Lecture 11 Spin 1 Quantization

Now, we come full circle – back to the EM field

Classically, we found

$$\pi^i(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_i}$$

$$\mathcal{L}(x) = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

π^i conjugate to A_i

where $F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$

and $\pi^\mu(x) = \pi^i(x) = -\dot{A}^i(x) - \frac{\partial A^0}{\partial x^i} = E^i$

$\pi^0(x) = 0$ $\longleftarrow$ led us to an original quantization for only space $\vec{A}$

To quantize: $\sum_n \longrightarrow \int dk$ no problem

$a_{n\lambda} \longrightarrow \hat{a}_\lambda(k)$ no problem.

demand $[\pi^i(x,t), A_j(x',t)] = -i\delta^i_j\,\delta(\vec{x}-\vec{x}')$ a problem

$$\left.\begin{aligned} [A^\mu, A^{\nu\prime}] &= 0 \\ [\pi^i, \pi^{j\prime}] &= 0 \\ [\pi^i, A^0] &= 0 \end{aligned}\right\} \text{ no problem.}$$

What's wrong? well

$$[\pi^i(x,t), A_j(x',t)] = -[\pi^i(x,t), A^j(x',t)]$$
$$= -[E^i, A^{j\prime}] = -i\delta^i_j\,\delta(\vec{x}-\vec{x}')$$

for no sources, $\vec{\nabla}\cdot\vec{E} = 0$, but

$$\left[\frac{\partial E^i(x,t)}{\partial x^i}, A^j(x',t)\right] = -i\delta^{ij}\frac{\partial}{\partial x^i}\delta(\vec{x}-\vec{x}') \neq 0$$

Consider covariant Lorentz gauge. (LG)

$$\frac{\partial A^\mu}{\partial x^\mu} = 0$$

This reduces the number of independent A^μ components from $4 \to 3$. Notice, for

$$A^\mu \to A^{\mu\prime} = A^\mu + \frac{\partial \theta}{\partial x_\mu}$$

now $A^{\mu\prime}$ also satisfies LG, if $\dfrac{\partial^2 \theta}{\partial x^\mu \partial x_\mu} = 0$

If θ is chosen such that $\dfrac{\partial \theta}{\partial x^0} = -\phi$

then $A^0 \to A^{0\prime} = A^0 + \dfrac{\partial \theta}{\partial x^0} \Rightarrow \phi' = 0$

Then, $$\frac{\partial A'^\mu}{\partial x^\mu} = \frac{\partial A'^0}{\partial x^0} + \frac{\partial A'^i}{\partial x^i} = 0 \quad (LG)$$

"gauged away"

$\Rightarrow \vec{\nabla} \cdot \vec{A}' = 0$ Coulomb or Radiation gauge as a special circumstance of L.G.

So — number of independent components further reduced $3 \to 2$.

This is how real EM fields <u>are</u>... so there is some satisfaction in using the Coulomb gauge for quantization. *However, it's not covariant.*

In this gauge, quantization is done this way – first note the commutation relation is sick a second time, $\nabla' \rightarrow$

$$-\left[E^i(x,t), \frac{\partial A^j}{\partial x^{j'}}(x',t)\right] = -i\,\delta^{ij}\frac{\partial}{\partial x^{j'}}\,\delta(\vec{x}-\vec{x}') \neq 0$$

Still sick!

should be ϕ in this gauge

So, modify . .

$$\underbrace{\left[\Pi^i(x,t), A_j(x',t)\right]}_{\text{must be 2nd rank tensor}} = \left[-i\,\eta^i{}_j \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\right]$$

$$\frac{\partial}{\partial x^i}\left[\;\right] = -i\eta^i{}_j \int \frac{d^3k}{(2\pi)^3}\, ik_i\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$ which we need to be $= 0$

so, require $\eta^i{}_j\, k_i = 0$

which is satisfied by $\eta^i{}_j = \delta^i{}_j - \frac{k^i k_j}{k^2}$

ie $\eta^i{}_j\, k_i = k_j - \frac{k^2 k_j}{k^2} = 0$

so, define

$$\delta^{tr\,i}{}_j(\vec{x}-\vec{x}') \equiv \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\left(\delta^i{}_j - \frac{k^i k_j}{k^2}\right)$$

and

$$\left[\Pi^i(x,t), A_j(x',t)\right] = -i\,\delta^{tr\,i}{}_j(\vec{x}-\vec{x}')$$

Now, we interpret the Fourier expansion in the standard way.

$$\vec{A}(x) = \int dK \sum_{\lambda=1}^{2} \vec{\epsilon}_\lambda(\vec{k}) \left[a_\lambda(\vec{k}) e^{-ik\cdot x} + a_\lambda^\dagger(\vec{k}) e^{ik\cdot x} \right]$$

and in this gauge $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \vec{k}\cdot\vec{\epsilon}_\lambda = 0$

→ only the transverse components of A^μ have been quantized... only the physical photon, like classical treatment.

Other gauges are sometimes used... the most common is the LG.

Let's postulate that what we want is this:

$$[\pi_\mu(x,t), A_\nu(x',t)] = -i g_{\mu\nu} \delta(\vec{x}-\vec{x}')$$ so A_0 commutes w/ π_0

since $\pi_0 = 0$ for our $\mathcal{L}$, $[0, A_\nu] = 0 = g_{0\nu} \Rightarrow \nu = 0$ for any ν, so this suggests that $\nu=0$ leads to A_0 being a c-number.

So, modify $\mathcal{L}$. (due to Fermi, actually) need to maintain $\Box A_\mu = 0$, the equation of motion.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2$$ "gauge-fixing" term.

$$\frac{\partial A^\mu}{\partial x^\mu} = g_{\mu\nu} \frac{\partial A^\mu}{\partial x_\nu} \Rightarrow \left\{ \begin{array}{l} (\partial_\mu A^\mu)^2 = \\ -\frac{1}{2} g_{\mu\nu} \frac{\partial A^\mu}{\partial x_\nu} \frac{\partial A^\alpha}{\partial x^\alpha} \end{array} \right.$$

so,

$$\frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathcal{L}}{\partial A^\beta/\partial x_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial A^\beta} = 0$$

now leads to $\qquad$ still $= 0$

$$\frac{\partial \mathcal{L}}{\partial(\partial A^\mu/\partial x_\nu)} = F_{\mu\nu} - g_{\mu\nu} \partial_\alpha A^\alpha$$

so $$\frac{\partial}{\partial x_\nu}\left(F_{\mu\nu} - g_{\mu\nu}\partial_\alpha A^\alpha\right) = \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} - g_{\mu\nu}\frac{\partial A^\alpha}{\partial x^\alpha}\right)$$

$$= \Box A_\mu - \frac{\partial^2 A_\nu}{\partial x_\nu \partial x^\mu} - \frac{\partial}{\partial x_\mu}\frac{\partial A^\alpha}{\partial x^\alpha}$$

$$= \Box A_\mu - \frac{\partial^2 A_\nu}{\partial x_\nu \partial x^\mu} + \frac{\partial A_\alpha}{\partial x_\mu \partial x^\alpha}$$

$$\Box A_\mu = 0 \quad \checkmark$$

The extra term is called the Gauge Fixing term...

sometimes this is made general

like a Lagrange multiplier – a constraint

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{2} (\partial_\mu A^\mu)^2$$

w/ equation of motion:

$$\Box A^\mu - (1-\lambda)\,\partial^\mu(\partial_\alpha A^\alpha) = 0 \qquad \text{(not Maxwellian)}$$

$\lambda = 1$ "Feynman Gauge" (or "Fermi gauge")

$ = 0$ "Coulomb Gauge"

<u>now</u> $$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} - 2\left(\tfrac{1}{2}\right)\left(\frac{\partial A^\alpha}{\partial x^\alpha}\right) \frac{\partial (\partial_\mu A^\mu)}{\partial(\partial_\nu A^\nu)}\, g^{\mu 0}$$

$$= F^{\mu 0} - \partial_\alpha A^\alpha g^{\mu 0}$$

$$\pi^0 = F^{00} - \partial_\alpha A^\alpha = -\partial_\alpha A^\alpha$$

<u>but</u>, this is <u>zero</u> in the L.G. The way out we'll see – ~~of that is to relax the L.G. as a strict operator identity~~* ~~to a statement about measurements~~ –

$$\langle \psi | \partial_\mu A^\mu | \psi \rangle = 0 \quad \longrightarrow \quad \text{L.G. and Maxwell's Eq still hold as a classical limit.}$$

real states

classical vector field

* worked for $\partial_\mu A^\mu = 0$ okay – operator, not okay

So, now we deal with a covariant expansion.

$$A_\mu(x) = \int dK \sum_{\lambda=0}^{3} \epsilon_{\mu(\lambda)}(k)\left[a_\lambda(k) e^{-ik\cdot x} + a_\lambda^\dagger(k) e^{ik\cdot x}\right] \quad *$$

all 4

convention: $\epsilon^\mu_{(0)}$ = timelike vector

$\epsilon^\mu_{(i)}$ = spacelike vector

orthonormality: $\epsilon_{(\lambda)} \cdot \epsilon_{(\lambda')} = \epsilon_{(\lambda)\mu} \cdot \epsilon^\mu_{(\lambda')}$

$= \epsilon_{(\lambda)\mu} g^{\mu\nu} \epsilon_{(\lambda')\nu}$

$= \epsilon_{(\lambda)0}\epsilon_{(\lambda)0} - \vec{\epsilon}_{(\lambda)} \cdot \vec{\epsilon}_{(\lambda')}$ and demand

$= g_{\lambda\lambda'} = -\rho_\lambda \delta_{\lambda\lambda'}$ where $\rho_0 = -1$, $\rho_i = +1$

completeness:

$$\sum_\lambda \rho_\lambda \epsilon^\mu_{(\lambda)} \epsilon^\nu_{(\lambda)} = -g^{\mu\nu}$$

How do these look?

presume momentum along 3 axis –

$k^\mu = (k, 0, 0, k)$

and we can define

$\epsilon_{(0)} = (1,0,0,0)$ $\quad \epsilon_{(1)} = (0,1,0,0)$ $\quad \epsilon_{(2)} = (0,0,1,0)$

$\underbrace{\epsilon_{(0)}}_{\text{scalar or timelike photon}}$ $\quad \underbrace{\epsilon_{(1)},\ \epsilon_{(2)}}_{\text{physical photon}}$

$\underbrace{\epsilon_{(3)} = (0,0,0,1)}_{\text{longitudinal photon}}$

The commutation relations for the a's we can limit at. - go back

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\dot{A}_\mu)} = F^{\mu 0} - g^{\mu 0}\left(\frac{\partial A^\nu}{\partial x^\nu}\right)$$

or $$\pi^0 = F^{00} - g^{00}\left(\frac{\partial A^\nu}{\partial x^\nu}\right) = 0 - \frac{\partial A^0}{\partial x^0} + \vec{\nabla}\cdot\vec{A} = -\partial_\mu A^\mu$$

$$\pi^i = F^{i0} - 0 = \frac{\partial A^0}{\partial x_i} - \frac{\partial A^i}{\partial x_0}$$

L.G $=0$! hmm

so want $[A_\mu(x,t), \pi_\nu(x',t)] = i g_{\mu\nu}\,\delta(\vec{x}-\vec{x}')$

from $\quad [A_\mu(x,t), A_\nu(x',t)] = 0$

$$\frac{\partial}{\partial x_i} \longrightarrow \left[\frac{\partial A_\mu}{\partial x_i}(x,t), A_\nu(x',t)\right] = 0$$

part of $[A,\pi]$ stuff

$\rightarrow$ spatial derivatives commute at equal time. ✓

So, from

$$[A_\mu(x,t), \pi_\nu(x',t)] = i g_{\mu\nu}\,\delta^3(\vec{x}-\vec{x}')$$

only the time derivatives of A inside π_ν won't commute, so

$$[A_\mu(x,t), \dot{A}_\nu(x',t)] = -i g_{\mu\nu}\,\delta^3(\vec{x}-\vec{x}')$$

substitute the expansions for A_μ and $\dot{A}_\nu$ --

...

$$[\, a_{(\lambda)}(k),\, a^{\dagger}_{(\lambda')}(k') \,] = -g_{\lambda\lambda'}\, 2k_0\, (2\pi)^3 \delta(\vec{k}-\vec{k}')$$

negative sign is problematic

consider a ~~timelike~~ scalar, 1 photon state,

$$|h_{(0)}\rangle = \int dK\, f_k(x)\, a^{\dagger}_{(0)}(k)|0\rangle$$

find the norm

$$\langle h_{(0)}|h_{(0)}\rangle = \int dK \int dK'\, f_k f^{*}_{k'} \langle 0|a_{(0)}(k')\, a^{\dagger}_{(0)}(k)|0\rangle$$

Use commutation relations -

$$= \int dK \int dK'\, f_k f^{*}_{k'} \langle 0| -2k_0 (2\pi)^3 \delta(\vec{k}-\vec{k}')|0\rangle$$

$$= -\int dK \int d^3k\, f_k f^{*}_{k'}\, \delta(\vec{k}-\vec{k}')\langle 0|0\rangle$$

$$= -\int dK\, |f_k|^2 \quad \rightarrow \text{ a negative norm!}$$

for 2 time-like photon state + norm

3 " − norm.

$\rightarrow$ indefinite!

To add insult to injury,

$$N_{(0)}(k) = -a^{\dagger}_{(0)}(k)\, a_{(0)}(k)$$

so that the timelike photon contribution to the energy is negative.

The way out is to demand a weakened "subsidiary condition"

$$\frac{\partial A^{(+)\mu}}{\partial x^\mu} |\psi\rangle = 0$$ (already satisfied for the vacuum)

Since then for $\langle\psi| \frac{\partial A^\mu}{\partial x^\mu} |\psi\rangle \overset{\text{want}}{=} 0$ ✓

$$= \langle\psi| \underbrace{\frac{\partial A^{(+)\mu}}{\partial x^\mu}}_{=0} + \frac{\partial A^{(-)\mu}}{\partial x^\nu} |\psi\rangle$$

$$= \langle\psi| \frac{\partial A^{(-)\mu}}{\partial x^\nu} |\psi\rangle = \langle\psi| \frac{\partial A^{(+)\mu}}{\partial x^\nu} |\psi\rangle^* = 0$$

Called Gupta-Bleuler quantization - 1950.

put expansion for $A^{(+)\mu}$ into G.B. condition above, from ✱

$$\sum_{\lambda=0}^{3} k^\mu \epsilon_{(\lambda)\mu}\, a_{(\lambda)}(k) |\psi\rangle = 0$$

all 4 ... need to make this happen

for γ in k_3 direction, for example,

$$k^\mu = (k, 0, 0, k) \;\ldots\; \text{so} \quad k \cdot \epsilon_{(1,2)} = 0 \qquad \checkmark$$

we get for G.B. condition:

$$\left[k^\mu \epsilon_{(0)\mu}\, a_{(0)}(k) + k^\mu \epsilon_{(3)\mu}\, a_{(3)}(k) \right] |\psi\rangle = 0$$

from the definition of ϵ_0 and ϵ_3,

$$k^\mu \epsilon_{(0)\mu} = - k^\mu \epsilon_{(3)\mu} \quad \text{so,}$$

$$k^\mu \epsilon_{(0)\mu} \left[a_{(0)}(k) - a_{(3)}(k) \right] \cdot |\psi\rangle = 0$$

$$\text{so} \qquad \left(a_{(0)}(k) - a_{(3)}(k) \right) |\psi\rangle = 0$$

serves as a <u>constraint</u> on the timelike and longitudinal pieces.

Physical states will be admixtures of these 2 so that the constraint holds. → there are no 1 particle physical states of only timelike photons

What does this mean practically.—

The Hamiltonian can be constructed in the standard fashion to be,

$$H = \sum_{\lambda=0}^{3} \int dK\, \zeta_\lambda\, a^+_{(\lambda)}(\vec{k})\, a_{(\lambda)}(\vec{k})\, k_0$$

Even though $\zeta_0 = -1$ .. the energy is positive definite

ie $$\int dK\, k_0 \left\{ \sum_{\lambda=1}^{3} a^+_{(\lambda)}(k)\, a_{(\lambda)}(k) - a^+_{(0)}(k)\, a_{(0)}(k) \right\}$$

What about $\langle \psi | H | \psi \rangle$?

$$= \langle \psi | \int dK\, k_0 \left\{ \sum_{\lambda=1}^{2} a^+_{(\lambda)}(k)\, a_{(\lambda)}(k) + a^+_{(3)}(k)\, a_{(3)}(k) - a^+_{(0)}(k)\, a_{(0)}(k) \right\} | \psi \rangle$$

The subsidiary condition says

$$a_{(0)}(k)|\psi\rangle = a_{(3)}(k)|\psi\rangle$$

its adjoint says $\langle \psi | a^+_{(0)}(k) = \langle \psi | a_{(3)}(k)$

so $$\langle \psi | a^+_{(0)} a_{(0)} | \psi \rangle = \langle \psi | a^+_{(0)} a_{(3)} | \psi \rangle$$

$$= \langle \psi | a^+_{(3)} a_{(3)} | \psi \rangle$$ cancels

so

$$\langle \psi | H | \psi \rangle = \langle \psi | \int dK\, k_0 \left\{ \sum_{\lambda=1}^{2} a^+_{(\lambda)}(k)\, a_{(\lambda)}(k) \right\} | \psi \rangle$$

$\longrightarrow$ only the physical states contribute: 2 dof

A word about massive spin 1 fields --

In your problem set you'll find that for spin 1 massive fields, B_μ, that $\frac{\partial B^\mu}{\partial x^\mu} = 0$

THIS IS NOT A GAUGE CONDITION -- only a constraint which comes from the equation of motion - Proca Equations

An expansion of the sort

$$B_\mu = \int dK \sum_{\lambda=0}^{3} \left[\epsilon_{(\lambda)\mu}(k)\, a_\lambda(k)\, e^{-ik\cdot x} + \epsilon_{(\lambda)\mu}\, a_\lambda^\dagger(k)\, e^{ik\cdot x} \right]$$

The constraint $\Rightarrow$ $\overset{4d}{k \cdot \varepsilon_{(\lambda)}} = 0$ 4 components $\to$ 3

From $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} M^2 B_\mu B^\mu$

we get $\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{B}_\mu} = \partial^\mu B^0 - \dot{B}^\mu$

now using $\partial^\mu B_\mu = 0$ $\Rightarrow$ we can eliminate a $B \to B^0$

so, $\pi^i = -\dot{B}^i$

and

$$[B^i(x\,t), \pi_j(x',t)] = i\,\delta^i{}_j\,\delta^3(\vec{x}-\vec{x}')$$

becomes,

$$[\dot{B}_i(\vec{x},t), B_j(\vec{x}'t)] = i\,g_{ij}\,\delta^3(\vec{x}-\vec{x}')$$

w/ expansion substituted,

$\vdots$

$$[a_{(\lambda)}(k), a_{(\lambda')}(k')] = \delta_{\lambda\lambda'}\, 2k^0 (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

no negative sign. $\rightarrow$ all is good because π_μ is different.

Polarization tensor is different. –

In the rest frame $k \cdot \epsilon = 0$ is satisfied by

$k^\mu = (M, 0, 0, 0)$

$\epsilon^\mu_{(1)} = (0, 1, 0, 0)$ usually spacial used $\vec{\epsilon}_{(+)} = -\sqrt{1/2}\,(1, i, 0)$

$\epsilon^\mu_{(2)} = (0, 0, 1, 0)$ $\vec{\epsilon}_{(-)} = \sqrt{1/2}\,(1, -i, 0)$

$\epsilon^\mu_{(3)} = (0, 0, 0, 1)$

called longitudinal, but refered to as $\epsilon^\mu_{(0)} \equiv (0, 0, 0, 1)$

$$k \cdot \epsilon = 0 = k^0 \epsilon_0 + \sum_i k^i \epsilon_i$$

rest frame: define $\epsilon_0 = 0 \Rightarrow k \cdot \epsilon = M(0) + \sum_i (0)\epsilon_i = 0$

Boost along 3-direction: $k \rightarrow k^\mu = (E, 0, 0, \vec{k})$

for ϵ:

$$\epsilon^\mu_{(\lambda)}(k_{rest}) = (0, \vec{\epsilon}_{(\pm)})$$

BOOST 3 $\longrightarrow$ $\epsilon^\mu_{(\pm)}(k) = \epsilon^\mu_{(\lambda)}(k_{rest}) = (0, \vec{\epsilon}_{(\pm)})$

what about $\lambda = 0$?

$$k \cdot \epsilon_{(0)} = E\,\epsilon^0_{(0)} - k^3\,\epsilon^3_{(0)} = 0$$

<u>construct</u> : $$\epsilon^\mu_{(0)} \equiv (k^3/M,\, 0,\, 0,\, E/M)$$
$$= (\gamma\beta,\, 0,\, 0,\, \gamma)$$

then, $$k \cdot \epsilon_{(0)} = E\,(k^3/M) - k^3\,(E/M) = 0$$

So, we summarize

$$\epsilon^\mu_{(\pm)} = \mp \frac{(0,\, 1,\, \pm i,\, 0)}{\sqrt{2}} \qquad \text{transverse}$$

$$\epsilon^\mu_{(0)} = (\gamma\beta,\, 0,\, 0,\, \gamma) \qquad \text{longitudinal}$$

both necessary for real spin 1 massive.