Collect all of our contributions —

electron self energy

$$\Sigma(p) = \Sigma^{(1)} + (\not p - m)\Sigma^{(2)} + (\not p - m)\Sigma^{(3)}$$

$$\Sigma^{(1)} = -\frac{3\alpha m}{8\pi^2}\frac{2}{\epsilon}$$

$$\Sigma^{(2)} = -\frac{\alpha}{4\pi}\left\{\frac{2}{\epsilon} + 2\ln(\lambda^2/m^2) + 9/4\right\}$$

$$= \Sigma^{(2)}_{\Lambda} + \Sigma^{(2)}_{\lambda}$$

vacuum polarization

$$\Pi_{\mu\nu} = (q_\mu q_\nu - q^2 g_{\mu\nu})\left[\Pi^{(1)} + q^2\Pi^{(2)}_f\right]$$

$$\Pi^{(1)} = \frac{\alpha}{3\pi}\frac{2}{\epsilon}$$

$$\Pi^{(2)}_f = \frac{\alpha}{15\pi}\frac{1}{m^2}$$

vertex correction

$$\Lambda_\mu = \gamma_\mu \Lambda^{(1)} + \gamma_\mu \Lambda^{(2)} + \Lambda^f_\mu$$

$$\Lambda^{(1)} = \frac{\alpha}{4\pi}\frac{2}{\epsilon}$$

$$\Lambda^{(2)} = -\frac{\alpha}{\pi}\, p\cdot p' \int dy\, \frac{1}{P^2}\ln(P^2/\lambda^2)$$

$$\Lambda^f_\mu = -\frac{\alpha m}{4\pi}\int dy\, \frac{1}{m^2 + q^2 y(y-1)}\, \sigma^{\mu\nu} q_\nu$$

(notice: $\Lambda^{(1)} = -\Sigma^{(2)}_{\Lambda}$)

We also have these order-$\alpha$ renormalization constants

$$\Sigma^{(2)}_\Lambda = 1 - \frac{1}{Z_2}$$

$$\Pi^{(1)} = 1 - Z_3$$

$$\Lambda^{(1)} = \left(\frac{1}{Z_1} - 1\right)$$

The coefficients in $\Lambda$ are often called $F_1$ and $F_2$ ..

$$\Lambda^\mu(p p') = F_1(q^2)\gamma^\mu + i\sigma^{\mu\nu}q_\nu F_2(q^2)$$

contains IR and UV divergent pieces ..

isolate the singular term by

$$F_1(0) + F_1(q^2) - F_1(0)$$

at $q^2$ & $q \to$ $(p \to p')$, $F_2 \to 0$ and.

$$\Lambda^\mu(pp') \xrightarrow{p \to p'} \Lambda^\mu(pp) = -\frac{i\alpha}{4\pi^3}\int \frac{d^d k}{k^2}\,\gamma^\alpha \frac{1}{\not p - \not k - m}\,\gamma^\mu \frac{1}{\not p - \not k - m}\,\gamma^\beta g_{\alpha\beta}$$

remember

$$-i\Sigma(p) = -\frac{\alpha}{4\pi^3}\int \frac{d^d k}{k^2}\,\gamma^\alpha \frac{1}{\not p - \not k - m}\,\gamma^\beta g_{\alpha\beta}$$

note that:

$$\frac{\partial}{\partial p_\mu}\frac{1}{\not p - m} = -\frac{1}{\not p - m}\gamma^\mu\frac{1}{\not p - m}$$

and

$$\frac{\partial}{\partial p_\mu}\frac{\not p + m}{p^2 - m^2} = \frac{\gamma^\mu}{p^2-m^2} + (\not p + m)\frac{\partial}{\partial p_\mu}\left(\frac{1}{p^2-m^2}\right)$$

$$= \frac{\gamma^\mu}{p^2-m^2} + (\not p - m)\frac{(-2p^\mu)}{(p^2-m^2)^2}$$

continuing

$$\frac{1}{\not p - m}\gamma^\mu \frac{1}{\not p - m} = \frac{(\not p + m)\gamma^\mu(\not p + m)}{(p^2-m^2)^2}$$ can also be made to be —

$$= \frac{\not p \gamma^\mu \not p + m^2\gamma^\mu + (\not p \gamma^\mu + \gamma^\mu \not p)m}{(p^2-m^2)^2}$$

$$= \frac{-\not p \not p \gamma^\mu + 2\not p p^\mu + m^2\gamma^\mu + 2p^\mu m}{(p^2-m^2)^2}$$

$$= \frac{(-p^2+m^2)\gamma^\mu + 2(\not p + m)p^\mu}{(p^2-m^2)^2}$$

$$= \frac{-\gamma^\mu}{(p^2-m^2)} + 2\frac{(\not p + m)p^\mu}{(p^2-m^2)}$$

$$= -\frac{\partial}{\partial p_\mu}\left(\frac{\not p + m}{p^2-m^2}\right)$$ (K inserted doesn't change this)

so:

$$-\frac{\partial}{\partial p_\mu}\Sigma(p) = \Lambda^\mu(p,p)$$ called the Ward Identity -- holds to all orders.

Remember

$$i\Sigma(p) = A + (\not p - m)B + \Sigma_f(p)$$

$$\frac{\partial i\Sigma}{\partial p_\mu} = \gamma^\mu B + \frac{\partial}{\partial p_\mu}\Sigma_f$$

Between spinors $\frac{\partial i\Sigma}{\partial p_\mu} = \gamma^\mu B = i\Lambda^\mu(p,p)$

so, $$\lim_{p'\to p} \bar u(p')\Lambda^\mu(pp')u(p) = -iB\,\bar u(p)\gamma^\mu u(p)$$

$$= F_1(0)\,\bar u(p)\gamma^\mu u(p) = -iB\,\bar u(p)\gamma^\mu u(p)$$

so, $F_1(0) = -iB$ , both are order $\alpha$.

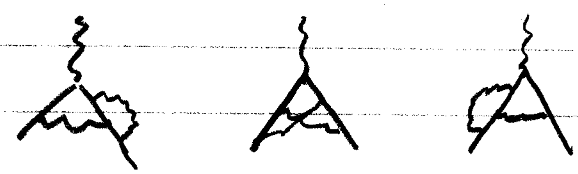
A more generalized statement called the Generalized Ward Identity is,

$$(p'-p)_\mu \Lambda^\mu(p,p') = -\left[\Sigma(p') - \Sigma(p)\right]$$

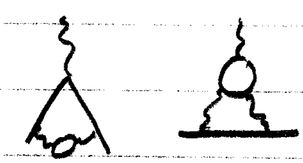
The utility is a labor saving device (but there is an underlying fundamental importance due to gauge invariance).

For example, suppose you calculate:

The Ward Identity gives you:



or:

 gives you 

etc.

Following through

$$\Lambda_\mu(p,p') = \left(\frac{1}{Z_1}-1\right)\gamma_\mu + \Lambda_\mu^f$$
$$\Sigma(p) = \delta m + \left(1-\frac{1}{Z_2}\right)(\not p - m) + \Sigma_f$$

From Ward Identity

$$(p-p')^\mu\left[\left(\frac{1}{Z_1}-1\right)\gamma_\mu + \Lambda_\mu^f\right] = -\left[\delta m + \left(1-\frac{1}{Z_2}\right)(\not p - m) + \Sigma_f\right] + \left[\delta m + \left(1-\frac{1}{Z_2}\right)(\not p' - m) + \Sigma_f\right]$$

$$(\not p - \not p')\left(\frac{1}{Z_1}-1\right) + (p-p')\Lambda^f = \left(\frac{1}{Z_2}-1\right)(\not p' - \not p) - \Sigma_f(p) + \Sigma_f(p')$$

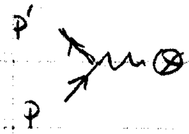
$$\left(\frac{1}{Z_1}-1\right) + \frac{1}{\not p - \not p'}(p-p')\Lambda^f = \left(\frac{1}{Z_2}-1\right) + \frac{1}{\not p - \not p'}\left[\Sigma_f(p') - \Sigma_f(p)\right]$$

as $\quad p \to p' \qquad \left.\begin{array}{l} \Lambda^f(p,p') \to 0 \\ \Sigma_f(p') - \Sigma_f(p) \to 0 \end{array}\right\}$ forward scattering

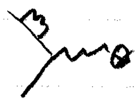
so $\quad \underbrace{Z_1 = Z_2}$ ... often another statement of Ward Id. (but really a consequence of it)

a direct consequence of current conservation and/or gauge invariance.

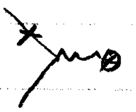
So, now let's put everything together — ignoring the IR divergence and concentrating on the forward scattering amplitude — using physical masses, now.



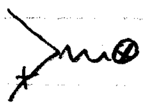

$$\bar{u}_0(p')(-ie_0\gamma_\mu)u_0(p)$$



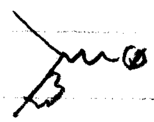
$$\bar{u}_0(p')\left[\frac{\Sigma^{(1)}}{\not{p}'-m} + \frac{(\not{p}'-m)\Sigma^{(2)}}{\not{p}'-m}\right](-ie_0\gamma_\mu)u_0(p)$$



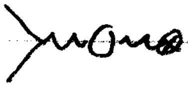
$$\bar{u}_0(p')\left[\frac{-\delta m}{\not{p}'-m}\right](-ie_0\gamma_\mu)u_0(p)$$



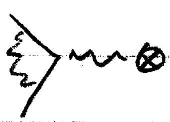
$$\bar{u}_0(p')(-ie_0\gamma_\mu)\left[\frac{-\delta m}{\not{p}-m}\right]u_0(p)$$



$$\bar{u}_0(p')(-ie_0\gamma_\mu)\left[\frac{\Sigma^{(1)}}{\not{p}-m} + \frac{(\not{p}-m)\Sigma^{(2)}}{\not{p}-m}\right]u_0(p)$$



$$\bar{u}_0(p')\left(\frac{ie_0}{q^2}\right)\left[q^2\gamma_\mu\Pi^{(1)} + q^4\,\Pi_f^{(2)}\right]u_0(p)$$



$$\bar{u}_0(p')(-ie_0)\left[\gamma_\mu\Lambda^{(1)} + \Lambda_\mu^{(f)}\right]u_0(p)$$

Feynman rules really are altered: the mass counterterm is used ($\delta m$ results) and renormalized wavefunctions are used $\Rightarrow$ insert appropriate $\sqrt{Z_2}$ and $\sqrt{Z_3}$

add them up, make the external legs renormalized (ie multiply by $\frac{1}{\sqrt{Z_2}}$ for each electron leg and $\frac{1}{\sqrt{Z_3}}$ for each photon leg)

$$\frac{(-ie_0)\,\bar{u}(p')}{Z_2\sqrt{Z_3}}\Big[\gamma_\mu + \frac{\Sigma^{(1)}}{\not p' - m} + \Sigma^{(2)} + \frac{\Sigma^{(1)}}{\not p - m} + \Sigma^{(2)}$$

$$- \frac{\delta m}{\not p' - m} - \frac{\delta m}{\not p - m} - \gamma_\mu \Pi^{(1)} - q^2 \Pi_f^{(2)}\gamma_\mu$$

$$+ \gamma_\mu \Lambda^{(1)} + \Lambda_\mu^{(f)}\Big]\, u(p)$$

remember: $\Sigma^{(1)} = \delta m$

$$\frac{(-ie_0)\,\bar{u}(p')}{Z_2\sqrt{Z_3}}\Big[\big(1 + 2\Sigma^{(2)} - \Pi^{(1)} + \Lambda^{(1)}\big)\gamma_\mu$$

$$+ \big(\Lambda_\mu^f - q^2\,\Pi_f^{(2)}\big)\gamma_\mu\Big]\, u(p)$$

postpone this

talked about this effect

now $\Sigma^{(2)} = 1 - \frac{1}{Z_2}$

$\Pi^{(1)} = 1 - Z_3$

$\Lambda^{(1)} = \left(\frac{1}{Z_1} - 1\right)$

$$= \frac{(-ie_0)\bar{u}(p')}{Z_2\sqrt{Z_3}} \left\{ \left[1 + 2(1 - Z_2^{-1}) - (1 - Z_3) + (Z_1^{-1} - 1)\right] \gamma_\mu u(p) \right.$$

to order $\alpha$

$$= \frac{(-ie_0)\bar{u}(p')}{Z_2\sqrt{Z_3}} \left\{ \frac{[1 + \overbrace{(Z_1^{-1} - 1)}^{\Theta(\alpha)}][1 + \overbrace{(Z_3 - 1)}^{\Theta(\alpha)}]}{[1 + \underbrace{(Z_2^{-1} - 1)}_{\Theta(\alpha)}]^2} \gamma_\mu u(p) \right.$$

$$= \frac{(-ie_0)\bar{u}(p')}{Z_2\sqrt{Z_3}} \left\{ Z_1^{-1} Z_3 Z_2^2 \right\} \gamma_\mu u(p)$$

$$= (-ie_0)\bar{u}(p') \left( \frac{Z_2\sqrt{Z_3}}{Z_1} \right) \gamma_\mu u(p)$$

$$= -ie_R \bar{u}(p') \gamma_\mu u(p)$$

where $e_R = \dfrac{Z_2}{Z_1}\sqrt{Z_3}\, e_0$ w/ Ward Identity $Z_1 = Z_2$

$$e_R = \sqrt{Z_3}\, e_0$$