Relativistic Quantum Mechanics.

Schrodinger's original attempt at QM was relativistic — and it failed.

Instead, he defined $\Psi(\vec{x},t)$ as a 1 particle wave function satisfying

$$\frac{\partial \Psi(\vec{x},t)}{\partial t} = \frac{-i}{\hbar} \mathcal{H}\Psi(\vec{x},t)$$

It has definate $\vec{p}$ and E as a free particle and solution are

$$\Psi(\vec{x},t) = N e^{i(\vec{p}\cdot\vec{x} - Et)/\hbar}$$

Using the wave number, $\vec{k} = \frac{2\pi}{\lambda}\hat{k} = \frac{|\vec{p}|}{\hbar}\hat{k} = \frac{\vec{p}}{\hbar}$

related to the de Broglie wavelength $\lambda = \frac{2\pi\hbar}{|\vec{p}|}$

$$E = h\nu = \hbar\omega$$

so,

$$\Psi(\vec{x},t) = N e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

"First Quantization" $\equiv$

$$\vec{p} \longrightarrow -i\hbar\vec{\nabla}$$

$$E \longrightarrow i\hbar\frac{\partial}{\partial t}$$

So, the NR Hamiltonian

$$\mathcal{H} = \frac{\vec{p}^{\,2}}{2m} \longrightarrow \frac{(-i\hbar\vec{\nabla})^2}{2m} = -\frac{\hbar^2}{2m}\vec{\nabla}^2$$

and

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{x},t) = -i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t)$$

The free particle solutions give

$$\frac{p^2}{2m}\psi = E\psi \qquad E \text{ is all kinetic}$$

The obvious relativistic extensions (Schrodinger, Gordon, Klein, Fock, Kudar, deDonder, Van Dungen) in 1926:

$$\mathcal{H}^2 = \vec{p}^{\,2}c^2 + m^2c^4 \qquad \text{and "1st Quantized"}$$

BUT: remember Born's interpretation of ψ:

$$P = |\psi|^2 d^3x \qquad \equiv \text{ probability of finding } \psi \text{ within the 3-volume element } d^3x$$

$$0 \leq P \leq 1$$

Probability flux density:

$$\vec{j} = -\frac{i\hbar}{2m}\left(\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

$$\psi^* \rightarrow \quad \mathcal{H}\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\psi \longrightarrow \quad \mathcal{H}^*\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t}$$

Subtract the results:

$$\psi^*\mathcal{H}\psi - \psi\mathcal{H}^*\psi^* = i\hbar\,\psi^*\frac{\partial\psi}{\partial t} + i\hbar\,\psi\frac{\partial\psi^*}{\partial t}$$

$$= i\hbar\frac{\partial}{\partial t}\left(\psi^*\psi\right)$$

Sub $\mathcal{H} = -\frac{\hbar^2\vec{\nabla}^2}{2m}$

$$\vec{\nabla}\cdot\left[-\frac{\hbar^2}{2m}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)\right] = i\hbar\frac{\partial}{\partial t}\left(\psi^*\psi\right)$$

which looks like a continuity equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

$\rho = \psi^*\psi$ prob density $\ni \frac{\partial}{\partial t}\int d^3x\,\rho = 0$

from Gauss' Theorem $\Rightarrow$
ρ is Constant of the Motion.

Covariant notation:

$$\phi(\vec{x},t) \rightarrow \phi(x^\mu)$$

4d dot product

for free solution $\qquad \phi(x^\mu) = N e^{-p\cdot x/\hbar}$

$$p\cdot x = p^\mu x_\mu = \frac{E}{c}ct - \vec{p}\cdot\vec{x} \qquad p^\mu = (E/c, \vec{p}) \quad x^\mu = (ct, \vec{x})$$

Want $\partial_\mu^2 \phi$

$$\vec{\nabla}\phi(x^\mu) = \vec{\nabla} N e^{-ip\cdot x/\hbar} = N\vec{\nabla} e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - Et)}$$

$$= \frac{i}{\hbar}\vec{p}\,\phi(x^\mu)$$

$$\vec{\nabla}^2\phi(x^\mu) = \left(\frac{i}{\hbar}\vec{p}\right)^2\phi(x^\mu) = -\frac{\vec{p}^{\,2}}{\hbar^2}\phi(x^\mu)$$

$$\frac{\partial\phi(x^\mu)}{\partial t} = -\frac{i}{\hbar}E\phi(x^\mu)$$

$$\frac{\partial^2\phi(x^\mu)}{\partial t^2} = \left(-\frac{i}{\hbar}E\right)\phi(x^\mu) = -\frac{E^2}{\hbar^2}$$

So, $$\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(x^\mu) = \left(-\frac{p^2}{\hbar^2} + \frac{E^2}{c^2\hbar^2}\right)\phi(x^\mu)$$

$$= \frac{1}{\hbar^2}\left(\frac{E^2}{c^2} - \vec{p}^{\,2}\right)\phi(x^\mu)$$

$$= \frac{1}{\hbar^2}(m^2c^2)\,\phi(x^\mu)$$

So, this fits since:

$$\mathcal{H}^2 \varphi = \left[c^2 (-i\hbar \vec{\nabla}^2)^2 + m^2c^4 \right] \varphi$$

$$= \left(-c^2\hbar^2 \vec{\nabla}^2 + m^2c^4 \right) \varphi$$

$$= \left(c^2\hbar^2 \frac{\vec{p}^2}{\hbar^2} + m^2c^4 \right) \phi$$

$$= \underbrace{\left(c^2 \vec{p} + m^2c^4 \right)}\varphi$$

$$\mathcal{H}^2 \varphi = E^2 \varphi$$

relativistic eigenvalue equation for free particle.

Going backwards a bit,

$$\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2} \right) \varphi(x^\mu) = 0$$

is the relativistic 2nd order wave equation

Klein Gordon Equation.

NOTHING WRONG HERE! all relativistic wavefunctions must solve this equation

Remember: $\partial^\mu = (\partial^0, -\vec{\nabla}) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$

$$i\hbar\partial^\mu = (i\hbar\partial^0, -i\hbar\vec{\nabla})$$

so, $\partial^\mu\partial_\mu \equiv \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}\cdot\vec{\nabla}$

and K.G.

$$\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi(x^\mu) = 0$$

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi(x^\mu) = 0$$

Okay, great. How about probability?

Make the continuity equation in the standard way

$$\phi^*\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\right)\phi - \phi\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\right)\phi^* = 0$$

$$\vdots$$

$$\vec{\nabla}\cdot\left[\varphi^*\vec{\nabla}\varphi - \varphi\vec{\nabla}\varphi^*\right] - \frac{1}{c^2}\frac{\partial}{\partial t}\left[\varphi^*\frac{\partial}{\partial t}\varphi - \varphi\frac{\partial\varphi^*}{\partial t}\right] = 0$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad \uparrow$$

$$\vec{\nabla}\cdot\vec{J} \qquad - \qquad \frac{1}{c^2}\frac{\partial}{\partial t}\rho_0 \qquad = 0$$

$\uparrow$

looks like before

ρ_0 is <u>not</u> positive-definite

$\Rightarrow$ probability density is out!

What about the eigenvalues of $\mathcal{H}$?

$$\mathcal{H}\varphi(x^\mu) = \pm E\varphi(x^\mu) \qquad \Rightarrow \text{ negative energies!}$$

TOO MUCH FOR SCHRODINGER!

Play with it... put back the plane wave solutions.

$$\partial^\mu \varphi = \partial^\mu N e^{-ip\cdot x/\hbar} = -\frac{ip^\mu}{\hbar} N e^{-ip\cdot x/\hbar}$$

$$= -\frac{ip^\mu}{\hbar}\varphi$$

So, if $j^\mu = (\rho, \vec{j}) = \frac{\hbar i}{2m}\left(\varphi^* \partial^\mu \varphi - \varphi \partial^\mu \varphi^*\right)$

and $$j^\mu = \frac{\hbar i}{2m}\left[-\frac{ip^\mu}{\hbar}\varphi^*\varphi - \varphi\varphi^*\left(+\frac{ip^\mu}{\hbar}\right)\right]$$

$$j^\mu = \frac{1}{2m} p^\mu |\varphi|^2$$

So, the $-E$ solutions require $\rho < 0$.

In 1928 Dirac took an entirely new approach:

E & ρ problems relate to $\frac{\partial^2}{\partial t^2} \leftarrow$ 2nd order

Sought a 1st order equation to avoid $-E$ & $-\rho$ issues

Assumed: $H = \sqrt{c^2 p^2 + m^2 c^4} = c\vec{\alpha}\cdot\vec{p} + \beta m c^2$

need to find $\vec{\alpha}$ and β.

$$H^2 \psi = (c^2 p^2 + m^2 c^4)\psi$$

$$= (c\vec{\alpha}\cdot\vec{p} + \beta m c)^2 \psi$$

so $$c^2 p^2 + m^2 c^4 = \vec{\alpha}\cdot\vec{p}\,\vec{\alpha}\cdot\vec{p}\,c^2 + c\vec{\alpha}\cdot\vec{p}\,mc^2\beta + \beta mc^2 c\,\vec{\alpha}\cdot\vec{p} + \beta^2 m^2 c^4$$

conclude: ① $p^2 = \vec{\alpha}\cdot\vec{p}\,\vec{\alpha}\cdot\vec{p}$

② $\beta^2 = 1$

③ $\beta\vec{\alpha} + \vec{\alpha}\beta = 0$ $\Rightarrow$ matrices

$$\vec{p}^2 = \vec{\alpha}\cdot\vec{p}\,\vec{\alpha}\cdot\vec{p} = (\alpha^1 p^1 + \alpha^2 p^2 + \alpha^3 p^3)(\alpha^1 p^1 + \alpha^2 p^2 + \alpha^3 p^3)$$

$$= \alpha^1 p^1 \alpha^1 p^1 + \alpha^2 p^2 \alpha^2 p^2 + \alpha^3 p^3 \alpha^3 p^3 + \alpha^1 p^1 \alpha^2 p^2 + \alpha^2 p^2 \alpha^1 p^1 + \text{etc.}$$

$$= \alpha^1\alpha^1 p^{1\,2} + \alpha^2\alpha^2 p^{2\,2} + \alpha^3\alpha^3 p^{3\,2} + (\alpha^1\alpha^2 + \alpha^2\alpha^1)p^1 p^2 + \text{etc.}$$

get rid of cross terms $\Rightarrow$ impose $\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}$

$\equiv \{\alpha^i, \alpha^j\}$

anticommutator

③ satisfied by $\{\beta, \alpha^i\} = 0$

NEW ALGEBRA – anti commuting – not so long after the newness of Heisenberg et al. new commuting algebra.

So,
$$H\psi = \frac{i}{\hbar}\frac{\partial\psi}{\partial t}$$
$$H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2$$
$$\vec{p} \rightarrow -i\hbar\vec{\nabla}$$
$\Big\}$ 1st quantization

$$-i\hbar c\,\vec{\alpha}\cdot\vec{\nabla}\psi + \beta mc^2\psi = \frac{i}{\hbar}\frac{\partial\psi}{\partial t}$$

The DIRAC EQUATION. $\vec{\alpha}, \beta$ are matrices → operators

Need a basis for $\vec{\alpha}$ & β to operate within.

$$-i\hbar c\,(\vec{\alpha}\cdot\vec{\nabla})_{ab}\psi_b + mc^2\beta_{ab}\psi_b = \frac{i}{\hbar}\frac{\partial\psi_b}{\partial t}$$

matrix indices ⇒ ψ is a column matrix.
a SPINOR.

Convention:
$$\beta = \begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & \ddots & & & \\ & 0 & & -1 & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix} \begin{matrix} \uparrow \\ N \\ \downarrow \end{matrix} \quad \begin{matrix} \updownarrow N/2 \\ \updownarrow N/2 \end{matrix}$$
$$\longleftarrow N \longrightarrow$$

diagonal & traceless

N= 4 was choosen - and is the smallest, unique choice

$$\beta = \begin{pmatrix} +1 & & & \\ & +1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

which connects to the Pauli Matrices!

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \text{e.g.} \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \text{etc}$$

Then ①, ②, and ③ are all satisfied!

So: $$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Done with $\hbar$ & c: $\hbar = c = 1$.

$$i \frac{\partial \psi}{\partial t} = -i\, \vec{\alpha} \cdot \vec{\nabla} \psi + \beta m \psi \quad \checkmark$$

How about probability?

$$i \frac{\partial \psi}{\partial t} + i\, \vec{\alpha} \cdot \vec{\nabla} \psi - \beta m \psi = 0$$

$$\frac{\partial \psi}{\partial t} + \vec{\alpha} \cdot \vec{\nabla} + i \beta m \psi = 0$$

(A) $$\frac{\partial \psi}{\partial t} + \alpha_j \frac{\partial \psi}{\partial x^j} + i\beta m \psi = 0$$

adjoint

(B) $$\frac{\partial \psi^+}{\partial t} + \frac{\partial \psi^+}{\partial x^j}\alpha_j^+ - im\psi^+\beta^+ = 0$$

note $\psi = \begin{pmatrix} \\ \end{pmatrix} \Rightarrow \psi^+ = (\quad)^*$

$\psi^+ \rightarrow$ (A) $+$ (B) $\leftarrow \psi$

$$\psi^+\frac{\partial \psi}{\partial t} + \psi^+\alpha_j\frac{\partial \psi}{\partial x^j} + im\psi^+\beta\psi = 0$$

$$\frac{\partial \psi^+}{\partial t}\psi + \frac{\partial \psi^+}{\partial x^j}\alpha_j^+\psi - im\psi^+\beta\psi = 0$$

2 spaces: E_3, coordinate space $\rightarrow j$
spinor space

$$\psi^+\beta\psi = \sum_{ab}\psi_a^+\beta_{ab}\psi_b$$

add:

$$\frac{\partial}{\partial t}\psi^+\psi + \frac{\partial \psi^+}{\partial x^j}\alpha_j^+\psi + \psi^+\alpha_j\frac{\partial \psi}{\partial x^j} + im(\psi^+\beta\psi - \psi^+\beta^+\psi) = 0$$

α & β are Hermitian, so

$$\frac{\partial}{\partial t}\underbrace{\psi^+\psi}_{\rho} + \frac{\partial}{\partial x^j}\underbrace{(\psi^+\alpha_j\psi)}_{\vec{J}} = 0$$

so $\quad \rho = \psi^\dagger \psi > 0$ definate!

$$\vec{j} = \psi^\dagger \vec{\alpha} \psi$$

Symmetries $\rightarrow$ constants of the motion.

$$\frac{\partial A}{\partial t} = i \left[H, A \right] = 0$$ $\Rightarrow$ A is a constant of the motion & in a group theoretical sense, A is a generator of a conserved quantum number.

1. ENERGY

$$[H, H] = [\vec{\alpha}\cdot\vec{p} + \beta m, \vec{\alpha}\cdot\vec{p} + \beta m] = 0 \quad \checkmark$$

2 MOMENTUM

$$[H, p_i] = [\vec{\alpha}\cdot\vec{p} + \beta m, p_i]$$

$$= [\alpha_j p_j + \beta m, p_i]$$

but $[p_i, p_j] = 0$ & α & β commute with p

$$\Rightarrow [H, p_i] = 0 \quad \checkmark$$

3. ORBITAL ANGULAR MOMENTUM

$\vec{L} = \vec{r} \times \vec{p} \longrightarrow L_i = \varepsilon_{ijk} r_j p_k$

$i[H, L_i] = i[\alpha_n p_n + \beta m, \varepsilon_{ijk} r_j p_k]$

$[r_j, p_n] = i\delta_{jn}$ and other stuff commutes

$$i[H, L_i] = i[\alpha_n p_n, \varepsilon_{ijk} r_j p_k]$$

$$= i(\alpha_n p_n \varepsilon_{ijk} r_j p_k - \varepsilon_{ijk} r_j p_k \alpha_n p_n)$$

$$= i\alpha_n [p_n, r_j] \varepsilon_{ijk} p_k$$

$$= -i\alpha_n i\delta_{nj} \varepsilon_{ijk} p_k$$

$$= \alpha_j \varepsilon_{ijk} p_k = \varepsilon_{ijk} \alpha_j p_k$$

so $\quad i[H, \vec{L}] = \vec{\alpha} \times \vec{p} = \frac{d\vec{L}}{dt} \neq 0 \qquad \times$

L is not a symmetry $\Rightarrow$ no conserved angular momentum

In regular Q.M. $\quad \psi \sim f(r) \cdot \underbrace{Y_\ell^m(\theta, \varphi)}_{\text{eigenstates of } L}$

not in Dirac theory

Search for some<u>thing</u> that's halfway related...

try $\quad \vec{c} \equiv \vec{\alpha} \times \vec{\alpha}$

$$i\,[H, \vec{\alpha}\times\vec{\alpha}]$$

Some useful identities: $\quad \{\beta, \alpha_i\} = 0 \Rightarrow \beta\alpha_i = -\alpha_i\beta$

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \Rightarrow \alpha_i\alpha_j = 2\delta_{ij} - \alpha_j\alpha_i$$

$$i\,[H, (\vec{\alpha}\times\vec{\alpha})_j] = i\,[H, \varepsilon_{jkl}\alpha_k\alpha_l]$$

$$= i\,[\alpha_i p_i + \beta m, \varepsilon_{jkl}\alpha_k\alpha_l]$$

$$= i\{(\alpha_i p_i + \beta m)\varepsilon_{jkl}\alpha_k\alpha_l - \varepsilon_{jkl}\alpha_k\alpha_l(\alpha_i p_i + \beta m)\}$$

$$= i\{\alpha_i p_i \varepsilon_{jkl}\alpha_k\alpha_l + \beta m\,\varepsilon_{jkl}\alpha_k\alpha_l - \varepsilon_{jkl}\alpha_k\alpha_l\alpha_i p_i - \varepsilon_{jkl}\alpha_k\alpha_l\beta m\}$$

$$\beta m\,\varepsilon_{jkl}\alpha_k\alpha_l = -m\,\varepsilon_{jkl}\alpha_k\beta\alpha_l = +m\,\varepsilon_{jkl}\alpha_k\alpha_l\beta$$

β terms cancel

$$= i\{\alpha_i p_i \varepsilon_{jkl}\alpha_k\alpha_l - \varepsilon_{jkl}\alpha_k\alpha_l\alpha_i p_i\}$$

$$= i\,\varepsilon_{jkl}\,p_i\{\alpha_i\alpha_k\alpha_l - \alpha_k\alpha_l\alpha_i\}$$

$$\{\ \} = \alpha_i\alpha_k\alpha_l + \alpha_k\alpha_i\alpha_l - 2\delta_{il}\alpha_k$$

$$= \alpha_i\alpha_k\alpha_l - \alpha_i\alpha_k\alpha_l + 2\delta_{ki}\alpha_l - 2\delta_{il}\alpha_k$$

$$= 2\delta_{ki}\alpha_l - 2\delta_{il}\alpha_k$$

$$i[H, (\alpha\times\alpha)_j] = 2i\,\varepsilon_{jkl}\,p_i\,\delta_{ki}\alpha_l - 2i\,\varepsilon_{jkl}\,p_i\,\delta_{il}\alpha_k$$

$$= 2i\,\varepsilon_{jkl}\,p_k\alpha_l - 2i\,\varepsilon_{jkl}\,p_l\alpha_k$$

$$= 4i\,\varepsilon_{jkl}\,p_k\alpha_l = 4i\,(\vec{p}\times\vec{\alpha})_j$$

$$i[H, (\vec{\alpha}\times\vec{\alpha})_j] = -4i\,(\vec{\alpha}\times\vec{p})_j = \frac{d}{dt}(\vec{\alpha}\times\vec{\alpha})_j$$

sorta like $\frac{d\vec{L}_j}{dt} = (\vec{\alpha}\times\vec{p})_j$

SO,

$$i\left[H, \vec{L} + \frac{1}{4i}(\vec{\alpha}\times\vec{\alpha})\right] = 0$$

$L + \frac{1}{4i}(\vec{\alpha}\times\vec{\alpha})$ is a constant of the motion ?

Look at this

$$(\vec{\alpha} \times \vec{\alpha})_i = \varepsilon_{ijk}\, \alpha_j \alpha_k = \varepsilon_{ijk} \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

$$= \varepsilon_{ijk} \begin{pmatrix} \sigma_j \sigma_k & 0 \\ 0 & \sigma_j \sigma_k \end{pmatrix}$$

$$= \varepsilon_{ijk} \begin{pmatrix} i\,\varepsilon_{jk\ell}\, \sigma_\ell & 0 \\ 0 & i\,\varepsilon_{jk\ell}\, \sigma_\ell \end{pmatrix}$$

$\uparrow$ from commutation relations for $\sigma \rightarrow$ the Lie Algebra of $SU(2)$

$$= i\, \varepsilon_{ijk}\, \varepsilon_{jk\ell} \begin{pmatrix} \sigma_\ell & 0 \\ 0 & \sigma_\ell \end{pmatrix}$$

$$\equiv i\, \underbrace{\varepsilon_{ijk}\, \varepsilon_{\ell jk}}_{2\delta_{i\ell}}\, \Sigma_\ell \qquad \Sigma_\ell \equiv \begin{pmatrix} \sigma_\ell & 0 \\ 0 & \sigma_\ell \end{pmatrix}$$

$$(\vec{\alpha} \times \vec{\alpha})_i = 2i\, \delta_{i\ell}\, \Sigma_\ell = 2i\, \Sigma_i$$

or. $$\frac{1}{4i} (\vec{\alpha} \times \vec{\alpha})_i = \frac{1}{2} \Sigma_i$$

$\uparrow$ "spin" - ish

$$S_i \equiv \frac{1}{2} \Sigma_i$$ The spin operator.

The constant of the motion is

$$\vec{J} = \vec{L} + \vec{S} \qquad \text{total angular momentum.}$$

SPIN comes for FREE in the Dirac Equation!

Notice $\qquad S^2 = \frac{1}{4}\Sigma_i \Sigma_i = \frac{1}{4}\left(\Sigma_1 \Sigma_1 + \Sigma_2 \Sigma_2 + \Sigma_3 \Sigma_3 \right)$

$$\Sigma_i \Sigma_i = \begin{pmatrix} \sigma_i^2 & 0 \\ 0 & \sigma_i^2 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \mathbb{1}$$

So,

$$S^2 = \frac{3}{4}\mathbb{1}$$

$$\parallel$$

$$\frac{1}{2}\left(1 + \tfrac{1}{2}\right) \Rightarrow s(s+1)$$

it really is SPIN -- S^2 is the Casimir Operator for IRR of SU(2)

SPIN ½ COMES FOR FREE in the Dirac Equation!

shut UP!!

Form: $\vec{p}\cdot\vec{J} = \cancel{\vec{p}\cdot\vec{L}}^{\,0} + \vec{p}\cdot\vec{S}$

$\vec{p}\cdot\vec{J} = \vec{p}\cdot\vec{S}$ ⟵ component of SPIN along direction of motion $\equiv$ HELICITY

So, obviously

$$i[H, \vec{p}\cdot\vec{S}] = i\left[H, \vec{p}\cdot\frac{(\vec{\alpha}\times\vec{\alpha})}{4i}\right] = -i\vec{p}\cdot\vec{\alpha}\times\vec{p} = 0$$

and we see that $\vec{p}\cdot\vec{S}$, HELICITY <u>is</u> a constant of the motion.

Helicity conservation is the real, relativistic analog to angular momentum conservation.

Need a covariant form for Dirac Equation

CONVENTION: $\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (4x4, right?)

$$\vec{\gamma} \equiv \beta\vec{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

so, $\gamma^\mu = (\gamma^0, \vec{\gamma})$

can raise and lower $\gamma_\mu = g_{\mu\nu}\gamma^\nu = (\gamma_0, -\vec{\gamma})$

The following are really useful:

$$\{\gamma_0, \gamma_i\} = \gamma_0\gamma_i + \gamma_i\gamma_0$$
$$= \beta\beta\alpha_i + \beta\alpha_i\beta = \beta\beta\alpha_i - \beta\beta\alpha_i = 0$$

$$\{\gamma_i, \gamma_j\} = \gamma_i\gamma_j + \gamma_j\gamma_i$$
$$= \beta\alpha_i\beta\alpha_j + \beta\alpha_j\beta\alpha_i$$
$$= -\beta\beta\alpha_i\alpha_j - \beta\beta\alpha_j\alpha_i$$
$$= -(\alpha_i\alpha_j + \alpha_j\alpha_i)$$
$$= -\{\alpha_i, \alpha_j\} = -2\delta_{ij}$$

so $$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

Also: $$\beta^+ = \beta = (\gamma^0)^+ = \gamma^0$$

$$\gamma_i^+ = (\beta\alpha_i)^+ = \alpha_i^+\beta^+ = \alpha_i\beta = -\beta\alpha_i = -\gamma_i$$

$\Rightarrow$ β is Hermitian
γ^i is antiHermitian

Useful: $$\gamma^{i+} = -\gamma^i \times \underset{\gamma^0\gamma^0}{\underset{\uparrow}{\mathbb{1}}}$$

$$\gamma^{i+} = -\gamma^i\gamma^0\gamma^0 = +\gamma^0\gamma^i\gamma^0$$

$$\gamma^{0+} = \gamma^0\gamma^0\gamma^0 = \gamma^0$$

so

$$\gamma^{\mu+} = \gamma^0\gamma^\mu\gamma^0$$

Now, the Dirac Equation becomes

$$\left(i\alpha_j \frac{\partial}{\partial x^j} + i\frac{\partial}{\partial t} - \beta m\right)\psi = 0$$

$$\left(i\beta\alpha_j \frac{\partial}{\partial x^j} + i\beta\frac{\partial}{\partial t} - \beta\beta m\right)\psi = 0$$

$$\left(i\gamma_j \frac{\partial}{\partial x^j} + i\gamma_0\frac{\partial}{\partial t} - m\right)\psi = 0$$

$$\left(i\gamma^0 \frac{\partial}{\partial t} - i\gamma^j\frac{\partial}{\partial x^j} - m\right)\psi = 0$$

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi = 0$$

or $$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$$

In momentum space: $$(\gamma^\mu p_\mu - m)\psi = 0$$

The conjugate equation will be important

$$\psi^\dagger\left(-i\overleftarrow{\partial}_\mu(\gamma^\mu)^\dagger - m\right) = 0$$

$$\psi^\dagger\left(-i\overleftarrow{\partial}_\mu\gamma^0\gamma^\mu\gamma^0 - m\right) = 0$$

$$\psi^\dagger\left(-i\overleftarrow{\partial}_\mu\gamma^0\gamma^\mu\gamma^0\gamma^0 - m\gamma^0\right) =$$

$$\psi^\dagger\gamma^0\left(-i\overleftarrow{\partial}_\mu\gamma^\mu - m\right) = 0$$

DEFINE $\bar{\psi} \equiv \psi^\dagger\gamma^0$ so we get:

$$\bar{\psi}\left(i\gamma^\mu\overleftarrow{\partial}_\mu + m\right) = 0 \quad \text{and} \quad \bar{\psi}\left(\gamma^\mu p_\mu + m\right) = 0$$

Feynman's shorthand notation – "slash notation"

$$B_\mu \gamma^\mu \equiv \not{B}$$

in LaTeX.. `\def\slash#1{#1\!\!\!\!/\!,\,}`

so that $\not{B} \rightarrow$ `\slash{B}`

no charge.

Then, $$(i\not{\partial} - m)\psi(x) = 0$$

$$\bar{\psi}(x)(i\overleftarrow{\not{\partial}} + m) = 0$$

$$(\not{p} - m)\psi(p) = 0$$

$$\bar{\psi}(p)(\overleftarrow{\not{p}} + m) = 0$$

Watch this:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$i\gamma^\nu \partial_\nu \rightarrow$

$$(-i\gamma^\nu \gamma^\mu \partial_\nu \gamma_\mu - i\gamma^\nu \partial_\nu m)\psi = 0$$

notice $m i\gamma^\nu \partial_\nu \psi = m^2 \psi$ from the D.E.

so $$(-\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu - m^2)\psi = 0$$

add in another one

$$(-\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - m^2 - \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi = 0$$

$$(-\gamma^\nu\gamma^\mu\partial_\mu\partial_\nu - m^2 - \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi =$$

$$[-(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)\partial_\mu\partial_\nu - 2m^2]\psi =$$

$$[-2g^{\mu\nu}\partial_\mu\partial_\nu - 2m^2]\mathbb{1}\psi = 0$$

look at a single component ψ_s $\psi = \Big(\;\Big) \ldots \psi_a$

$$(\partial^\nu\partial_\nu + m^2)\psi_a = 0$$

$$(\Box + m^2)\psi_a = 0$$

a Klein-Gordon Equation!
for every ψ component.

This had to happen .. arguments for K.G.E. were completely general.

SO: probability okay
still have negative energies.