Lorentz Transformation Properties.

Suppose we make some transformation on the x^μ

$$x' = \Lambda x$$

Any function of x, $\psi(x)$ would change its form as well to $\psi'(x')$

For a Lorentz transformation

$$x^\mu \xrightarrow{L.T.} x^{\mu'} = \Lambda^\mu{}_\nu x^\nu \qquad \psi(x) \xrightarrow{L.T.} \psi'(x')$$

Define an operation that does this in spinor space

$$\psi(x) \longrightarrow \psi'(x') = S\psi(x)$$

$$\psi'(\Lambda x) = S\psi(x)$$

S will be a matrix $\qquad \psi'_n(\Lambda x) = S_{nm}(\Lambda)\psi_m(x)$

or $\qquad \psi'_n(x') = S_{nm}(\Lambda)\psi_m(\Lambda^{-1}x')$

same x'

on both sides.

Assume:

1. $\psi^\dagger\psi$ transforms like 0^{th} component of 4-vector
2. D.E. is same form in all frames.
3. S is non-singular $\Rightarrow$ S^{-1} exists

So, in O: if $\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0$

$$\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + im\right)\psi(x) = 0$$

then in O':

$$\left(\gamma^\mu \frac{\partial}{\partial x^{\mu\prime}} + im\right)\psi'(x') = 0$$

note: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu$$

so $$\frac{\partial}{\partial x'^\mu} = \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{(\Lambda^{-1})^\nu{}_\mu} \frac{\partial}{\partial x^\nu}$$

and

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^\nu} (\Lambda^{-1})^\nu{}_\mu$$

So,

$$\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + im\right)\psi'(x') = 0$$

$$\gamma^\alpha (\Lambda^{-1})^\nu{}_\alpha \frac{\partial}{\partial x^\nu} S\psi(x) + im\, S\psi(x) = 0$$

$$S^{-1}\gamma^\alpha (\Lambda^{-1})^\nu{}_\alpha \frac{\partial}{\partial x^\nu} S\psi(x) + im\, S^{-1}S\psi(x) = 0$$

S – combination of γ's

$$S^{-1}\gamma^\alpha S(\Lambda^{-1})^\nu{}_\alpha \frac{\partial \psi}{\partial x^\nu} + im\psi(x) = 0$$

$$\| $$

$$-\gamma^\nu \frac{\partial \psi(x)}{\partial x^\nu} \quad \text{from D.E}$$

$$S^{-1}\gamma^\alpha S \Lambda^{-1\,\nu}{}_\alpha \frac{\partial \psi}{\partial x^\nu} - \gamma^\nu \frac{\partial \psi}{\partial x^\nu} = 0$$

$$\left[S^{-1}\gamma^\alpha S \Lambda^{-1\,\nu}{}_\alpha - \gamma^\nu \right] \frac{\partial \psi}{\partial x^\nu} = 0$$

so.

$$S^{-1}\gamma^\alpha S \Lambda^{-1\,\nu}{}_\alpha = \gamma^\nu$$

$$\gamma^\alpha (\Lambda^{-1})^\nu{}_\alpha = S\gamma^\nu S^{-1} \qquad (1)$$

Look at an example: $\left.\begin{array}{l} x'^0 = x^0 \\ x'^i = -x^i \end{array}\right\}$ space inversion

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \quad \text{so,} \quad \Lambda^{-1} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$S \to S_P$ for parity

$$S_P \gamma^\nu S_P^{-1} = (\Lambda^{-1})^\nu{}_\alpha \gamma^\alpha$$

$$S_{P\,mn} \gamma^\nu_{nl} S^{-1}_{P\,lq} = (\Lambda^{-1})^\nu{}_\alpha \gamma^\alpha_{mq}$$

For $\quad \nu = 0 \qquad S_P \gamma^0 S_P^{-1} = (\Lambda^{-1})^0{}_\alpha \gamma^\alpha \quad = 0$ unless $\alpha = 0$

so

$$= (\Lambda^{-1})^0{}_0 \gamma^0 = \gamma^0$$

$$\nu = i \qquad S_P \gamma^i S_P^{-1} = (\Lambda^{-1})^i{}_i \gamma^i = -\gamma^i \qquad i = 1,2,3$$

so,

$$\left. \begin{aligned} S_P \gamma^0 &= \gamma^0 S_P \\ S_P \gamma^i &= -\gamma^i S_P \end{aligned} \right\} \quad S_P \propto \gamma^0$$

and

$$\psi(x^\mu) \xrightarrow{P} \psi'(x'^\mu) = \psi'(x_\mu) = S_P \psi(x^\mu)$$

$$= \gamma^0 \psi(x^\mu)$$

or

$$\psi'(t, -\vec{x}) = \gamma^0 \psi(t, \vec{x})$$

Remember $\quad \psi = \begin{pmatrix} \psi^1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

$$\begin{aligned} \psi'^{(1)} &= \psi^{(1)} \\ \psi'^{(2)} &= \psi^{(2)} \\ \psi'^{(3)} &= -\psi^{(3)} \\ \psi'^{(4)} &= -\psi^{(4)} \end{aligned}$$

$$\uparrow \qquad\qquad \uparrow$$

$$(t, -\vec{x}) \qquad (t, \vec{x})$$

So. for the transformed D.E. O'

$$\Big(\gamma^i \underbrace{\frac{\partial}{\partial x'^i}}_{-\frac{\partial}{\partial x^i}} + \gamma^0 \underbrace{\frac{\partial}{\partial x^{0\prime}}}_{\frac{\partial}{\partial x^0}} + im\Big)\gamma^0 \psi(x) = 0$$

$$\Big(-\gamma^i \frac{\partial}{\partial x^i} + \gamma^0 \frac{\partial}{\partial x^0} + im\Big)\gamma^0 \psi(x) = 0$$

$$\gamma^0 \underbrace{\Big(\gamma^i \frac{\partial}{\partial x^i} + \gamma^0 \frac{\partial}{\partial x^0} + im\Big)\psi(x) = 0}_{\text{DE in } O}$$

If I'd started with $\quad \Big(\gamma^\mu \frac{\partial}{\partial x^\mu} + im\Big)\psi(x) = 0$

and demanded $\quad \Big(\gamma^\mu \frac{\partial}{\partial x'^\mu} + im\Big)\psi'(x') = 0$

would have gotten

$$S^{-1}\gamma^\nu S = \Lambda^\nu{}_\mu \gamma^\mu \qquad (2)$$

in addition to

$$S\gamma^\nu S^{-1} = (\Lambda^{-1})^\nu{}_\alpha \gamma^\alpha \qquad (1)$$

Adjoint: $\psi'(x') = S\psi(x)$

$$\psi'^{+}(x') = \psi^{+}(x) S^{+}$$

$$\bar{\psi}'(x') \equiv \psi'^{+}(x)\gamma^{0} = \psi^{+}(x) S^{+}\gamma^{0}$$

$$= \psi^{+}(x)\gamma^{0}\gamma^{0} S^{+}\gamma^{0}$$

$$\bar{\psi}'(x') = \bar{\psi}(x)\gamma^{0} S^{+}\gamma^{0}$$

and can transform the conjugate equations

$$\bar{\psi}'(x')(\overleftarrow{\not{\partial}}' - im) = 0$$

form independent from
$\bar{\psi}(x)(\overleftarrow{\not{\partial}} - im) = 0$

$$\underbrace{\frac{\partial}{\partial x'^{\mu}}}_{(\Lambda^{-1})^{\nu}{}_{\mu}\frac{\partial}{\partial x^{\nu}}} \bar{\psi}'(x')\gamma^{\mu} - im\,\bar{\psi}'(x') = 0$$

$$\quad \& \quad \psi^{+\prime}(x') = \psi^{+}(x) S^{+}$$

$$(\Lambda^{-1})^{\nu}{}_{\mu}\frac{\partial}{\partial x^{\nu}} \psi^{+}(x) S^{+}\gamma^{0}\gamma^{\mu} - \underbrace{im\,\bar{\psi}(x)\gamma^{0}S^{+}\gamma^{0}}_{\displaystyle \overset{\|}{\frac{\partial\bar{\psi}}{\partial x^{\nu}}(x)\gamma^{\nu}\gamma^{0}S^{+}\gamma^{0}}} = 0$$

D.E.

$$\frac{\partial}{\partial x^{\nu}} \bar{\psi}(x)\gamma^{0}S^{+}\gamma^{0}(\Lambda^{-1})^{\nu}{}_{\mu}\gamma^{\mu} = \frac{\partial\bar{\psi}}{\partial x^{\nu}}\gamma^{\nu}\gamma^{0}S^{+}\gamma^{0}$$

or as an operator equation

$$\gamma^0 S^\dagger \gamma^0 (\Lambda^{-1})^\nu{}_\mu \gamma^\mu = \gamma^\nu \gamma^0 S^\dagger \gamma^0$$

$$(\Lambda^{-1})^\nu{}_\mu \gamma^\mu = (\gamma^0 S^\dagger \gamma^0)^{-1} \gamma^\nu (\gamma^0 S^\dagger \gamma^0)$$

$$\Vert$$

1

$$S \gamma^\nu S^{-1} =$$

So, $S^{-1} = \gamma^0 S^\dagger \gamma^0 \neq S^\dagger$ — S not unitary in general.

Then,

$$\bar\psi'(x') = \bar\psi(x)\gamma^0 S^\dagger \gamma^0$$
$$= \bar\psi(x) S^{-1}$$

Now, this is important. We can consider the L.T. properties of a variety of $\bar\psi - \psi$ forms.

For example

- $\bar\psi\psi \xrightarrow{L.T.} \bar\psi'(x')\psi'(x') = \bar\psi(x) S^{-1} S \psi(x)$
$= \bar\psi(x)\psi(x)$

  so, $\bar\psi\psi$ transforms like a Lorentz Scalar.

- How about $\bar\psi\gamma^\mu\psi$?

$$\bar\psi\gamma^\mu\psi \xrightarrow{L.T.} \bar\psi'(x')\gamma^\mu\psi'(x') = \bar\psi(x)S^{-1}\gamma^\mu S\psi(x)$$

remember ②

$$S^{-1}\gamma^\mu S \rightarrow \Lambda^\mu{}_\nu\gamma^\nu$$

$$= \bar\psi(x)\Lambda^\mu{}_\nu\gamma^\nu\psi(x)$$

so $$\bar\psi'(x')\gamma^\mu\psi'(x') = \Lambda^\mu{}_\nu\bar\psi(x)\gamma^\nu\psi(x)$$

and remember the definition.

anything that transforms like:

$A'^\mu = \Lambda^\mu{}_\nu A^\nu$ is a 4-vector.

so $\bar\psi\gamma^\mu\psi$ transforms like a Lorentz 4-vector.

- Parity $S \rightarrow S_P$ as before

$$S_P^{-1}\gamma^\mu S_P = \gamma^0\gamma^\mu\gamma^0 \begin{cases} \gamma^0 & \mu=0 \\ -\gamma^i & \mu=i \end{cases}$$

so

$$\bar\psi(x)\gamma^\mu\psi(x) \xrightarrow{P} \bar\psi'(x')\gamma^\mu\psi'(x') = \begin{cases} \bar\psi(x)\gamma^0\psi(x) \\ -\bar\psi(x)\gamma^\mu\psi(x) \end{cases}$$

like a <u>polar</u> vector ... like $\vec{E}$ or $\vec{p}$

So, the probability current $j^\mu = \bar\psi\gamma^\mu\psi$ is a Lorentz 4-vector and a polar vector.

There are 5 independent quantities of the sort

$$\bar{\psi}\,\Gamma\,\psi$$

which have well-defined Lorentz Transformation properties. With one more definition, we can enumerate them

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) = \frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]$$

and

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

w/ $\gamma^{5\dagger} = \gamma^5$ $\qquad \overline{\gamma_5} = -\gamma_5$

$\gamma^5\gamma^5 = \mathbb{1}$

$$\{\gamma^5,\gamma^\mu\} = 0$$

BILINEAR COVARIANTS:

	Proper L.T.	Parity	name
$\bar{\psi}\psi$	S	+	scalar
$\bar{\psi}\gamma^\mu\psi$	V	−	vector
$\bar{\psi}\sigma^{\mu\nu}\psi$	T	mixed	tensor
$\bar{\psi}\gamma^5\psi$	S	−	pseudoscalar
$\bar{\psi}\gamma^\mu\gamma^5\psi$	V	+	axial vector

High Energy Limit

Define $\psi_R \equiv P_R \psi$ $\qquad$ $\psi_L \equiv P_L \psi$

where $P_R \equiv \frac{1+\gamma_5}{2}$ $\qquad$ $P_L \equiv \frac{1-\gamma_5}{2}$

projection operators: $P_{R,L}^2 = P_{R,L}$

$$P_R + P_L = 1$$

$$P_R P_L = P_L P_R = 0$$

$$\psi_R + \psi_L = 1$$

So, $P_R \gamma^\mu = \gamma^\mu P_L$

$$P_L \gamma^\mu = \gamma^\mu P_R$$

$$\bar{P}_R = P_L \quad \text{and} \quad \bar{P}_L = P_R$$

Zero mass D.E.:

$$i\gamma^\mu \partial_\mu \psi = 0$$

Look at ψ_R:

$$i\gamma^\mu \partial_\mu \psi_R = i\gamma^\mu P_R \partial_\mu \psi = i P_L \gamma^\mu \partial_\mu \psi = 0$$

but the mass term would have spoiled this

Note: $[\gamma^5, S] = 0$ (not shown)

$$[P_{R,L}, S] = [\tfrac{1 \pm \gamma^5}{2}, S] = 0$$

So, $\psi_{R,L}$ don't mix under Proper L.T.

$$\psi_R'(x') = P_R \psi'(x') = P_R S \psi(x) = S P_R \psi(x)$$
$$\psi_R'(x') = S \psi_R(x)$$

and separately

$$\psi_L'(x') = S \psi_L(x)$$

The "R" and "L" have meaning

Look at $P_L = \frac{1}{2}(1 - \gamma_5)$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

and try.

$$P_L u_+(p) = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_A \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} u_A \end{pmatrix} \sqrt{E+m}$$

$$P_L u_+(p) = \tfrac{1}{2}\sqrt{E+m}\begin{pmatrix} u_A - \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,u_A \\ -u_A + \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,u_A \end{pmatrix}$$

<u>now</u> look at the massless (extreme Relativistic) limit

$$m \to 0 \qquad E \to p$$

$$P_L u_+(p) = \tfrac{1}{2}\sqrt{|p|}\begin{pmatrix} u_A - \frac{\vec{\sigma}\cdot\vec{p}}{|p|}\,u_A \\ -u_A + \frac{\vec{\sigma}\cdot\vec{p}}{|p|}\,u_A \end{pmatrix}$$

$$\frac{\vec{\sigma}\cdot\vec{p}}{|p|} = \vec{\sigma}\cdot\hat{p} = \hat{h}$$

so, if $\hat{h}\,u_A = +u_A$ then $P_L u_+(p) = 0$

if $\hat{h}\,u_A = -u_A$ then $P_L u_+(p) = \sqrt{|p|}\begin{pmatrix} u_A \\ u_A \end{pmatrix}$

$$P_L u_+ \to u_L \qquad P_R u_+ \to u_R$$

and

$$P_L v_+ = \frac{1}{2}\sqrt{|p|} \begin{pmatrix} h v_A + v_A \\ h v_A + v_A \end{pmatrix}$$

$$h v_A = - v_A$$

$$h v_A = + v_A$$

$$P_L v_+ = \sqrt{|p|} \begin{pmatrix} v_A \\ v_A \end{pmatrix} \qquad \text{for} \qquad h v_A = + v_A$$

$$P_L v_+ = v_R \qquad\qquad P_R v_+ = v_L$$

So:

$$u_L = \frac{1-\gamma_5}{2} u \qquad \bar{u}_L = \bar{u} \frac{(1+\gamma_5)}{2}$$

$$u_R = \frac{1+\gamma_5}{2} u \qquad \bar{u}_R = \bar{u} \frac{(1-\gamma_5)}{2}$$

$$v_L = \frac{1+\gamma_5}{2} v \qquad \bar{v}_L = \bar{v} \frac{(1-\gamma_5)}{2}$$

$$v_R = \frac{1-\gamma_5}{2} v \qquad \bar{v}_R = \bar{v} \frac{(1+\gamma_5)}{2}$$