Since components satisfy K.G., can write

$$\psi_i(x) \sim f_i(p)\, e^{-ip\cdot x} \qquad \text{for free particles.}$$

$$= N u_i(p) e^{-ip\cdot x} \qquad i = 1,2,3,4$$

Consider rest frame

$$\psi_i(x) = N u_i(0)\, e^{-imx_0}$$

DE:
$$i\frac{\partial \psi}{\partial t} = (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi \xrightarrow{\text{R.F.}}$$

$$i\frac{\partial}{\partial x_0} N u_i(0) e^{-imx_0} = i(-im)\overbrace{N u_i(0) e^{-imx_0}}^{\psi} = \beta m \psi$$

$$m\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\psi = m\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\psi$$

or
$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_3 \\ -\psi_4 \end{pmatrix} \Rightarrow \psi_3 = \psi_4 = 0 \text{ in rest frame}$$

$\Rightarrow$ a natural segregation into upper and lower components in Dirac spinors

call them u_A and u_B

$$u(p) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

So, before: $m\psi = \beta m \psi$

$$mu = \beta m u$$

$$m\begin{pmatrix} u_A \\ u_B \end{pmatrix} = m\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = m\begin{pmatrix} u_A \\ -u_B \end{pmatrix}$$

$$\Rightarrow u_B = 0$$

Back to general form

$$\psi(x) = \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix} e^{-ip\cdot x}$$

get used to this 2x2 shorthand for the real 4x4 spinor space?

into D.E.

$$i(-ip_0)u(p) = -i(i\vec{p})\cdot\vec{\alpha}\, u(p) + \beta m u(p)$$

$$p_0 u(p) = -i(i\vec{p})\cdot\vec{\alpha}\, u(p) + \beta m u(p)$$

$$= (\vec{\alpha}\cdot\vec{p} + \beta m)u(p)$$

$$\begin{pmatrix} p_0 & 0 \\ 0 & p_0 \end{pmatrix} u = \underbrace{\left[\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} m\right]}_{\mathcal{H} = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & m \end{pmatrix}} u$$

and $\mathcal{H}u = Eu$.

Actually:

$$\mathcal{H} = \begin{pmatrix} m & 0 & p_3 & p_1 - ip_2 \\ 0 & m & p_1 + ip_2 & -p_3 \\ p_3 & p_1 - ip_2 & -m & 0 \\ p_1 + ip_2 & -p_3 & 0 & -m \end{pmatrix}$$

Here: $\mathcal{H} u = p_0 u$

$$\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} p_0 & 0 \\ 0 & p_0 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\begin{pmatrix} m u_A + \vec{\sigma}\cdot\vec{p}\, u_B \\ \vec{\sigma}\cdot\vec{p}\, u_A - m u_B \end{pmatrix} = \begin{pmatrix} p_0 u_A \\ p_0 u_B \end{pmatrix}$$

coupled set of 4 equations

$$\begin{pmatrix} (p_0 - m) & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & (p_0 + m) \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

The condition that a solution exists is $\det\left(\quad\right) = 0$

$$(p_0 - m)(p_0 + m) - (-\vec{\sigma}\cdot\vec{p})(-\vec{\sigma}\cdot\vec{p}) = 0$$

remember $\quad \vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B} = \vec{A}\cdot\vec{B} - i\,\vec{A}\cdot\vec{\sigma}\times\vec{B}$

so,
$$p_0^2 - m^2 - p^2 = 0$$
$$p_0^2 = \vec{p}^{\,2} + m^2$$

For a given $\vec{p}$, the energy eigenvalues must be:

$$p_0 = \pm\sqrt{\vec{p}^{\,2} + m^2} \equiv \pm E \qquad E > 0$$

choose $p_0 = +E$

$$\begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & E+m \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$(E-m)u_A - \vec{\sigma}\cdot\vec{p}\, u_B = 0$$
$$-\vec{\sigma}\cdot\vec{p}\, u_A + (E+m)u_B = 0$$

$$\underbrace{\begin{aligned} u_A &= \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\, u_B \\ u_B &= \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\, u_A \end{aligned}}_{\text{combine}}$$

$$(E^2 - m^2)u_A = \vec{p}^{\,2} u_A \qquad \text{Ⓐ}$$

choose $p_0 = -E$

$$\begin{pmatrix} -E-m & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & E+m \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$\vdots$$

$$\left.\begin{aligned} u_A &= -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\, u_B \\ u_B &= -\frac{\vec{\sigma}\cdot\vec{p}}{E-m}\, u_A \end{aligned}\right\} \quad (E^2 - m^2)u_B = \vec{p}^{\,2} u_B \qquad \text{Ⓑ}$$

Ⓐ and Ⓑ tell us that within u_A: u_1 and u_2 are arbitrary

within u_B: u_3 and u_4 are arbitrary

From Ⓐ: fix $u_A \Rightarrow u_B$ determined

Ⓑ: fix $u_B \Rightarrow u_A$ determined

Ⓐ: choose $u_1 = 1 \quad u_2 = 0$

or

$u_1 = 0 \quad u_2 = 1$

up to an overall constant

any other choice? linear combination.
Same for Ⓑ.

$p_0 = +E \qquad u_B = \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\, u_A$

So

$$u(p)_+ = N \begin{pmatrix} u_A \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\, u_A \end{pmatrix}$$

call the 2 solutions

$$u_A(\text{solution } 1) \equiv \chi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_A(2) \equiv \chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

so,

$$\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\chi^1 = \frac{1}{E+m}\begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{E+m}\begin{pmatrix} p_3 \\ p_1 + ip_2 \end{pmatrix}$$

and

$$\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\chi^2 = \frac{1}{E+m}\begin{pmatrix} p_1 - ip_2 \\ -p_3 \end{pmatrix}$$

So,

$\boxed{p_0 = +E}$

$$u(p)_+^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ p_3 \\ \frac{p_1 + ip_2}{E+m} \end{pmatrix}$$

$$u(p)_+^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_1 - ip_2}{E+m} \\ \frac{-p_3}{E+m} \end{pmatrix}$$

abbreviate

$$u(p)_+^{1,2} = N \begin{pmatrix} \chi^{1,2} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{1,2} \end{pmatrix}$$

$p_0 = -E$

$\vdots$

$$u(p)_- = N \begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} u_B \\ u_B \end{pmatrix}$$

$$u_B(\text{solution } 1) \equiv \chi^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_B(2) \equiv \chi^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\boxed{p_0 = -E}$ $$u(p)_-^{3,4} = N \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{3,4} \\ \chi^{3,4} \end{pmatrix}$$

Normalization:

might think ... $\psi^\dagger\psi = 1$. But then probability would not be Lorentz invariant

$$P = \int \psi^\dagger\psi \, d^3x$$

transforms like $\frac{1}{\gamma} d^3x$

so, want something like — Lorentz γ of course

$\psi^\dagger\psi$ transform like γ

Look at D.E.

$\bar{u}(p)\gamma^\mu \longrightarrow$ $$(\not{p} - m)u(p) = 0$$

$$\bar{u}(p)(\not{p} - m) = 0 \qquad \longleftarrow \gamma^\mu u(p)$$

and add $$\bar{u}(p)\left[\gamma^\mu\not{p} - \gamma^\mu m + \not{p}\gamma^\mu - m\gamma^\mu\right]u(p) = 0$$

$$\bar{u}(p)\left[\gamma^\mu\not{p} + \not{p}\gamma^\mu\right]u(p) = 2m\,\bar{u}(p)\gamma^\mu u(p)$$

$$\bar{u}(p)\,p_\nu\Big[\underbrace{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu}_{2g^{\mu\nu}}\Big]u(p) =$$

$$2p^\mu\, \bar{u}(p)u(p) = 2m\, \bar{u}\gamma^\mu u$$

try $\mu = 0$
$$p^0 \bar{u}u = m u^+ \gamma^0\gamma^0 u$$
$$= m\, u^+ u$$
$$u^+u = \frac{p^0}{m}\, \bar{u}(p)u(p)$$

could choose $\quad u^+u = \gamma = \dfrac{p_0}{m} \quad \Rightarrow \quad \bar{u}u = 1$

Lorentz!

But, as a normalization $\dfrac{p_0}{m}$ isn't convenient for massless particles. It <u>is</u> used, but not for me.

Use $\quad \bar{u}u = 2m$

so

$$u^+u = \frac{p_0}{m}\, 2m = 2p^0 = 2m\gamma$$

cheap

$p^0 = +E \qquad \bar{u}_i(p)u_j(p) = 2m\,\delta_{ij} \qquad i,j = 1,2$

$p^0 = -E \qquad \bar{u}_i(p)u_j(p) = -2m\,\delta_{ij} \qquad i,j = 3,4$

N: look at 1,2 states.

$$\bar{u}_i u_i = N^2 \left(\chi^+ , \chi^+ \left(\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \right)^+ \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \chi \\ -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi \end{pmatrix} = 2m$$

$\vdots$ problem

$$2m = N^2 \, 2m \frac{[m+E]}{(E+m)^2}$$

$$\Rightarrow N = \sqrt{E+m}$$

$$u(p)_+^{(1,2)} = \sqrt{E+m} \begin{pmatrix} \chi^{1,2} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{1,2} \end{pmatrix}$$

$$u(p)_-^{(3,4)} = \sqrt{E+m} \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{3,4} \\ \chi^{3,4} \end{pmatrix}$$

Define $p_+ \equiv p_1 + i p_2$

$p_- \equiv p_1 - i p_2$

$$\psi_1 = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_+}{E+m} \end{pmatrix} e^{-iEt + i\vec{p}\cdot\vec{x}}$$

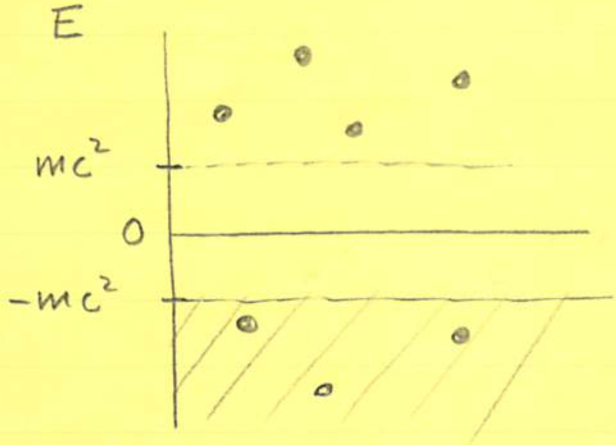
$$\psi_2 = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ \frac{-p_3}{E+m} \end{pmatrix} e^{-iEt + i\vec{p}\cdot\vec{x}}$$

$$\psi_3 = \sqrt{E+m} \begin{pmatrix} \frac{-p_3}{E+m} \\ \frac{-p_+}{E+m} \\ 1 \\ 0 \end{pmatrix} e^{iEt + i\vec{p}\cdot\vec{x}}$$

$$\psi_4 = \sqrt{E+m} \begin{pmatrix} \frac{-p_-}{E+m} \\ \frac{p_3}{E+m} \\ 0 \\ 1 \end{pmatrix} e^{iEt + i\vec{p}\cdot\vec{x}}$$

negative energies : Dirac's idea

– energies okay!

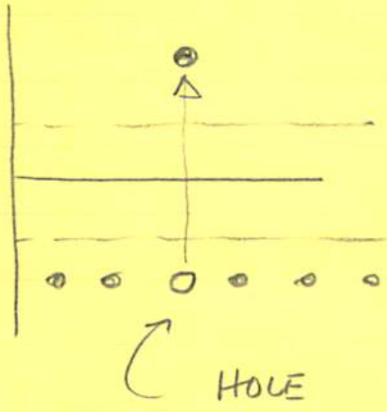


← +E electrons
$E > mc^2$
few.

← −E electrons
VACUUM = FULL

By Pauli Exclusion Principle
no −E electron can
go to −∞ energy.

Suppose one −E electron gains $\geq 2mc^2$



$$Q_{left} = Q_{VACUUM} - Q_{removed} = Q_{VACUUM} + |e|$$

so, $$Q_{left} - Q_{VACUUM} = Q_{HOLE} = +|e|$$

&

$$E_{left} = E_{VACUUM} - E_{removed} = E_{VACUUM} + |E_{negative}|$$

$$E_{HOLE} = E_{left} - E_{VACUUM} = +|E_{negative}|$$

Absence of $-E$ particle = Presence of $+E$ hole

+Q antiparticle

$$\gamma \rightarrow e^+ e^-$$

Dirac proposed this 1930. Thought holes = protons.

nope $m_{hole} = m_{electron}$, so proposed positron.

1932 — found by Anderson in cosmic rays.

Whole business makes no sense as simply a theory of single electrons.

N.R. Q.M. $\rightarrow$ single particle theory.

R.Q.M. $\rightarrow$ NECESSARILY multiparticle theory.

a relativistic effect.

Know this anyhow.

position – uncertain!

$\leftarrow \Delta L \rightarrow$

$$\Delta p = \frac{\hbar}{\Delta L}$$

suppose you "look" with Compton-length precision

$$\Delta L \sim \lambda_c = \frac{h}{mc}$$

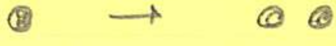
$$\Delta p \sim \frac{h}{h/mc}$$

$$\Delta p \sim mc$$

$$\Delta E \sim \Delta p c \sim mc^2$$

as much as the rest mass just from uncertainty.

squeeze new particles from the vacuum



$-E$, Feynman.

$$j^\mu = (\rho, \vec{j})$$

$$\rho = 2\rho \qquad \vec{j} = 2\vec{p} \qquad \Big\} \quad j^\mu = 2p^\mu$$

probability 4-current.

It's an electric charge 4-current by

$$j^\mu_{EM,e^-} = Q_e \, j^\mu$$

$$= -2ep^\mu = -2e(p^0, \vec{p}) \qquad p^0 = \sqrt{\vec{p}^2 + m^2} = E$$

For <u>anti</u>particle state

$$j^\mu_{EM,e+} = Q_{positron} \, j^\mu = +2e(p^0, \vec{p})$$

$$= -2e(-p^0, -\vec{p})$$

$$j^\mu_{EM,e+}(p^\mu) = j^\mu_{EM,e^-}(-p^\mu)$$

$$e^{-ip_0 t} = e^{-i(-p_0)(-t)}$$

+E wavefunction

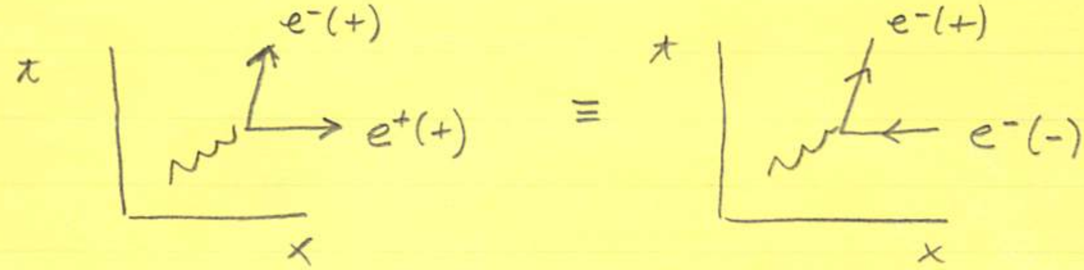
−E wavefunction AND backwards in time

Feynman concluded:

−E particles going backwards in time

$$\equiv$$

+E antiparticles going forward in time



Time to interpret the solutions.

Imagine a $U(1)$ phase rotation

$$|\psi(x)\rangle \xrightarrow{U(1)} |\psi'(x)\rangle = R(\theta)|\psi(x)\rangle = e^{iq\theta}|\psi(x)\rangle$$

θ: a constant "GLOBAL" phase. What happens?

D.E. $\quad (i\not{\partial} - m)\psi(x) = 0$

$$\downarrow U(1)$$

$$0 = (i\not{\partial} - m)\psi'(x) = (i\not{\partial} - m)e^{iq\theta}\psi(x) = 0$$

trivially the same as $(i\not{\partial} - m)\psi(x) = 0$

$\Rightarrow$ D.E. invariant w.r.t. a global $U(1)$ transformation

What about a "LOCAL" $U(1)$ transformation $\Rightarrow \theta = \theta(x^\mu)$

$$|\psi(x)\rangle \longrightarrow |\psi'(x)\rangle = R(\theta(x))|\psi(x)\rangle = e^{iq\theta(x)}|\psi(x)\rangle$$

D.E. $\longrightarrow \quad (i\not{\partial} - m)\psi'(x) = 0$

$$(i\not{\partial} - m)e^{iq\theta(x)}\psi(x) = 0$$

$$i\partial_\mu\gamma^\mu e^{iq\theta(x)}\psi(x) - me^{iq\theta(x)}\psi(x) = 0$$

$$i\left[iq(\partial_\mu\theta(x))\gamma^\mu e^{iq\theta(x)}\psi(x) + \partial_\mu\psi(x)e^{iq\theta(x)}\gamma^\mu\right] - me^{iq\theta(x)}\psi = 0$$

$$\left\{ -q\left[\partial_\mu \theta(x)\right]\gamma^\mu + i\partial_\mu\gamma^\mu - m \right\}\psi(x) = 0$$

$$i(\not\partial - m)\psi(x) - q\not\partial\theta\,\psi = 0$$

spoils the invariance!

SUPPOSE YOU MUST HAVE THE INVARIANCE AT ALL COST!

Do what you have to do...

Old trick: covariant derivative: $D^\mu \equiv \partial^\mu + X^\mu$

some 4-vector

Find X^μ so that

$$(i\not\partial - m)\psi(x) = 0 \qquad \text{is invariant.}$$

$$= (i\not D - m)\psi'(x)$$

$$\left[i\gamma_\mu(\partial^\mu + X^\mu) - m\right]\psi e^{iq\theta(x)} = 0$$

$$i\gamma_\mu\, iq\left[\partial^\mu\theta(x)\right]\psi e^{iq\theta(x)} + i\gamma_\mu(\partial^\mu\psi)e^{iq\theta(x)}$$

$$+ i\gamma_\mu X^\mu \psi e^{iq\theta(x)} - m\psi e^{iq\theta(x)} = 0$$

$$-q\not\partial\theta(x)\psi + i\not\partial\psi + i\not X\psi - m\psi = 0 \qquad \star$$

not quite right → make it so

Suppose the U(1) phase transformation

$$\psi \to \psi' = e^{i\theta(x)}\psi(x) \qquad ①$$

goes along with an additional transformation

$$X^\mu \to X^{\mu\,\prime} = X^\mu - iq\,\partial^\mu\theta(x) \qquad ②$$

that is vector is undetermined up to a gradient of a scalar. FAMILIAR??

Then ✱ would be, including ②:

$$i\gamma_\mu\left[\partial^\mu + X'^\mu + iq\partial^\mu\theta\right]\psi - m\psi = 0$$

$$i\gamma_\mu\left[\partial^\mu + X^\mu - iq\partial^\mu\theta + iq\partial^\mu\theta\right]\psi - m\psi = 0$$

$$i\gamma_\mu\left[\partial^\mu + X^\mu\right]\psi - m\psi = 0$$

$$\left[i\not{D} - m\right]\psi = 0 \qquad ✓$$

So, what's gained by adding X^μ?

Remember the mysterious "minimum coupling rule"?

$$\vec{p} \longrightarrow \vec{p} - q\vec{A}$$

$$E \to E - q\phi$$

is the classical way to add electromagnetism to an equation covariantly : $\quad p^\mu \to p^\mu - qA^\mu$

Quantize $\quad i\frac{\partial}{\partial t} \to i\frac{\partial}{\partial t} - q\phi \quad \to \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + iq\phi$

$$-i\vec{\nabla} \to -i\vec{\nabla} - q\vec{A} \quad \to -\vec{\nabla} \to -\vec{\nabla} + iq\vec{A}$$

remember $\quad \partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$

So, $\quad \partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \to \left(\frac{\partial}{\partial t} + iq\phi, -\vec{\nabla} + iq\vec{A}\right)$

or $\quad \partial^\mu \to \partial^\mu + iqA^\mu \quad \equiv D^\mu.$

Identify $\quad X^\mu \equiv iqA^\mu \quad$ and ② becomes.

$$\cancel{iq}A^\mu \to \cancel{iq}A^\mu - \cancel{iq}\partial^\mu\theta$$

$$A^\mu \to A^\mu - \partial^\mu\theta \qquad \textit{Gauge Invariance}$$

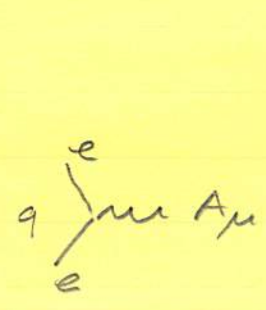
So, the Gauge Invariant Dirac Equation is

$$(i\not{\partial} - m)\psi = 0$$

$$i\gamma_\mu(\partial^\mu + iqA^\mu)\psi - m\psi = 0$$

$$(i\not{\partial} - q\not{A} - m)\psi = 0$$

$$\underbrace{(i\not{\partial} - m)\psi}_{\text{free}} = \underbrace{q\not{A}\psi}_{\text{source}}$$



Writing it out ... (problem)

$$i\frac{\partial\psi}{\partial t} = (\vec{\alpha}\cdot\vec{p} + \beta m + e\vec{\alpha}\cdot\vec{A} - e\phi)\psi$$

So, electric charge, $q=-e$ has snuck in

If it's charge, then should be able to change from $-e$ equation (electron) to $+e$ equation (positron) via some CHARGE CONJUGATION OPERATOR

$$(i\not{\partial} + e\not{A} - m)\psi(x) = 0 \qquad \text{D.E.}$$

$$\uparrow C'$$

$$(i\not{\partial} - e\not{A} - m)\psi(x) = 0 \qquad \text{the "positron" D.E.}$$

Take $*$ and note that $A^{\mu *} = A^\mu$

$$(-i\not{\partial}^* - e\not{A}^* - m^*)\psi^* = 0$$

$$[(-i\partial_\mu - eA_\mu)\gamma^{\mu *} - m]\psi^* = 0$$

$$C' \rightarrow [\ ]\psi^* = 0$$

$$[(i\partial_\mu + eA_\mu)C'\gamma^{\mu *} + C'm]\psi^* = 0$$

$$[(i\partial_\mu + eA_\mu)C'\gamma^{\mu *}C'^{-1}C' + C'mC'^{-1}C']\psi^* = 0$$

$$[(i\partial_\mu + eA_\mu)\, c'\gamma^{\mu *} c'^{-1} + m]\, c'\psi^* = 0$$

$\widetilde{}$
want this to be $\psi_c \equiv c'\psi^*$

Remember $\quad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$

so,

$$\gamma^{\mu *} = (\gamma^{\mu\dagger})^T = \gamma^0\gamma^{\mu T}\gamma^0 \qquad \gamma^{0T} = \gamma^0$$

we need $\quad c'\gamma^{\mu *}c'^{-1} = c'\gamma^0\gamma^{\mu T}\gamma^0 c'^{-1} = -\gamma^\mu$

so $\quad \{[-(i\partial_\mu + eA_\mu)\gamma^\mu + m\}\psi = 0$

$$[i\cancel{\partial} + e\cancel{A} - m]\,\psi(x) = 0 \qquad \text{the electron DE}$$

Define $\quad C \equiv c'\gamma^0$

$$-\gamma^\mu = C\gamma^{\mu T}C^{-1}$$
$$-\gamma^\mu C = C\gamma^{\mu T}$$

$$0 = C\gamma^{\mu T} + \gamma^\mu C$$

what's γ^T?

if $\quad \gamma^T = \gamma : \quad \{\gamma, C\} = 0$

$\quad\;\; \gamma^T = -\gamma : \quad [\gamma, C] = 0$

Look at them one by one

$$\gamma^{\mu *} = \gamma^0 \gamma^{\mu T} \gamma^0$$
$$\gamma^0 \gamma^{\mu *} \gamma^0 = \gamma^{\mu T}$$

$$\gamma^0 \gamma^{1 *} \gamma^0 = \gamma^0 \gamma^1 \gamma^0 = -\gamma^1$$

⋮ PROBLEM

$$\left.\begin{aligned} \gamma^{2T} &= \gamma^2 \\ \gamma^{3T} &= -\gamma^3 \\ \gamma^{0T} &= \gamma^0 \end{aligned}\right\} \quad \begin{aligned} \{C, \gamma^\mu\} &= 0 \quad i = 2, 0 \\ [C, \gamma^\mu] &= 0 \quad i = 1, 3 \end{aligned}$$

Try $\quad C \propto \gamma^2 \gamma^0 \longrightarrow$ conventionally $\quad C \equiv -i\gamma^2\gamma^0$

Since
$$\psi_c = C' \psi^*$$
$$= C\gamma^0 \psi^* = -i\gamma^2\gamma^0\gamma^0\psi^* = -i\gamma^2\psi^*$$

or,
$$\psi_c = (-i)\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \psi^* = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \psi$$

Again, one by one

$$\psi_{c_1} = -i\gamma^2\psi_1^* = \sqrt{E+m}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_+^*}{E+m} \end{pmatrix} e^{+iEt - i\vec{p}\cdot\vec{x}}$$

$p_+^* = p_-$

$$= \sqrt{E+m}\begin{pmatrix} & & & \\ \end{pmatrix}\begin{pmatrix} \\ \end{pmatrix} e^{ip\cdot x}$$

$p_0 = +E = +\sqrt{\vec{p}^2 + m^2}$

$$= \sqrt{E+m}\begin{pmatrix} \frac{-p_-}{E+m} \\ \frac{p_3}{E+m} \\ 0 \\ -1 \end{pmatrix} e^{ip\cdot x}$$

$$= -\sqrt{E+m}\begin{pmatrix} \frac{p_-}{E+m} \\ \frac{-p_3}{E+m} \\ 0 \\ 1 \end{pmatrix} e^{ip\cdot x}$$

Remember

$$\psi_4 = \sqrt{E+m}\; u_4(\vec{p})\, e^{iEt + i\vec{p}\cdot\vec{x}} = \sqrt{E+m}\begin{pmatrix} \frac{-p_-}{E+m} \\ \frac{p_3}{E+m} \\ 0 \\ 1 \end{pmatrix} e^{iEt + \vec{p}\cdot\vec{x}}$$

reverse momentum direction

$$= \sqrt{E+m}\; u_4(-\vec{p})\, e^{iEt - i\vec{p}\cdot\vec{x}}$$

$$= \sqrt{E+m}\begin{pmatrix} \frac{p_-}{E+m} \\ \frac{-p_3}{E+m} \\ 0 \\ 1 \end{pmatrix} e^{ip\cdot x}$$