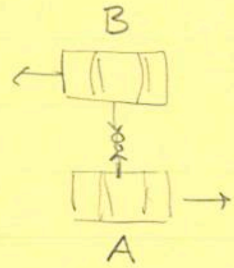
$$c(1+\beta) = u(1+\beta) \quad \Rightarrow \quad u = c \text{ also.}$$

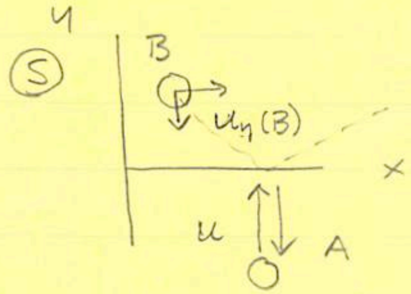
now $u'_y \neq u_y$ if the object moves in the (5') plane

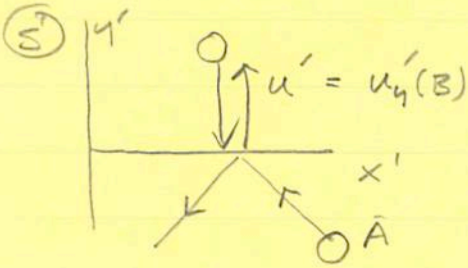
$$u'_y = \frac{dy'}{dt'} = \frac{dy}{dt'} = \frac{dy}{dt}\frac{dt}{dt'} = u_y \frac{dt}{dt'}$$

$$u_y' = \frac{u_y}{\gamma(1 - \frac{\beta}{c} u_x)}$$

Colliding billiard balls between co-moving frames.







identical balls, perfect recoil $\quad |u| = |u'|$

(S)
$$u_y(B) = \frac{u_y'(B)}{\gamma\left(1 - \beta \frac{u_x'(B)}{c}\right)}$$

$$= \frac{u_y'(B)\sqrt{1-\beta^2}}{1 - \beta \frac{u_x'(B)}{c}} \qquad \frac{u_x'(B)}{c} = \emptyset$$

$$u_y(B) = u'\sqrt{1-\beta^2}$$

demand momentum conservation

(S) y comp before = y comp after (S)

$$m_A u - m_B u_y(B) = \text{reverse}$$

m_A & m_B measured in (S)

$$m_A u - m_B u'\sqrt{1-\beta^2} = -m_A u + m_B u'\sqrt{1-\beta^2}$$

$u = u'$

$$m_A - \frac{m_B}{\gamma} = -m_A + \frac{m_B}{\gamma}$$

$$m_A = \frac{m_B}{\gamma}$$

$$m_B = \gamma m_A$$

if we let u be very small, then the collision is just glancing, then $m_A \sim m_0$ ... the mass of A in Ⓢ rest frame.

$$m_B \longrightarrow m(v) = \gamma m_0$$

In the general sense in which $\vec{p} = m\vec{v}$

$$\vec{p} = m_0 \gamma \vec{v}$$

Here, there is a point of view: m_0 is a universal quantity called the rest mass .. an invariant

Now, $\vec{F} = \frac{d}{dt}\vec{p}$ still

$$= m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} = m\vec{a} + \vec{v}\frac{dm}{dv}\frac{dv}{dt}$$

$$\vec{F} = m_0\gamma\left(\frac{d\vec{v}}{dt} + \frac{v\, dv/dt}{c^2 - v^2}\vec{v}\right)$$

when $\vec{v}$ is not in the same direction as $\vec{F}$, this becomes a very complicated tensorial formula.

Work done $= \Delta KE$

$$KE = \int_{v=0}^{v=v} F\,ds$$

$$= \int_{v=0}^{v=v} \frac{d}{dt}(mv)\frac{ds}{dt}\,dt$$

$$= \int v \frac{d}{dt}(mv)\,dt = \int v\,d(mv)$$

$$= \int v\,d\left[\frac{m_0 v}{\sqrt{1-v^2/c^2}}\right]$$

$$= m_0 \int v \left[\frac{1}{\sqrt{1-v^2/c^2}} + \frac{v^2/c^2}{(1-v^2/c^2)^{3/2}}\right] dv$$

$$= m_0 \int_0^v \frac{v\,dv}{(1-v^2/c^2)^{3/2}}$$

$$= m_0 c^2 \frac{1}{(1-v^2/c^2)^{1/2}} \Big|_0^v$$

$$K = m_0 c^2 \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) = m_0 c^2 (\gamma - 1)$$

$$K = mc^2 - m_0 c^2$$

new approach to energy:

$$mc^2 = E = K + m_0 c^2 = m_0 \gamma c^2$$

and $p = m_0 \gamma v$ connect -- like the interval

$$E^2 = \gamma^2 m_0^2 c^4 = \frac{m_0^2 c^4}{1 - v^2/c^2}$$

$$= \frac{m_0^2 v^2 c^2}{1 - v^2/c^2} + m_0^2 c^4$$

$$E^2 = m_0^2 \gamma^2 v^2 c^2 + m_0^2 c^4$$

$$E^2 = p^2 c^2 + m_0^2 c^4$$

or

$$m_0^2 c^4 = E^2 - p^2 c^2$$

just like $\quad s^2 = (ct)^2 - x^2 \quad$ (see next page)

Relativity usually deals in infinitesimals —

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

and a tensorial description...

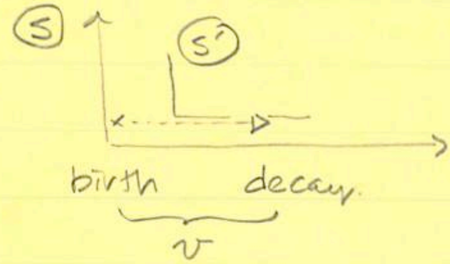
$$x^1 \equiv x \qquad x^0 \equiv ct$$
$$x^2 \equiv y$$
$$x^3 \equiv z$$

$$ds^2 = dx^{0\,2} - dx^{1\,2} - dx^{2\,2} - dx^{3\,2}$$

just as $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$

so too $\begin{pmatrix} E'/c \\ p \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ p \end{pmatrix}$

Suppose we're considering the decay of an unstable particle



time to decay in S ("lab") $\equiv T$

time to decay in S' (rest) $\equiv \tau$

S $\quad t_d - t_b = T$

$$\sum_{i=1}^{3} \left(x_d^i - x_b^i \right)^2 = v^2 T^2$$

S' $\quad t'_d - t'_b = \tau$

$$\sum_{i=1}^{3} \left(x_d'^i - x_b'^i \right)^2 = 0$$

Postulate 2 implies $\quad s^2 = s'^2$

$$s^2 = c^2 T^2 - \sum \left(x_d^i - x_b^i \right)^2 = c^2 T^2 - v^2 T^2$$

$$s'^2 = c^2 \tau'^2 = c^2 T^2 - v^2 T^2$$

$$\tau'^2 = T^2 (1 - v^2/c^2)$$

$$T = \gamma \tau$$ the dilation formula

This length is not Euclidean... but can be thought of as a rotation

like rotation matrix

Id: $$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$x = ax' + bct'$$
$$ct = bx' + act'$$

$$s^2 = c^2t^2 - x^2$$
$$= (c^2t'^2 - x'^2)(a^2 - b^2) = s'^2(a^2 - b^2)$$

$\Rightarrow \quad a^2 - b^2 = 1$ looks a bit like $\cos^2\theta + \sin^2\theta = 1$

But, it's imaginary.

$$a = \cosh\xi$$
$$b = \sinh\xi$$
some imaginary angle

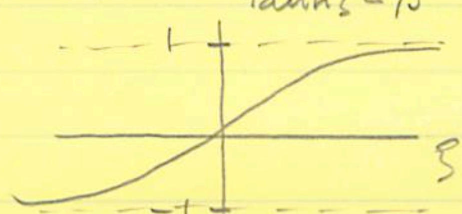
$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

For the decaying particle

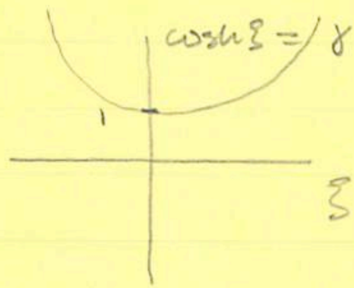
$$x = ct'\sinh\xi \qquad (x' = 0)$$
$$ct = ct'\cosh\xi$$

$$\frac{x}{ct} = \tanh\xi = \frac{v}{c} = \beta$$

$\tanh\xi = \beta$

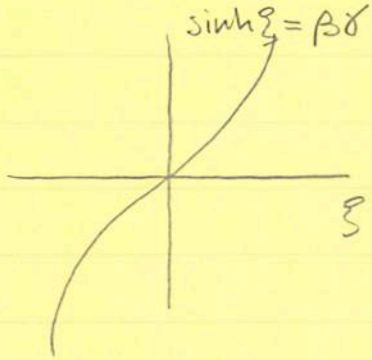


$t = t' \cosh \xi$ or $\dfrac{t}{t'} = \cosh \xi = \dfrac{1}{\sqrt{1 - v^2/c^2}} = \gamma$



and $x = ct' \sinh \xi$

$$\frac{x}{ct'} = \frac{x}{ct}\,\frac{t}{t'} = \frac{x}{ct}\,\gamma = \frac{v\gamma}{c} = \beta\gamma = \sinh \xi$$



and we get
$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

or
$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

the Lorentz Transformations

$\xi = \tanh^{-1} \beta = \frac{1}{2} \ln \left(\dfrac{1+\beta}{1-\beta} \right)$ called RAPIDITY

since it's an angle .. it's additive under boosts.

ASIDE...

Notice $p = m\gamma v = m\gamma\beta c$

$E = m\gamma c^2$

$$\frac{p}{E} = \frac{m\gamma\beta c}{m\gamma c^2} = \frac{\beta}{c}$$

$$\beta = \frac{pc}{E}$$

$$\frac{1+\beta}{1-\beta} = \frac{1+\frac{pc}{E}}{1-\frac{pc}{E}} = \frac{E+pc}{E-pc}$$

if p in the direction of the boost. If not, then this is $\frac{E+p_L c}{E-p_L c}$.

So, another expression for rapidity is

$$\xi \equiv y = \frac{1}{2}\ln\left(\frac{E+p_L c}{E-p_L c}\right)$$

usually

Now, if particles are all very relativistic

$E \sim pc$ and $p_L = p\cos\theta$

$$y \to \eta = \frac{1}{2}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right) = -\ln\left(\tan\theta/2\right)$$

called the PSEUDORAPIDITY

The interval is written,

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}\, dx^\mu dx^\nu$$

repeated indices: Einstein convention.

$g_{\mu\nu}$ — depends on coordinate system.

$$ds^2 = g_{00}\, dx^0 dx^0 + g_{11}\, dx^1 dx^1 + g_{22}\, dx^2 dx^2 + g_{33}\, dx^3 dx^3$$

so $\quad g_{00} = 1 \qquad g_{ij} = -\delta_{ij} \qquad i,j = 1,2,3$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \qquad \text{"signature"} = (1,-1,-1,-1)$$

CONVENTION: refer to spacetime coordinates as CONTRAVARIANT four-vectors

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

$$= [x_0, \vec{x}]$$

notice $\quad g_{\mu\nu} x^\nu = \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu = \underset{\mu = 0-3}{(ct, -x, -y, -z)} \neq x^\mu$

A^μ is a contravariant 4-vector

A_μ is a covariant 4-vector

$$x_\mu = (x_0, x_1, x_2, x_3) = [x_0, -\vec{x}]$$

so $\quad x_\mu = g_{\mu\nu} x^\nu$

and a role that $g_{\mu\nu}$ plays is evident - lowering an index

Notice $\quad g_{\mu\nu} = g^{\mu\nu}$

and

$$g^{\mu\nu} g_{\mu\nu} = 1$$

So, $\quad g^{\mu\nu} g_{\nu\alpha} = g^\mu{}_\alpha = \delta^\mu{}_\alpha \quad$ Kronecker delta $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

4d dot product:

$$\begin{aligned} x^\mu y_\mu &= x^\mu g_{\mu\nu} y^\nu \\ &= x^0 (1) y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 \\ &= x^0 y^0 - \vec{x} \cdot \vec{y} \\ &\equiv x \cdot y \end{aligned}$$

Suppose

$x'^\mu = f^\mu (x^\mu) \quad$ one set defined in terms of the other.

$$\begin{aligned} dx'^\mu &= \frac{\partial f^\mu}{\partial x^\nu}(x^\mu)\, dx^\nu \\ &= \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \end{aligned}$$

$dx'^\mu \equiv \Lambda^\mu{}_\nu \, dx^\nu \quad$ defines the transformation of a Lorentz vector - tensor, rank 1

$$A^\mu \longrightarrow A'^\mu = \Lambda^\mu{}_\nu A^\nu$$ anything that does this is a Lorentz-vector

Suppose a scalar function - $\phi(x^\mu)$

gradient:

$$A_\nu \equiv \frac{\partial \phi(x^\mu)}{\partial x^\nu}$$

$$A'_\nu = \frac{\partial \phi(x^\mu)}{\partial x'^\nu} = \frac{\partial \phi(x^\mu)}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial x^{\nu\prime}}$$

$$A'_\nu = \frac{\partial x^\sigma}{\partial x'^\nu} A_\sigma$$

not $\Lambda^\sigma{}_\nu$

anything that transforms like a gradient — COVARIANT vector

How about the contraction of 2 vectors -

$$A_\mu B^\mu \longrightarrow A'_\mu B'^\mu = A_\nu \frac{\partial x^\nu}{\partial x^{\mu\prime}} B^\alpha \frac{\partial x'^\mu}{\partial x^\alpha}$$

$$= A_\nu B^\alpha \frac{\partial x^\nu}{\partial x^{\mu\prime}} \frac{\partial x'^\mu}{\partial x^\alpha}$$

$$= A_\nu B^\alpha \frac{\partial x^\nu}{\partial x^\alpha} = A_\nu B^\alpha \delta^\nu{}_\alpha$$

$$A'_\mu B'^\mu = A_\nu B^\nu$$

same form $\Rightarrow$ invariant $\Rightarrow$ SCALAR

The Lorentz transformation operator has 16 components
BUT

$$ds'^2 = ds^2$$

$$g_{\mu\nu} x'^{\mu} dx'^{\nu} = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta}$$

$$g_{\mu\nu} \Lambda^{\mu}{}_{\rho}\, dx^{\rho}\, \Lambda^{\nu}{}_{\xi}\, dx^{\xi} = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta}$$

$$g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\xi}\, dx^{\rho} dx^{\xi} = g_{\rho\xi}\, dx^{\rho} dx^{\xi}$$

so $$\Lambda^{\mu}{}_{\rho}\, g_{\mu\nu} \Lambda^{\nu}{}_{\xi} = g_{\rho\xi} = \Lambda^{T}{}_{\rho}{}^{\mu}\, g_{\mu\nu} \Lambda^{\nu}{}_{\xi} = g_{\rho\xi}$$

which is an equation of constraint ⇒ 10 conditions.

⇒ # independent parameters = 6

3 of relative velocity

3 to orient coordinate axes.

Define tensorial properties by how objects transform

trans → no change ⇒ SCALAR

$$A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu} \quad \Rightarrow \text{VECTOR, tensor rank 1}$$

$$A'^{\mu\nu} = \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} A^{\mu\nu} \quad \Rightarrow \text{tensor rank 2}$$

$$A'^{\mu}{}_{\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} A^{\rho}{}_{\sigma}$$

$$= \Lambda^{\mu}{}_{\rho} A^{\rho}{}_{\sigma} (\Lambda^{-1})^{\sigma}{}_{\nu} \quad \Rightarrow \text{mixed rank 2}$$

Aside - aside -

$$A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}$$

$$\frac{\partial A'^{\mu}}{\partial x'^{\rho}} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial A^{\nu}}{\partial x'^{\rho}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x'^{\rho}} A^{\nu}$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\alpha}}{\partial x'^{\rho}} \frac{\partial A^{\nu}}{\partial x^{\alpha}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\alpha}} \frac{\partial x^{\sigma}}{\partial x'^{\rho}} A^{\nu}$$

if not zero, $\frac{\partial A^{\mu}}{\partial x'^{\rho}}$ is

not a tensor

related to the affine connection in non-Euclidean geometries

$\boxed{\text{Standard tensors}}$

⓪ rank ϕ scalar $\quad A^{\mu} B_{\mu} = A^0 B_0 - \vec{A} \cdot \vec{B}$

① metric tensor $\quad$ useful: $\quad g_{\mu\nu} T^{\nu\sigma}{}_{\rho} = T_{\mu}{}^{\sigma}{}_{\rho}$ etc.

② gradient $\quad \frac{\partial}{\partial x^{\mu}} \equiv \partial_{\mu} = [\partial/\partial t, \vec{\nabla}]$

$$\frac{\partial}{\partial x_{\mu}} = \partial^{\mu} = [\partial/\partial t, -\vec{\nabla}]$$

so, $\quad \frac{\partial A_{\mu}}{\partial x_{\mu}} = \partial^{\mu} A_{\mu} = \frac{\partial A_0}{\partial t} + (-\vec{\nabla}) \cdot (-\vec{A}) = \frac{\partial A_0}{\partial t} + \vec{\nabla} \cdot \vec{A}$

③ D'Alembertian operator

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} \equiv \partial_\mu \partial^\mu \equiv \Box = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

④ Antisymmetric tensors

$$\epsilon_{123} = -\epsilon^{123} = -1$$

$$\epsilon_{0123} = -\epsilon^{0123} = 1 \quad \text{where} \quad \epsilon_{\mu\nu\alpha\beta} = -\epsilon_{\nu\mu\alpha\beta}$$

etc

$$\epsilon_{\mu\nu\alpha\beta} S^{\alpha\beta} = 0$$

$\boxed{\text{Calculations}}$

$a + b \rightarrow c + d$

$p_a = (E_a, \vec{p}_a)$

$p_b = (E_b, \vec{p}_b)$

Two likely reference frames.

Center of Momentum $\boxed{CM}$

$$\vec{p}_a^{\,*} = -\vec{p}_b^{\,*}$$

$$\longrightarrow \ \longleftarrow$$

fixed target system $\boxed{TS}$

$$\vec{p}_a \qquad p_b = 0$$

$$\longrightarrow \quad \circ$$

CM: $p_a^{\mu *} = (E_a^*, 0, 0, P_a^*)$
$p_b^{\mu *} = (E_b^*, 0, 0, -P_a^*)$

TS: $p_a^{T\mu} = (E_a^T, 0, 0, P_a^T)$
$p_b^{T\mu} = (m_b c^2, 0, 0, 0)$

notation – $c = 1$ and $\hbar = 1$ used everywhere –
$1f = 10^{-15}m$ $\hbar c = 197.33$ MeV·fm $\hbar = 6.582 \times 10^{-22}$ MeV·s

$E = mc^2 \rightarrow E = m$ so mass and energy have same dimensions.

$[J] = Nm = kg\,m^2/s^2$

use eV's
$1\ TeV = 10^3\ GeV = 10^{12}\ eV$
$1\ GeV = 10^3\ MeV = 10^9\ eV$
$1\ MeV = 10^3\ keV$
$1\ keV = 10^3\ eV$

$1\ GeV = 1.602 \times 10^{-10}\ J$
$1\ erg = 10^{-7}\ J = 624\ GeV$
$1\ GeV = 1.7827 \times 10^{-27}\ kg.$

$m(\text{proton}) = 1.673 \times 10^{-27}\ kg = 0.938\ GeV$

Compton wavelength $\lambda = \frac{\hbar}{mc} \rightarrow \frac{1}{m}$ length $\propto$ energy^{-1}

$\frac{1}{GeV} = 0.19733\ fm = 0.19733 \times 10^{-15}\ m$
$= 6.5822 \times 10^{-25}\ s$

To go from one frame to the other, need to transform according to the whole system.

total CM energy: $E^* = E_a^* + E_b^*$

System 4 vectors:

$$P^\mu_{CM} = \left(E^*/c,\ 0,\ 0,\ 0 \right)$$

$$P^\mu_{TS} = \left(\frac{E_a^T + m_b c^2}{c},\ 0,\ 0,\ p_a^T \right)$$

To transform from one to the other need the γ and β for CM relative to TS

$$\begin{pmatrix} E_{CM}/c \\ p_{CM} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E_{TS}/c \\ p_{TS} \end{pmatrix} \quad ①$$

$$\begin{pmatrix} E_{TS}/c \\ p_{TS} \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E_{CM}/c \\ p_{CM} \end{pmatrix} \quad ②$$

From ②

$$\frac{E_{TS}}{c} = \gamma \frac{E_{CM}}{c} + \beta\gamma \cancel{p_{CM}}^{\,0}$$

$$\frac{E_a^T + mc^2}{c} = \gamma \frac{E^*}{c} \qquad E^* = \frac{E_a^T + mc^2}{\gamma}$$

$$\boxed{\gamma_{CM} = \frac{E_a^T + mc^2}{E^*}}$$

$$p_{TS} = \gamma \left(\beta \frac{E_{CM}}{c} + \cancel{p_{CM}}^{\,0} \right) = p_a^T$$

$$p_a^T = \frac{\gamma\beta}{c}\frac{E^*}{c}$$

$$\beta = \frac{p_a^T c^2}{\gamma}\frac{1}{cE^*} = \frac{p_a^T c^2}{\gamma c}\left(\frac{\gamma}{E_a^T + m_b c^2}\right)$$

$$\boxed{\beta_{CM} = \frac{p_a^T c}{E_a^T + m_b c^2}}$$

jargon: $\text{"}s\text{"} = (p_1^\mu + p_2^\mu)^2$ invariant, right?

$$= (p_1^\mu + p_2^\mu)\cdot(p_{1\mu} + p_{2\mu})$$

$$= p_1^\mu \cdot p_{1\mu} + p_2^\mu \cdot p_{2\mu} + 2p_2^\mu \cdot p_{1\mu}$$

any 4-vector $A^\mu A_\mu = A\cdot A = A^{02} - \vec{A}\cdot\vec{A}$

momenta: $p\cdot p = p^{02} - \vec{p}\cdot\vec{p}$

$$= \frac{E^2}{c^2} - |p|^2$$

$$(p\cdot p)c^2 = E^2 - (pc)^2$$

recognize

$$E^2 = (p\cdot p)c^2 + (pc)^2$$

$p\cdot p = m^2$ "invariant mass"

So, for any 2 momenta

$$s = m_1^2 + m_2^2 + 2\underbrace{p_1 \cdot p_2}_{\text{4 vector dot product.}}$$

$$p_1 \cdot p_2 = p_1^0 p_2^0 - \vec{p}_1 \cdot \vec{p}_2$$

$$s = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2$$

another way:

$$s = E_1^2 - \vec{p}_1 \cdot \vec{p}_2 + E_2^2 - \vec{p}_2 \cdot \vec{p}_2 + 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2$$

$$s = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

CM $\quad s = (E_a^* + E_b^*)^2 + (0)$

we speak of "root-s" as a characterization of a colliding beam accelerator.

$$\sqrt{s} = E_a^* + E_b^*$$

@ Fermilab Tevatron $\quad E_p = E_{\bar{p}} = 1 \text{ TeV}$

so

$\sqrt{s} = 2 \text{ TeV}$ "in the center of mass"

@ LHC $\quad E_p = E_p = 7 \text{ TeV}$

$$\sqrt{s} = 14 \text{ TeV}$$

TS:
$$S = (p_a + p_b)^2$$
$$= \left(\underbrace{E_a^T + E_b^T}_{E_b^T = m_b}\right)^2 - \left(\vec{p}_a^{\,T} + \underbrace{\vec{p}_b^{\,T}}_{=0}\right)^2$$

$$s = (E_a^T + m_b)^2 - p_a^{T2}$$

$$= E_a^{T2} + m_b^2 + 2E_a^T m_b - p_a^{T2}$$
$$E_a^{T2} - p_a^{T2} = m_a^2$$

$$S = m_a^2 + m_b^2 + 2E_a^T m_b$$

why colliding beams? How to get 2 TeV in TS...

$$S = 2000 = 1^2 + 1^2 + 2E_a^T(1)$$

$$E_a^T = 1998 \text{ GeV.} \quad \text{okay.}$$

But, what is $\sqrt{s}$ for TS for extracted Tevatron beam?

$$s = 1^2 + 1^2 + 2(1000)(1)$$
$$s = 2002$$
$$\Rightarrow \sqrt{s} = 44.7 \text{ GeV.}$$
$$= 0.0447 \text{ TeV.}$$

Here's the fun part. What's the energy of the E_a^* relative to E_T?

Could go through the whole Lorentz Transformation. Almost never need the L.T. Just be clever with invariants.

$$p_a^T \cdot (p_a^T + p_b^T) = \text{invariant quantity} - \text{scalars.}$$

same in any frame. So:

$$= p_a^* \cdot (p_a^* + p_b^*)$$

$$p_a^T \cdot p_a^T + p_a^T \cdot p_b^T = p_a^* \cdot p_a^* + p_a^* \cdot p_b^*$$

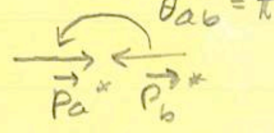
$$\cancel{m_a^2} + p_a^T \cdot p_b^T = \cancel{m_a^2} + p_a^* \cdot p_b^*$$

$$E_a^T \underset{\uparrow\ =m_b}{E_b^T} - \vec{p}_a \cdot \underset{\uparrow\ =\phi}{\vec{p}_b} = E_a^* E_b^* - 2\vec{p}_a \cdot \vec{p}_b$$

$$E_a^T m_b = E_a^* E_b^* - 2\,|\vec{p}_a^*|\,|\vec{p}_b^*| \cos\theta_{ab}^*$$

$$= E_a^* E_b^* + 2\,p_a^* p_b^*$$

$$E_a^T m_b = E_a^* E_b^* + 2p^*$$



$|\vec{p}_a^*| = |\vec{p}_b^*|$

since m_a and m_b need not be equal E_a^* and E_b^* could be different.

Usually write these things in terms of the invariant s

$$s = m_a^2 + m_b^2 + 2E_a^T m_b$$

$$E_a^T m_b = \frac{s - m_a^2 - m_b^2}{2} = E_a^* E_b^* + E_a^{*2} - m_a^2$$

$$= E_a^*(E_b^* + E_a^*) - m_a^2$$

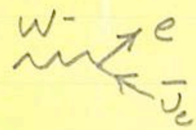
$$= E_a^*(\sqrt{s}) - m_a^2$$

So,

$$E_a^* = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}$$

and $$E_b^* = \frac{s + m_b^2 - m_a^2}{2\sqrt{s}}$$

How about a decay.. $W^- \to e\bar{\nu}_e$



Use 4-momentum conservation

$$p_W^\mu = p_e^\mu + p_\nu^\mu$$ standard trick.

square it $$(p_W^\mu)^2 = (p_e + p_\nu)^2$$ 4 vectors.

$$\| $$

$$M_W^2 = m_e^2 + m_\nu^2 + 2\, p_e \cdot p_\nu$$

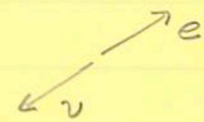
work in limit in which m_e and m_ν are small.

$m_e \sim 0.511$ MeV $\qquad m_\nu \sim$ fraction eV.

$M_W \sim 80$ GeV

$$M_W^2 = 2p_e \cdot p_\nu$$
$$= 2E_e E_\nu - 2\vec{p}_e \cdot \vec{p}_\nu$$

in rest frame of W:



$\vec{p}_e = -\vec{p}_\nu$ and, since massless, $E_e = E_\nu = E$

$$M_W^2 = 2E^2 + 2E^2 = 4E^2$$

$E = \frac{1}{2} M_W$ which you expect.

another, even more clever way

$p_W = p_e + p_\nu$ keep m_e

ignore $m_\nu = 0$

$$p_\nu = p_W - p_e$$

$p_\nu^2 = (p_W - p_e)^2$ no neutrino terms left.

$$\begin{aligned} &\shortparallel \\ &0 = M_W^2 + m_e^2 - 2p_W \cdot p_e \end{aligned}$$

$$0 = M_W^2 + m_e^2 - 2E_W E_e - 2\vec{p}_W \cdot \vec{p}_e$$

again, go to rest frame of W $\Rightarrow \vec{p}_W = 0$

$$0 = M_W^2 + m_e^2 - 2M_W E_e$$

$$E_e = \frac{M_W^2 + m_e^2}{2M_W}$$

In general, could heavy m_ν

$$p_\nu^2 = (p_W - p_e)^2$$

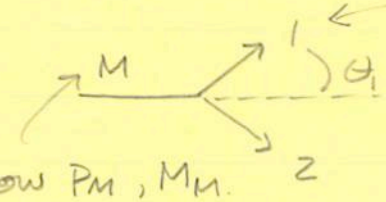
$$m_\nu^2 = M_W^2 + m_e^2 - 2M_W E_e$$

still no neutrino variables...

$$E_e = \frac{M_W^2 - m_e^2 - m_\nu^2}{2M_W}$$

measure P_1, know m_1

In lab



know P_M, M_M.

suppose you know M and 1 and don't know 2 which you cannot measure

what's m_2? missing mass measurement.

$$P_M = P_1 + P_2$$

don't measure any "2" variables... get rid of them

$$P_2 = P_M - P_1$$

$$P_2^2 = m_2^2 = (P_M - P_1)^2 = M_M^2 + m_1^2 - 2P_M \cdot P_1$$

$$m_2^2 = M_M^2 + m_1^2 - 2E_M E_1 - 2\,\vec{P}_M \cdot \vec{P}_1$$

$$m_2^2 = M_W^2 + m_1^2 - 2\sqrt{M_M^2 - P_M^2}\,\sqrt{m_1^2 - P_1^2} - 2P_M P_1 \cos\theta_1$$