Lecture 10 Relation to 1st Quantization

We can make touch with regular quantum mechanics --

Consider the object

$$\langle 0|\phi(x)|\vec{k}'\rangle = \int dK \langle 0| a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} |\vec{k}'\rangle$$

$$= \int dK \langle 0| (a(k) e^{-ik\cdot x} + a^\dagger(k) e^{ik\cdot x}) a^\dagger(\vec{k}')|0\rangle$$

$$= \int dK \langle 0| a(k)a^\dagger(k') e^{-ik\cdot x} + \underbrace{a^\dagger(k)a^\dagger(k') e^{ik\cdot x}|0\rangle}_{=0}$$

$$= \int dK \langle 0|a(k)a^\dagger(k')|0\rangle e^{-ik\cdot x}$$

$$= \int dK \langle \vec{k}|\vec{k}'\rangle e^{-ik\cdot x}$$

$$= \int dK (2\pi)^3 2\omega_k \delta(\vec{k}-\vec{k}') e^{-ik\cdot x}$$

$$= \int d^3k\, \delta(\vec{k}-\vec{k}') e^{-ik\cdot x}$$

$$\langle 0|\phi(x)|\vec{k}'\rangle = e^{-i\omega_{k'}t + i\vec{k}'\cdot\vec{x}}$$

a plane wave in +x direction in time for a particle with

$$\omega_{k'}^2 = \vec{k}'^2 + m^2$$

- a deBroglie wave - or regular wave function - of a free particle

$$\text{So } \psi_{\vec{k}}(x) = \langle 0|\hat{\phi}(x)|0,0,\dots\vec{k}'\dots 0\dots\rangle$$

change of rep from Fock to coord.

normalization aside

$$\langle k|k'\rangle = (2\pi)^3 2\omega_k \delta(\vec{k}-\vec{k}')$$

the single particle wavefunction -

$$\psi(x) = \langle 0|\varphi(x)|k\rangle = e^{-ik\cdot x}$$

Calculate the resultant single particle wave function normalization

$$\int \psi^+(x)\psi(x)\,dx \equiv N$$

$$= \int d^3x \; \langle k|\varphi^+(x)|0\rangle\langle 0|\varphi(x)|k\rangle$$
$$= \int d^3x \int dP \int dP' \; \langle k|p\rangle\langle p'|k\rangle \, e^{ip\cdot x} \, e^{-ip'\cdot x}$$

$$= \int dP \int dP' \; \langle k|p\rangle\langle p'|k\rangle \, \delta(\vec{p}-\vec{p}')$$
$$= \int dP \; \langle k|p\rangle\langle p|k\rangle$$

$|p\rangle$ are a complete set, so

$$\int \psi^+\psi\, d^3x = \langle k|k\rangle = (2\pi)^3 2\omega_k \delta(0)$$

what is $\delta(0)$?

$$\delta(k) \equiv \lim_{L\to\infty} \iiint_{-L/2}^{L/2} dx\,dy\,dz\, \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^3}$$

$$\Rightarrow \quad \delta(0) = \lim_{L\to\infty} \iiint \frac{d^3x}{(2\pi)^3} = \lim_{L\to\infty} \frac{L^3}{(2\pi)^3}$$

call $\lim_{L\to\infty} L^3 \equiv V$

so $$\langle k|k\rangle = (2\pi)^3\, 2\omega_k \frac{V}{(2\pi)^3} \quad \Rightarrow$$

$$\frac{\langle k|k\rangle}{V} = 2\omega_k = \frac{N}{V}$$ # particles per unit volume.

Remember density of states?

The number of states per energy interval usually

$$\rho(E)dE = \frac{V\,d^3k}{(2\pi)^3}$$ for 1 particle / unit volume

we have $2E$ particles / unit volume, so the phase space that we use is

$$\rho(E)dE = \frac{V\,d^3k}{(2\pi)^3\, 2E}$$

which makes $\int \psi^\dagger \psi\, d^3x$ Lorentz invariant.

For a complex scalar field, remember we had,

$$\phi = \sqrt{1/2}\,(\varphi_1 + i\phi_2)$$

$$\mathcal{L} = \left(\frac{\partial\phi}{\partial x^\mu}\right)^*\left(\frac{\partial\varphi}{\partial x_\mu}\right) - m^2\varphi^*\varphi$$

now:

$$\hat{\phi}(x) = \int dK\,\left[a(\vec{k})e^{-ik\cdot x} + b^\dagger(\vec{k})e^{ik\cdot x}\right]$$

$$\hat{\varphi}^\dagger(x) = \int dK\,\left[b(\vec{k})e^{ik\cdot x} + a^\dagger(\vec{k})e^{-ik\cdot x}\right]$$

These operators have the separate commutation rules,

$$[a(k), a^\dagger(k)] = (2\pi)^3\, 2\omega_k\, \delta(\vec{k}-\vec{k}')$$

$$[b(k), b^\dagger(k)] = (2\pi)^3\, 2\omega_k\, \delta(\vec{k}-\vec{k}')$$

The Hamiltonian is constructed normally.

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$$

where $\pi(x) = \dot{\phi}^\dagger = i\int dK\, i\omega_k\left[b(k)e^{-ik\cdot x} - a^\dagger(k)e^{ik\cdot x}\right]$

By now we know how to write it down —

$$:H: = \int dK\, \omega_k \left[ a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})\, b(\vec{k}) \right]$$

Let's look at the Noether current and the symmetry.

$$J^\mu = -i \frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^\mu)} \quad f(\Phi)$$

For classical 2d complex fields and $U(1)$ transformations $f(\varphi) = \varphi$

$$\mathcal{L} = \partial^\mu \varphi^* \partial_\mu \varphi - m^2 \varphi^* \varphi$$

$$\rightarrow \quad J^\mu(x) = i\,(\varphi^* \partial^\mu \varphi - \varphi\, \partial^\mu \varphi^*)$$

So in the quantized theory we would have

$$:J^\mu(x): \; = \; i :\hat{\varphi}^\dagger \partial^\mu \hat{\varphi} - \hat{\varphi}\, \partial \hat{\varphi}^\dagger:$$

The constant of the motion comes from

$$J^0(x) = i: \varphi^\dagger \dot{\varphi} - \varphi\, \dot{\varphi}^\dagger :$$

Call the "charge" associated with this $U(1)$ symmetry "$T$":

$$\left.\begin{aligned} T &= \int d^3x\, J^0(x) \\ &= i \int d^3x\, :\varphi^\dagger \dot{\varphi} - \varphi\, \dot{\varphi}^\dagger: \end{aligned}\right\} \quad U(\alpha) = e^{i\alpha T}$$

This is like all of the others that we've done -- the result is

$$T = \int dK \left[ a^\dagger(k)a(k) - b^\dagger(k)b(k) \right]$$
$$= \int dK \left[ N_a(k) - N_b(k) \right]$$

→ $T$ is an operator, the generator of this $U(1)$ transformation

note: · there is no weighting of this integral by anything dimensionful -- like $\hbar\omega$ ⇒ regardless of the kinematics of the $\varphi$, each quantum counts the same

→ such a characteristic is called "internal"

The application of $T$ to a state counts the net amount of this characteristic. and labels the fields created by $a^\dagger$ with the opposite sign but same magnitude as those created by $b^\dagger$.

Remember: before we did the transformation as

$$\varphi \to \varphi' = e^{i\alpha T}\varphi$$

but now $\varphi$ is an operator and that's not how operators transform. Rather,

$$\varphi \to \varphi' = U(\alpha)\,\varphi\, U^{-1}(\alpha)$$

$$\varphi' = e^{i\alpha T}\,\varphi\, e^{-i\alpha T}$$
$$= (1 + i\alpha T + \tfrac{1}{2}(i\alpha T)^2 + \cdots)\,\varphi\,(1 - i\alpha T + \tfrac{1}{2}(-i\alpha T)^2 + \cdots)$$
$$= \varphi + i\alpha(T\varphi - \varphi T) + \tfrac{1}{2}(i\alpha)^2(TT\varphi - 2T\varphi T) + \cdots$$

$$\varphi' = \varphi + i\alpha[T,\varphi] + \tfrac{1}{2}(i\alpha)^2[T,[T,\varphi]] + \cdots$$

So, we need $[T,\varphi]$

$$[T,\varphi] = \int dK' \int dK \left[a^+(k)c(k) - b^+(k)b(k),\ a(k')e^{-ik'\cdot x} + d^+(k')e^{ik'\cdot x}\right]$$

; problem assignment

$[T,\varphi] = -\varphi$ and $[T,\varphi^+] = \varphi^+$

also, $[T,a] = -a$ $\quad [T,b^+] = -b^+$
$\quad\quad [T,a^+] = a^+$ $\quad [T,b] = b$

~~so~~ $\varphi' = \varphi + i\alpha(-\varphi) + \tfrac{1}{2}(i\alpha)^2[T,-\varphi] \cdots$
$\quad = \varphi - i\alpha\varphi - \tfrac{1}{2}\alpha^2\varphi \cdots$

$\varphi' = e^{-i\alpha}\varphi$ showing that for $U(1)$ this is still a simple phase rotation.

Suppose we have

$$|\psi\rangle = |n_a(k_1), n_a(k_2) \cdots, n(k_i), \cdots;\ n_b(k_1), n_b(k_2) \cdots n_b(k_i) \cdots\rangle$$

$$T|\psi\rangle = \int dK\ [N_a(k) - N_b(k)]\ |\psi\rangle$$

$$= \int dK\ (\underbrace{n_a(k) - n_b(k)}_{\text{net }\#})\ |\psi\rangle$$

how about $T\,a^\dagger(k_i)|\psi\rangle$

$$T\,a^\dagger(k_i)|\psi\rangle = \int dK\ [N_a(k) - N_b(k)]\ a^\dagger(k)|\psi\rangle$$

$$= \int dK\ [N_a(k) - N_b(k)]\sqrt{n(k_i)}\,|\ldots, n_a(k_i)+1, \ldots; \ldots n_b(k_i)\ldots\rangle$$

not useful, or an eigenvalue equation.

Instead use commutator.

$$[T, a^\dagger(k_i)] = +a^\dagger(k_i)$$

$$T a^\dagger(k_i) - a^\dagger(k_i)\,T = +a^\dagger(k_i)$$

$$T a^\dagger(k_i) = a^\dagger(k_i)\,T + a^\dagger(k_i)$$

$$T\,a^\dagger(k_i)|\psi\rangle = (a^\dagger T + a^\dagger(k_i))|\psi\rangle$$

$$= a^\dagger(k_i)\int dK\ [N_a - N_b]|\psi\rangle + a^\dagger(k_i)|\psi\rangle$$

$$= a^\dagger(k_i)\int dK\ (n_a - n_b)|\psi\rangle + a^\dagger(k_i)|\psi\rangle$$

$$= \int dK\,(n_a - n_b)\,a^\dagger(k_i)|\psi\rangle + 1\,a^\dagger(k_i)|\psi\rangle$$

$$= \left\{\int dK\ [n_a(k) - n_b(k)] + 1\right\} a^\dagger(k_i)|\psi\rangle$$

so $a^\dagger(k_i)|\psi\rangle$ contains 1 more quantum of charge +1 than $|\psi\rangle$

likewise $a(k)|\Psi\rangle$ contains 1 less.

Also, $T\, b^{+}|\Psi\rangle = \left\{ \int dk \left[ n_a(k) - n_b(k) \right] - 1 \right\} b^{+}|\Psi\rangle$

so this supports the idea of antiparticles having opposite internal quantum characteristics. ("charges").

| | | | |
|---|---|---|---|
| $a^{+}(\vec{k})$ | creates a spin 0 particle of | $\vec{k}$ and | $T=+1$ |
| $b^{+}(\vec{k})$ | " " " " " " | $\vec{k}$ " | $T=-1$ |
| $a(\vec{k})$ | annihilates " | $\vec{k}$ " | $T=+1$ |
| $b(\vec{k})$ | " " | $\vec{k}$ " | $T=-1$ |

in our "frequency" description

$$\underset{\substack{\text{annihilates}\\ \text{particle}}}{\varphi = \varphi^{(+)}} + \underset{\substack{\text{creates}\\ \text{antiparticle.}}}{\varphi^{(-)}}$$

$$\underset{\substack{\text{creates}\\ \text{particle}}}{\varphi^{+} = \varphi^{(+)+}} + \underset{\substack{\text{annihilates}\\ \text{antiparticle}}}{\varphi^{(-)+}}$$

Spin 1/2 Fields

Recall,

$$\mathcal{L}(x) = \bar{\psi}(x)\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x)$$

which gave $\quad \mathcal{H} = \pi\dot{\psi} - \mathcal{L}$

$$\pi = i\psi^+$$

$$\mathcal{H} = i\psi^+ \partial^0\psi - \bar{\psi}\,(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\psi$$
$$= i\bar{\psi}\gamma^0\partial_t\psi - i\bar{\psi}\gamma^0\partial_t\psi - i\bar{\psi}\gamma^i\partial_i + \bar{\psi}m\psi$$
$$= \psi^+\gamma^0(-i\gamma^i\partial_i + m)\psi$$

from Dirac equation, the stuff in $(\,) = i\gamma^0\partial_t$

$$\mathcal{H} = \psi^+\gamma^0\,(i\gamma^0\partial_t\psi)$$
$$\mathcal{H} = i\psi^+\frac{\partial\psi}{\partial t}$$

Quantization proceeds as usual, with an important exception.

Expand, component by component,

$$\psi_j(x) = \sum_{i=1,2}\int dK\left[\,a^{(i)}(k)\,u_j^{(i)}(k)\,e^{-ik\cdot x} + b^{(i)+}(k)\,v_j^{(i)}(k)\,e^{ik\cdot x}\right]$$
$$\psi^{(+)} + \psi^{(-)}$$

where

$$u^{(i)}(k) = \sqrt{E+m}\begin{pmatrix} \chi^{(i)} \\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi^{(i)} \end{pmatrix} \qquad \begin{array}{l} i=1 \Rightarrow \text{"spin"} +1/2 \\ \;=2 \Rightarrow \quad -1/2 \end{array}$$

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$v^{(i)}(k) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\chi^{(i)} \\ \chi^{(i)} \end{pmatrix} \qquad \begin{array}{l} i=1 \Rightarrow \text{"spin"} -1/2 \\ \;=2 \Rightarrow \quad +1/2 \end{array}$$

and $\quad u^{+(i)}u^{(j)} = 2E\,\delta_{ij} \Rightarrow \bar{u}^{(i)}u^{(j)} = 2m\,\delta_{ij}$

$\quad\quad v^{+(i)}v^{(j)} = 2E\,\delta_{ij} \qquad \bar{v}^{(i)}v^{(j)} = -2m\,\delta_{ij}$

plus $\quad (\not{k}-m)u(k) = 0 \qquad \bar{u}(k)(\not{k}-m) = 0$

$\quad\quad (\not{k}+m)v(k) = 0 \qquad \bar{v}(k)(\not{k}+m) = 0$

So,

$$\bar{\psi}_j(x) = \sum_{i=1,2}\int dK\left[\bar{u}_j^{(i)}(k)a^{(i)+}(k)e^{ik\cdot x} + \bar{v}_j^{(i)}b^{(i)}(k)e^{-ik\cdot x}\right]$$

$$\bar{\psi}^{(-)} + \bar{\psi}^{(+)}$$

The Hamiltonian provides the surprise. -

$$H = \int d^3x\, \mathcal{H}(x)$$

$$= \int d^3x\, i\,\psi^+\frac{\partial\psi}{\partial x^0} \quad \leftarrow \text{matrix equation.}$$

$$= i\int d^3x \sum_{i=1,2}\sum_{n=1,2}\int\frac{d^3k}{(2\pi)^3 2E}\int\frac{d^3k'}{(2\pi)^3 2E'}\left\{[\psi^+]\frac{\partial}{\partial t}[\psi]\right\}$$

$$\sum_i \sum_n \{ \} = \sum_i \sum_n \left[ a^{+(n)} u^{+(n)} e^{ih\cdot x} + b^{(i)} v^{(i)+} e^{-ih\cdot x} \right]$$

$$\times \left[ (-iE') a^{(n)} u^{(n)} e^{-ih'\cdot x} + (iE') b^{+(n)} v^{(n)} e^{ih\cdot x} \right]$$

∴ problem.

$$H = \int dK \sum_i E \left[ a^{+(i)}(h) a^{(i)}(h) - b^{(i)}(h) b^{+(i)}(h) \right]$$

care is now required!

Suppose : $[b, b^+] = \rho\delta()$ like spin 0, then
$$bb^+ = b^+b + c\delta(0)$$

substitute as before,
$$[a^+a - b^+b - c\delta(0)]$$

get rid of it

BUT $-b^+b$ would still result in non positive definite energy eigenvalues of $H$.

must postulate anticommutators –

$$\{ a^{(i)}(h), a^{+(j)}(h') \} = \{ b^{(i)}(h), b^{+(j)}(h') \} = (2\pi)^3 2E\delta_{ij}\delta^3(\vec{h}-\vec{h}')$$

$$\{b, b'\} = \{a, a'\} = \{a^+, a^+\} = \{b^+, b^+\} = 0$$

Then we have a sensible theory – with a new Normal ordering

$$H = \int dK \; : i \psi^{+} \partial_t \psi :$$

which amounts to using $\{\}$ to $\qquad \{b, b^{+}\} = c\,\delta(\;)$

$$b b^{+} = -b^{+} b + c\,\delta(\;)$$

$$a^{+}a - b\, b^{+} \longrightarrow a^{+}a + b^{+}b - c\,\delta(0)$$

ah. $\qquad$ forget it — just like before

so, now spin 0 & $\frac{1}{2}$: $\qquad$ P = # fermion permutations.

$$: ABCD : \equiv (-1)^{P} [\underbrace{\;---\;}]$$

ABCD arranged: creation - LEFT
annihilation - RIGHT

so. $\qquad : \psi \bar{\psi}' : = : (\psi^{+} + \psi^{-})(\bar{\psi}^{+} + \bar{\psi}^{-})$

$$= : \underset{a}{\psi^{+}} \underset{b}{\bar{\psi}^{+}} + \underset{a}{\psi^{+}} \underset{a^{+}}{\bar{\psi}^{-}} + \underset{b^{+}}{\psi^{-}} \underset{b}{\bar{\psi}^{+}} + \underset{b^{+}}{\psi^{-}} \underset{a^{+}}{\bar{\psi}^{-}} :$$

$$= \psi^{+}\bar{\psi}^{+} - \bar{\psi}^{-}\psi^{+} + \psi^{-}\bar{\psi}^{+} + \psi^{-}\bar{\psi}^{-}$$

Can get the commutation relations for field operators by, like before, inverting the expansions to get a and b..

form $\qquad u_j^{(n)+}(k')\, e^{i k' \cdot x} \cdot \psi_j(x)$

integrate $\int d^3x$ and $\sum_j$

$$\sum_j \int d^3x\, e^{ik'\cdot x} u_j^{(n)\dagger}(k')\psi_j(x) =$$

$$\int d^3x \sum_j \sum_i dK \left\{ \left(a^i(k)\, u_j^{(n)\dagger}(k')\, u_j^{(i)}(k)\right) e^{i(k'-k)\cdot x} + \left(b^{(i)\dagger}(k)\, u_j^{(n)\dagger}(k')\, v_j^{(i)}(k)\right) e^{i(k'+k)\cdot x} \right\}$$

$$= \sum_i \sum_j \int dK \left\{ (\quad)(2\pi)^3 \delta(\vec{k}-\vec{k}')\, e^{i(E'-E)t} + (\quad)(2\pi)^3 \delta(\vec{k}+\vec{k}') \right\} e^{i(E'+E)t} \}$$

use orthogonality $= 2E\,\delta_{ni}$

use $\delta(k-k')$ & $E^2 = k^2 + m^2$ $\Big\}$ P

use orth $\Rightarrow \emptyset$

so, all collapses to

$$= a^{(n)}(k)$$

likewise, $a^{(n)\dagger}(k) = \sum_j \int d^3x\, e^{-ik\cdot x} \psi_j^\dagger\, u_j^{(n)}(k)$

$$b^{(n)\dagger}(k) = \sum_j \int d^3x\, e^{-ik\cdot x} v_j^{(n)\dagger}(k)\, \psi_j(x)$$

$$b^{(n)}(k) = \sum_j \int d^3x\, e^{ik\cdot x} \psi_j^\dagger(x)\, v_j^{(n)}(k)$$

So, the required anticommutation relations for $a$'s, $b$'s result in

$$\{ \psi_i(\vec{x},t), \psi_j^\dagger(\vec{x}',t) \} = \delta_{ij}\, \delta(\vec{x}-\vec{x}')$$

$$\{ \psi, \psi \} = \{ \psi^\dagger, \psi^\dagger \} = 0$$

So, were now too deep in to back out now! We must reinterpret our 2nd quantization conditions to mean

$$[\, \chi, \Pi' \,] = i\, \delta(x-x') \qquad \text{bosons.}$$

$$\{ \chi, \Pi \} = i\, \delta(x-x') \qquad \text{fermions.}$$

same physical interpretation:

$a^\dagger$ and $b^\dagger$ create a & b-type quanta (fermions & antifermion)
$a$ and $b$ annihilate them

The charge operator becomes.

$$T = \int d^3x \, : \bar{\psi}\gamma^0\psi : \qquad \text{remember } j^\mu = \bar{\psi}\gamma^\mu\psi$$
$$= \int d^3x \, : \psi^\dagger\psi :$$
$$= \int dK \sum_i \left[ a^{(i)\dagger}(k)\, a^{(i)}(k) - b^{(i)\dagger}(k)\, b^{(i)}(k) \right]$$

b-type has opposite "charge" from a-type.

Consider a state like $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$ an isodoublet

$$\mathcal{L} = \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi$$

investigate the consequences of the $U(1)$ transformation

$$\psi \to \psi' = e^{i\alpha}\psi \quad \text{.. on both p and n states.}$$

The Noether current is still of the form

$$j^\mu(x) = -i\,\frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial x^\mu)}\,\psi = \bar{\psi}\gamma^\mu\psi$$

charge
$$\begin{aligned} B &= \int d^3x : \psi^\dagger\psi : \\ &= \int d^3x : (p^\dagger n^\dagger)\begin{pmatrix} p \\ n \end{pmatrix} : \\ &= \int d^3x : p^\dagger p + n^\dagger n : \end{aligned}$$

p and n are each separate spinors, each with its own expansion,

$$B = \int d^3x \left[ N_p(x) + N_n(x) - N_{\bar{p}}(x) - N_{\bar{n}}(x) \right]$$

net B charge .. obviously not electric charge, but rather Baryon Number charge. a quantity conserved in elementary particle physics reactions ie, you never see

$$p \to e^+\nu_e \quad \text{etc.}$$