The finite effect from the vertex correction comes from

$$\bar{u}(p')\,(\gamma_\mu + \Lambda^f_\mu)\,u(p) \equiv \bar{u}(p')\,\Gamma_\mu\, u(p)$$

where $\Lambda^f_\mu \equiv F_2(q^2)\,\sigma^{\mu\nu} q_\nu$

Remember this occurs inside $\int d^4x \ldots$

$$\int d^4x \,\langle p' | \mathcal{L}_I | p\rangle \sim e\int d^4x\, e^{iq\cdot x}\, \bar{u}(p')\,\Gamma_\mu\, u(p)\, A^\mu_{ext}$$

in which the second term is

$$e\int d^4x\; F_2(q^2)\,\bar{u}(p')\,\sigma^{\mu\nu} q_\nu\, u(p)\, A^{ext}_\mu\, e^{iq\cdot x}$$

$$\longrightarrow\; e\,F_2(q^2)\;\bar{u}(p')\,\sigma^{\mu\nu} u(p)\int d^4x\; q_\nu A^{ext}_\mu\, e^{iq\cdot x} \qquad *$$

(keep this in mind)

Let F.T. of A_μ be $\tilde{A}_\mu$ so

$$A^{ext}_\mu(x) = \frac{1}{(2\pi)^4}\int d^4q\; e^{iq\cdot x}\,\tilde{A}^{ext}_\mu(q)$$

$$\frac{\partial A_\mu}{\partial x^\nu} = \frac{1}{(2\pi)^4}\int d^4q\;(iq_\nu)\,e^{iq\cdot x}\,\hat{A}^{ext}_\mu(q)$$

ie $\dfrac{\partial A_\mu}{\partial x^\nu}$ and $(iq_\nu)\,\tilde{A}_\mu(q)$ are F.T. of each other

The interpretation of this is interesting -- Consider the Dirac Equation having a minimal electromagnetic interaction,

$$\left(i \frac{\partial}{\partial x_\mu} \gamma^\mu - e \not{A} - m \right) \psi(x) = 0$$

operate $\quad (i \partial_\mu \gamma^\mu - e \not{A} + m) \longrightarrow$

$$\left\{ (i\partial_\mu - eA_\mu)^2 + \frac{1}{2i} \sigma^{\mu\nu} [i\partial_\mu - eA_\mu , i\partial_\nu - eA_\nu] - m^2 \right\} \psi(x) = 0$$

becomes

$$= \left\{ (i\partial_\mu - eA_\mu)^2 - \frac{e^2}{2} \sigma^{\mu\nu} F_{\mu\nu} - m^2 \right\} \psi(x)$$

or. $$\left[(i\partial_\mu - eA_\mu)^2 - \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu} \right] \psi(x) = m^2 \psi(x)$$

$$\left[\frac{1}{2m} (i\partial_\mu - eA_\mu)^2 - \underbrace{\frac{1}{2m} \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu}}_{} \right] \psi = \frac{m}{2} \psi$$

$$\frac{1}{2} |\vec{\mu}_D| \sigma^{\mu\nu} F_{\mu\nu} \qquad \mu_D = \left| \frac{e\sigma}{2m} \right|$$

which becomes $\quad = \mu_D \vec{\Sigma} \cdot \vec{B} - i \mu_D \vec{\alpha} \cdot \vec{E}$

relativistic effect

in NR limit, decomposes to σ

so $\quad \sigma^{\mu\nu} F_{\mu\nu}$ *looks* like a magnetic interaction.

continue $\quad \frac{\mu_D}{2} \sigma^{\mu\nu} F_{\mu\nu} = \frac{\mu_D}{2} \sigma^{\mu\nu} \left[\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \right]$

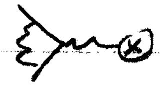
$$= \frac{\mu_D}{2} \left[\sigma^{\mu\nu} \frac{\partial A_\mu}{\partial x^\nu} - \sigma^{\mu\nu} \frac{\partial A_\nu}{\partial x^\mu} \right]$$

$$= \frac{\mu_D}{2} \left[\sigma^{\mu\nu} - \sigma^{\nu\mu} \right] \frac{\partial A_\mu}{\partial x^\nu}$$

$$= \mu_D \sigma^{\mu\nu} \frac{\partial A_\mu}{\partial x^\nu}$$

The F.T. of this is $\quad i\mu_D \sigma^{\mu\nu} q_\nu \tilde{A}_\mu(q)$

.. which is what we have as the second term of the Γ_μ matrix element — but not μ_D

$\Rightarrow$ looks like  induces a magnetic moment interaction.

In non-relativistic QM. language,

$$\vec{B} = \vec{\nabla} \times \vec{A} \xrightarrow{\text{F.T.}} i\vec{q} \times \vec{A}$$

$$\vec{\mu}_D = \frac{e\vec{\sigma}}{2m} = \frac{e\vec{S}}{2m}\, g \qquad \text{where} \qquad \vec{S} = \tfrac{1}{2}\vec{\sigma}$$

$g = 2$ Dirac electron

remember, this came for free in Dirac Theory.

$$\vec{\mu} \cdot \vec{B} = \vec{\mu} \cdot \vec{\nabla} \times \vec{A} \xrightarrow{\text{F.T.}} i\,\vec{\mu} \cdot \vec{q} \times \vec{A}$$

$$= g\,\frac{ie}{4m}\,\vec{\sigma} \cdot \vec{q} \times \vec{A}$$

The Ad-Hoc introduction of magnetic interactions by Pauli was

$$H = H^{KE} + H^{SPIN}$$

$$= \frac{p^2}{2m} - \frac{e\hbar}{2mc}\,\vec{\sigma} \cdot \vec{B}$$

write $\quad H^{KE} = \dfrac{(\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p})}{2m}$

minimally couple $\vec{A}$ --

$$H = \frac{1}{2m}\,\vec{\sigma}\cdot\left(\vec{p} - \frac{e}{c}\vec{A}\right)\vec{\sigma}\cdot\left(\vec{p}\cdot\frac{e}{c}\vec{A}\right)$$

$$= \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{2mc}\,\vec{\sigma}\cdot\vec{B}$$

adding the scalar potential gives the complete interaction ~~Hamiltonian~~,

$$\mathcal{H}_{int} = \frac{-e}{mc}\,\vec{p}\cdot\vec{A} - \underbrace{\frac{e\hbar}{2mc}\,\vec{\sigma}\cdot\vec{B}}_{-\,\vec{\mu}_D\cdot\vec{B}} + e\Phi$$

Now let's look at our Λ^f_μ "interaction" in the non-relativistic limit

Choose Coulomb gauge $\quad \vec{\nabla}\cdot\vec{A}=0 \quad \cap \quad \vec{q}\cdot\hat{\vec{A}}=0$

$$\Rightarrow \quad \vec{p}'\cdot\hat{\vec{A}} = \vec{p}\cdot\hat{\vec{A}}$$

Look at N.R. expansion of Γ_μ --

$\bar{u}(p')\gamma_\mu u(p)$: $\quad \mu=0 \quad \bar{u}(p')\gamma_0 u(p) = u^+(p')u(p)$

$$= \sqrt{E'+m}\sqrt{E+m}\ \chi^{+'}\left(1, \frac{\vec{\sigma}\cdot\vec{p}'}{E'+m}\right)\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}\chi$$

✓ $$= 2m\ \chi^+\chi + \theta(\beta^2)$$

$\mu=i \quad \bar{u}(p')\vec{\gamma}\,u(p)$

$$= 2m\ \chi^{+'}\left(1, -\frac{\vec{\sigma}\cdot\vec{p}'}{2m}\right)\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0\end{pmatrix}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{2m}\end{pmatrix}\chi + \theta(\beta^2)$$

$$= \chi^{+'}\left[\vec{\sigma}\,\vec{\sigma}\cdot\vec{p} + \vec{\sigma}\,\vec{\sigma}\cdot\vec{p}'\right]\chi + \theta(\beta^2)$$

✓ $$= \chi^{+'}\left[\vec{p}+\vec{p}' + i\,\vec{\sigma}\times\vec{q}\right]\chi + \theta(\beta^2)$$

$\bar{u}(p')\sigma^{\mu\nu}u(p)$: $\quad$ write $\quad \sigma_{ij} = \varepsilon_{ijk}\begin{pmatrix}\sigma_k & 0 \\ 0 & \sigma_k\end{pmatrix}$

$$\sigma_{0j} = \begin{pmatrix} 0 & i\sigma_j \\ i\sigma_j & 0\end{pmatrix}$$

✓ $\mu=0 \quad \bar{u}(p')\sigma_{0i}u(p) = i2m\ \chi^{+'}\left(1, -\frac{\vec{\sigma}\cdot\vec{p}'}{2m}\right)\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0\end{pmatrix}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{2m}\end{pmatrix}\chi + \theta(\beta^2)$

$$= \theta(\beta^2)$$

$\mu = i \quad \bar{u}(p')\sigma_{ij}\, u(p) = \varepsilon_{ijk}\, 2m\, \chi^{\dagger\prime}\left(1, \frac{-\vec{\sigma}\cdot\vec{p}'}{2m}\right)\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{2m} \end{pmatrix} + \mathcal{O}(\beta^2)$

✓ $= \varepsilon_{ijk}\, 2m\, \chi^{\dagger\prime}\sigma_k \chi + \mathcal{O}(\beta^2)$

Look at static electric and magnetic fields for $\mathcal{M} = \bar{u}(p')\Gamma_\mu u(p) A^\mu_{ext}$ in momentum space:

<u>Electric</u> $\quad A^\mu_{ext} = e\Phi \qquad \mu = 0$

$\qquad\qquad = 0 \qquad \mu = i$

so $\mathcal{M}_E = \bar{u}(p')\Gamma_\mu u(p)\, \tilde{A}^\mu_{ext} \xrightarrow[\text{electric}]{\text{static}} \bar{u}(p')\Gamma_0 u(p)\, e\Phi$

$\downarrow$ N.R.

$= e\, 2m\, \chi^{\dagger\prime}\chi\, \Phi + \mathcal{O}(\beta^2)$ ✓

<u>Magnetic</u> $\quad A^\mu_{ext} = 0 \qquad \mu = 0$

$\qquad\qquad = eA^i \qquad \mu = i$

so $\mathcal{M}_M = \bar{u}(p')\Gamma_\mu u(p)\, \tilde{A}^\mu_{ext} \xrightarrow[\text{mag}]{\text{static}} -e\bar{u}(p')\vec{\Gamma}\, u(p)\cdot \tilde{\vec{A}}_{ext}$

$\downarrow$ N.R.

Coulomb gauge ($p \cdot A = p' \cdot A$)

$$= -e\, \tilde{\vec{A}} \cdot \chi^{\dagger\prime}\left[(2\vec{p} + i\vec{\sigma}\times\vec{q}) \right]\chi$$
$$- e F_2(q^2)\, \underbrace{q^j \tilde{A}^i \varepsilon_{ijk}}_{\vec{\sigma}\cdot\tilde{\vec{A}}\times\vec{q}}\, 2m\, \chi^{\dagger\prime}\sigma^k \chi$$

$$= \chi^{\dagger\prime}\left[-2e\, \tilde{\vec{A}}\cdot\vec{p} - ie\, \tilde{\vec{A}}\cdot\vec{\sigma}\times\vec{q} - 2m F_2(q^2) e\, \vec{\sigma}\cdot\tilde{\vec{A}}\times\vec{q}\right]\chi$$
$$= \chi^{\dagger\prime}\left[-2e\, \tilde{\vec{A}}\cdot\vec{p} + ie\, \tilde{\vec{A}}\times\vec{q}\cdot\vec{\sigma} - 2m F_2(q^2)\, \vec{\sigma}\cdot\tilde{\vec{A}}\times\vec{q}\right]\chi$$

$$= \chi^{\dagger\prime}\left[(-2e\, \tilde{\vec{A}}\cdot\vec{p}) + ie\vec{\sigma}\cdot(1 - 2m F_2(q^2))\, \tilde{\vec{A}}\times\vec{q} \right]\chi$$

Divide by $\frac{1}{2m}$

$$\chi^{+\prime}\left[-\frac{e}{m}\hat{A}\cdot\vec{p}+\frac{ie}{2m}\vec{\sigma}\cdot\left(1-2mF_2(q^2)\right)\hat{A}\times\vec{q}\right]\chi$$

$$=\chi^{+\prime}\left[-\frac{e}{m}\hat{A}\cdot\vec{p}+\underbrace{\frac{e}{2m}\vec{\sigma}\cdot\left(1-2mF_2(q^2)\right)\vec{B}}_{\vec{\mu}}\right]\chi \quad \checkmark$$

ie. the 2nd gives us a total magnetic moment

$$\vec{\mu}=\frac{e\vec{\sigma}}{2m}\left[1-2mF_2(q^2)\right]$$

$$=\frac{e\vec{\sigma}}{4m}g$$

so $\frac{g}{2}=1-2mF_2(q^2) = \frac{g_D}{2}+\frac{g_{ANAMOLOUS}}{2}$

Remember

$$F_2(q^2)=-\frac{\alpha m}{4\pi}\int_0^1 dy\,\frac{1}{m^2+q^2(y-1)y}$$

in limit $q^2\to 0$

$$F_2(0)=-\frac{\alpha m}{4\pi m^2}=-\frac{\alpha}{4\pi m}$$ called the Schwinger Term

and.

$$\vec{\mu}=\frac{e\vec{\sigma}}{2m}\left[1+\frac{\alpha}{2\pi}\right]=\frac{e\vec{\sigma}}{2m}g$$

so $g - g_D = g_{\text{ANAMOLOUS}}$

called "$g-2$" $= \frac{\alpha}{2\pi} \simeq \frac{1}{6}\frac{1}{137} \sim 1.2 \times 10^{-3}$

Presently

$$\frac{1}{2}(g-2)_{\text{electron}} = \frac{1}{2}\frac{\alpha}{\pi} - 0.32848\left(\frac{\alpha}{\pi}\right)^2 + 1.49\left(\frac{\alpha}{\pi}\right)^3 + \dots$$

experiment $0.001159652193 \pm 0.000000000010$

theory $0.001159652140 \pm 0.000000000028$

the most precisely measured quantity and remarkable confirmation of QED and the whole renormalization approach. The largest error in the theoretical calculation is knowledge of α.

$\alpha = 7.297\ 352\ 533(27) \times 10^{-3}$ @ $q^2 = 0$. 3.7 ppb.

So we would conclude that the electron-potential scattering matrix element, radiatively corrected to order α should be written

$$\bar{u}(p')\left[\gamma_\mu - \frac{e}{2m}\left(1+\frac{\alpha}{2\pi}\right) i\sigma^{\mu\nu}q_\nu\right]u(p)\,\hat{A}^{ext}_\mu(q)$$

We still have not dealt with the infrared divergence associated with the vertex correction

$$\Lambda^\mu(pp')_{IR} = \gamma^\mu - \frac{\alpha}{2\pi}\, p\cdot p' \int dy\, \frac{1}{P^2} \ln\left(P^2/\lambda^2\right)\gamma^\mu$$

$P^\mu \equiv p^\mu y + p'^\mu(1-y)$ $\quad py + p' - p'y$

$\quad p + p'(1-y)$

Following through with this would give us the cross section

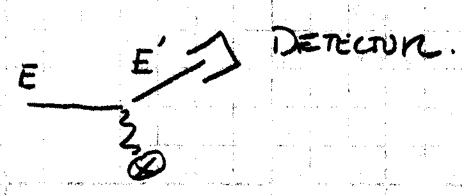
$$\cancel{\frac{d\sigma}{d\Omega_{el}} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 + \frac{2\alpha}{\pi}\right.}$$

$$\frac{d\sigma}{d\Omega_{El}} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi}\, p\cdot p' \int dy\, \frac{1}{P^2} \ln\left(P^2/\lambda^2\right)\right] A$$

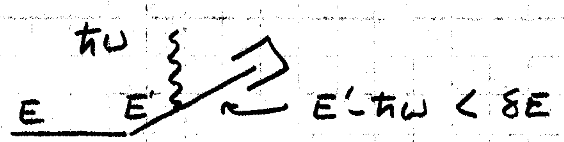
where ↗ is the lowest order (radiatively corrected) Coulomb scattering cross section, the Mott Formula.

We must ask a sensible question—
Experiments cannot resolve electron energies to infinitesimal precision, but rather to some finite precision δE

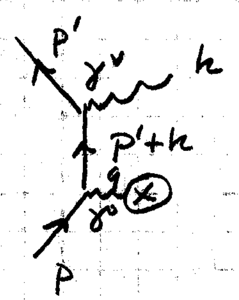
Therefore, experiments cannot tell the difference between a scatter that does this:

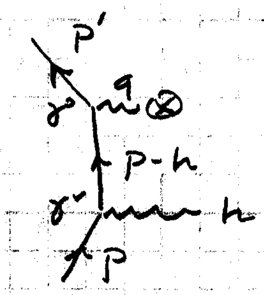


and this:



This means that in order to get a completely accurate and sensible prediction we must consider the high-order process which we skipped, Bremsstrahlung... for soft photon emission.
We already sketched out the WKB expansion





The T-Matrix contribution from these diagrams is

$$T = -i\left\{\bar{u}(p')\,\frac{i\,\not{\epsilon}\,e^2}{\not{p}'-\not{k}-m}\,\gamma^0 u(p) + \bar{u}(p')\,\gamma^0\,\frac{i\,\not{\epsilon}\,e^2}{\not{p}-\not{k}-m}\,\not{\epsilon}\,u(p)\right\}\frac{Ze}{\vec{q}^{\,2}}$$

$$\vec{q} = \vec{p} - \vec{p}\,' - \vec{k}$$

A

The cross section is $\quad (k^0 = \omega)$

$$d\sigma = \frac{Z^2 e^6}{2\omega|p|EE'} \Sigma\Sigma|T|^2 2\pi\,\delta(E-E'-\omega)\frac{d^3p'}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}$$

$$\Sigma\Sigma|T|^2 = \frac{1}{2}\frac{\Sigma\Sigma}{q^4}\,|\bar{u}(p')\Gamma u(p)|^2$$

$$|\ | = \bar{u}(p')\left[\frac{\not{\varepsilon}\,(\not{p}'+\not{k}+m)\gamma_0}{(p'+k)^2-m^2} + \frac{\gamma_0(\not{p}-\not{k}+m)\not{\varepsilon}}{(p-k)^2-m^2}\right]u(p)$$

between spinors $\quad \not{\varepsilon}\not{p}' = -\not{p}'\not{\varepsilon} + 2\varepsilon\cdot p'$

so $\quad \bar{u}(p')\not{p} = \bar{u}(p')m$

$\not{p}\not{\varepsilon} = -\not{\varepsilon}\not{p} + 2\varepsilon\cdot p$

so $\quad \not{p}u(p) = m u(p)$

$\not{\varepsilon}\not{k} = -\not{k}\not{\varepsilon} + 2\varepsilon\cdot k$ not zero since massive photons.

The denominator $\quad (p'+k)^2-m^2$

$$= p'^2 + k^2 + 2p'\cdot k - m^2$$

$$= \lambda^2 + 2p'\cdot k.$$

& $\quad (p-k)^2 - m^2 = \lambda^2 + 2p\cdot k$

so,

$$\Gamma = \frac{[(-\not{p}'-\not{k}+m)\not{\varepsilon} + 2\varepsilon\cdot k]\gamma^0}{\lambda^2+2p'\cdot k} - \frac{\gamma_0[-2\varepsilon\cdot k + \not{\varepsilon}(-\not{p}-\not{k}+m)]}{\lambda^2+2p\cdot k}$$

$$+ \frac{\varepsilon\cdot p'\gamma^0}{\lambda^2+2p'\cdot k} - \frac{\gamma^0\,\varepsilon\cdot p}{\lambda^2+2p\cdot k}$$

Using the Dirac equation

$$\Gamma = -\left[\frac{\not k \not \varepsilon \gamma^0 + 2\varepsilon\cdot k}{\lambda^2 + 2p'\cdot k} + \frac{\gamma^0 \not\varepsilon \not k + 2\varepsilon\cdot k}{\lambda^2 + 2p\cdot k} \right]$$

$$+ \underbrace{\left[\frac{\varepsilon\cdot p'}{\lambda^2 + k\cdot p'} - \frac{\varepsilon\cdot p}{\lambda^2 + k\cdot p} \right]}_{\equiv\, L} \gamma^0$$

we're interested in low energy brems.— so the first term can be neglected relative to the second.

$$\Gamma = L\gamma^0$$

From the standard

$$\frac{1}{2}\sum\sum |\bar{u}\,\Gamma\, u|^2 \frac{1}{q^4} = \frac{1}{2} \mathrm{Tr}\left[(\not p' + m)\Gamma(\not p + m)\bar{\Gamma} \right] \frac{1}{q^4}$$

$$\bar{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0 = -\left[\frac{\varepsilon\cdot p'}{\lambda^2 + k\cdot p'} - \frac{\varepsilon\cdot p}{\lambda^2 + k\cdot p} \right]\gamma^0$$

so

$$= \left(\frac{\varepsilon\cdot p'}{\lambda^2 + k\cdot p'} - \frac{\varepsilon\cdot p}{\lambda^2 + k\cdot p} \right)^2 \underbrace{\frac{1}{2} \mathrm{Tr}\left[(\not p' + m)\gamma^0(\not p + m)\gamma^0 \right] \frac{1}{q^4}}_{\text{same as for the no-radiation cross section}}$$

factors, ultimately:

$$(\text{BREM}) \quad \left.\frac{d\sigma}{d\Omega}\right|_0$$

$$= (\ )^2 \tfrac{1}{2} \, \mathrm{Tr} \left[\not{p}' \gamma^0 \not{p} \gamma^0 + m^2 \right]$$

$$= (\ )^2 \tfrac{1}{2} \cdot 4 \left[p_0' p_0 - p_\mu' p^\mu + p_0' p_0 + m^2 \right]$$

$$= (\ )^2 \, 2 \left[\vec{p}' \cdot \vec{p} + E'E + m^2 \right]$$

from phase space $\quad d^3p' = p' E' dE' d\Omega$

so we get,

$$\frac{d\sigma}{d\Omega'} = \frac{Z^2 e^6}{(2\pi)^3}\left(\frac{p'}{p}\right)\frac{1}{|\vec q|^4}\overset{\delta(E-E'-\omega)}{\wedge}\frac{1}{\omega}\left(\;\right)^2\left(\vec p\cdot\vec p' + EE' + m^2\right) d^3k\, dE'$$

in the soft photon limit $\quad dE' \Rightarrow E' = E - \omega \simeq E$

and $|\vec p'| \simeq |\vec p|$ (like elastic)

$$\frac{d\sigma}{d\Omega'} = \frac{Z^2 e^6}{(2\pi)^3}\left(\frac{p'}{p}\right)\frac{1}{|\vec q|^4}\frac{1}{\omega}\left(\;\right)^2\left(\vec p\cdot\vec p' + EE' + m^2\right) d^3k$$

$$|\vec q|^4 = |\vec p - \vec p'|^4 = 16 p^4 \sin^4\theta/2$$

$$\vec p\cdot\vec p' = p^2\cos\theta$$

$$m^2 = E^2 + p^2$$

so

$$\vec p\cdot\vec p' + m^2 + EE' = 2\left(E^2 - p^2\sin^2\theta/2\right)$$

Since, radiationless ---

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{1}{4}\,\frac{Z^2 e^4}{p^4\sin^4\theta/2}\left(E^2 - p^2\sin^2\theta/2\right)$$

A'

we now get

sum over photon spins

$$\left.\frac{d\sigma}{d\Omega}\right|_{inelastic} = \left(\frac{d\sigma}{d\Omega}\right)_0 \sum_\gamma \frac{1}{2}\,\frac{e^2}{(2\pi)^3}\,\frac{k^2 dk\, d\Omega_k}{k}\left(\frac{\epsilon\cdot p'}{\underset{\lambda^2 +}{k\cdot p'}} - \frac{\epsilon\cdot p}{\underset{\lambda^2 +}{k\cdot p}}\right)^2$$

$\sim \dfrac{dk}{k}$ characteristic of Brems.

A

write as $\quad \frac{\alpha}{\pi} \frac{d\Omega_k}{4\pi} \frac{k^2 dk}{\omega} = \varepsilon_\mu(\lambda) B^\mu \varepsilon_\nu(\lambda) B^\nu$

where $\quad B^\mu = \dfrac{p^{\mu\prime}}{\lambda^2 + k\cdot p'} - \dfrac{p^\mu}{\lambda^2 + k\cdot p}$

The polarization sum extends $\displaystyle\sum_{\lambda=1}^{3} \varepsilon_\mu(\lambda)\,\varepsilon_\nu(\lambda) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{\lambda^2}$

(bad notation! helicity ≠ mass !

so, $\quad \dfrac{\alpha}{\pi} \dfrac{d\Omega_k}{4\pi} \dfrac{k^2 dk}{\omega}\left(-g_{\mu\nu} + \dfrac{k_\mu k_\nu}{\lambda^2}\right)\left(\dfrac{p'^\mu}{\lambda^2+k\cdot p'} - \dfrac{p^\mu}{\lambda^2+k\cdot p}\right)$

$$\times\left(\frac{p'^\nu}{\lambda^2+k\cdot p'} - \frac{p^\nu}{\lambda^2+k\cdot p}\right)$$

⋮

$$-\frac{\alpha}{\pi}\frac{d\Omega_k}{4\pi}\frac{k^2dk}{\omega}\left[\frac{4m^2-\lambda^2}{(2p'\cdot k+\mu^2)^2} + \frac{4m^2-\lambda^2}{(2p\cdot k-\mu^2)^2}\right.$$

$$\left. + \frac{-8\,p\cdot p' + 2\mu^2}{(2p'\cdot k+\lambda^2)(2p\cdot k+\lambda^2)}\right]$$

Look at a general term of the first pair. –

sufficient to work in $\lambda^2 \to 0$ limit

$$\int \frac{k^2dk}{\omega}\int d\Omega \frac{1}{(k\cdot p)^2} = 2\pi\int\frac{k^2dk}{\omega}\int_{-1}^{1} d(\cos\theta)\,\frac{1}{(k_0p_0 - |\vec{k}||\vec{p}|\cos\theta)^2}$$

$$= 2\pi\int\frac{k^2dk}{\omega}\int_{-1}^{1} dx\,\frac{1}{(p_0\omega - |\vec{p}||\vec{k}|x)^2}$$

A

$$= 4\pi \int dh \frac{h^2}{\omega(\omega^2 p_0^2 - \vec{p}^2 \vec{h}^2)}$$

change to $d\omega$

$$= 4\pi \int \frac{d\omega}{\omega} \frac{h\omega}{(m^2\omega^2 + \lambda^2 \vec{p}^2)}$$

$$= 4\pi \int d\omega \frac{(\omega^2 - \lambda^2)^{\frac{1}{2}}}{(m^2\omega^2 + \lambda^2 \vec{p}^2)}$$

expand numerator $\omega(1 - \lambda^2/\omega^2)^{1/2} = \omega(1 - 1/2\, \lambda^2/\omega^2)$

$$\simeq \omega - 1/2\, \lambda^2/\omega$$

The $d\omega$ limits go from λ to the resolution limit ΔE

A'

$$4\pi \int_{\lambda}^{\Delta E} d\omega \frac{\omega}{m^2\omega^2 + \lambda^2 \vec{p}^2}$$

change variables $y^2 = m^2\omega^2$

and get --- $\frac{4\pi}{m^2} \cdot \frac{1}{2} \ln\left(\frac{\Delta E^2}{\lambda^2}\right) + \mathcal{O}(m^2/E^2)$

likewise for the other term

The other term requires a Feynman parameter,

$$(8p\cdot p' - 2\lambda^2)\frac{\alpha}{\pi}\int_\lambda^{\Delta E} d\omega\sqrt{\omega^2-\lambda^2}\int\frac{d\Omega_k}{4\pi}\int dy\,\frac{1}{2k\cdot(p'y+p(1-y))-\lambda^2(1-2y))}$$

$$\sim \quad 8p\cdot p'\frac{\alpha}{\pi}\int_\lambda^{\Delta E}\sqrt{\omega^2-\lambda^2}\,d\omega\int dy\,\frac{1}{(2P_0\omega-\lambda^2(1-2y))^2-4k^2\vec{P}_0^{\,2})}$$

where $P_0 = yE' + (1-y)E$, $\vec{P} = y\vec{p}\,' + (1-y)\vec{p}$ $\Big\}$ $P = p^\mu y + p'^\mu(1-y)$, $P^2 = q^2(y^2-y)+m^2$

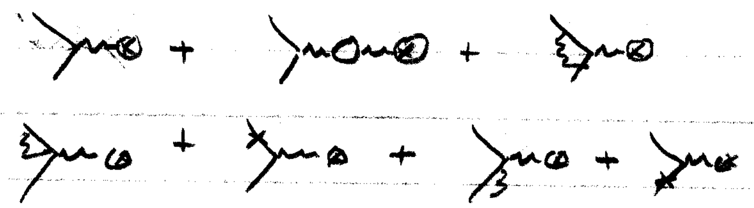
doing the ω integration and keeping the leading terms gives,

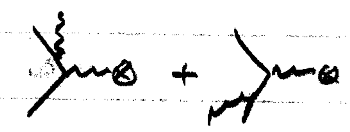
$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{inelastic}} = \left.\frac{d\sigma}{d\Omega}\right|_0 \cdot 2p\cdot p'\frac{\alpha}{4\pi^2}2\pi\int dy\,\frac{1}{P^2}\ln\left[\frac{P^2}{\lambda^2}\frac{\Delta E^2}{P_0^2}\right]$$

$$= \left.\frac{d\sigma}{d\Omega}\right|_0 \frac{\alpha}{\pi}p\cdot p'\int dy\,\frac{1}{P^2}\ln\left[\frac{P^2}{P_0^2}\frac{\Delta E^2}{\lambda^2}\right] \quad \checkmark$$

IR divergent.

However, this is to be combined with the $O(\alpha)$ radiatively corrected pieces from the previously work which resulted in Λ_2^μ

ie $\Big|$  $\Big|^2$ elastic

$+ \displaystyle\int_{\omega < \Delta E} d^3k$ $\Big|$  $\Big|^2$ inelastic

so may add as amplitudes squared.

$$\frac{d\sigma}{d\Omega}\bigg|_{obs} = \frac{d\sigma}{d\Omega}\bigg|_0 \left[1 - \frac{\alpha}{\pi}\, p\cdot p' \int dx\, \frac{1}{P^2} \ln\left(P^2/\lambda^2\right) + \frac{\alpha}{\pi}\, p\cdot p' \int dx\, \frac{1}{P^2} \ln\left(\frac{P^2 \Delta E^2}{P_0^2 \lambda^2}\right)\right]$$

$$= \frac{d\sigma}{d\Omega}\bigg|_0 \left[1 + \frac{\alpha}{\pi}\, p\cdot p' \int dx\, \frac{1}{P^2} \left[\ln\left(\lambda^2/P^2\right) + \ln\left(\frac{P^2}{P_0^2}\,\frac{\Delta E^2}{\lambda^2}\right)\right]\right]$$

$$= \frac{d\sigma}{d\Omega}\bigg|_0 \left[1 + \frac{\alpha}{\pi}\, p\cdot p' \int dx\, \frac{1}{P^2} \ln\left[\frac{\cancel{\lambda^2}}{\cancel{P^2}} \cdot \frac{\cancel{P^2}}{P_0^2}\, \frac{\Delta E^2}{\cancel{\lambda^2}}\right]\right]$$

$$= \frac{d\sigma}{d\Omega}\bigg|_0 \left[1 + \frac{\alpha}{\pi}\, p\cdot p' \int dx\, \frac{1}{P^2} \ln\left(\frac{\Delta E^2}{P_0^2}\right)\right]$$

the photon mass is gone.

Depending on the kinematics, the integral can be done for particular experimental conditions.

Evaluation of radiative corrections is an industry — especially at electron accelerators — an an experiment-dependent circumstance.

THE END