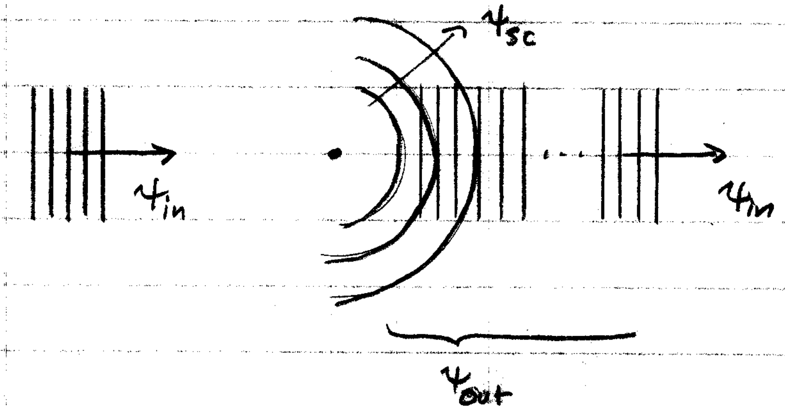
Lecture 13 Scattering I

We can say quite a lot about quantum mechanical scattering with rather general ideas. Let's consider a scattering process like this:



$$\psi_{in} = N e^{ikz} \qquad \psi_{sc} = N f(\theta,\phi) \frac{e^{ikr}}{r}$$

$$\psi_{out} = \underbrace{N e^{ikz}}_{\text{unscattered}} + \underbrace{N f(\theta,\phi) \frac{e^{ikr}}{r}}_{\text{scattered}} \quad \Big\} \text{ General } \text{~~Solution~~}$$

The incoming current density is from

$$\vec{J} = \frac{\hbar(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*)}{2mi}$$

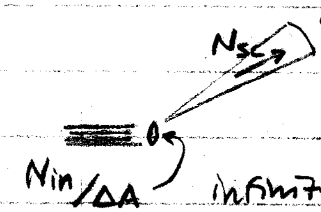
$$\vec{J}_{in} = \frac{N^2 \hbar \vec{k}}{m} = N^2 \hbar \vec{v}_{in}$$

$$\vec{J}_{sc} = N^2 |f(\theta,\phi)|^2 \left[\frac{e^{-ikr}}{ir} \left(ik \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r^2} \right) \right] \frac{\hbar}{m} \hat{r}$$

$$= N^2 \frac{|f(\theta,\phi)|^2}{r^2} \frac{k}{m} \hat{r} \qquad \text{(having taken real part)}$$

$$\vec{J}_{sc} = j_{in} \frac{|f(\theta,\phi)|^2}{r^2} \frac{v_{out}}{v_{in}} \hat{r}$$

The differential cross section — another way to look at it —



$\frac{N_{sc}}{\Delta\Omega}$ infinitesimally small → independent of detector size.

infinitesimally small → independent of beam current and target size.

$$\frac{d\sigma}{d\Omega} \equiv \lim_{\Delta\to 0} \frac{\left(\frac{N_s(\theta,\phi)}{\Delta\Omega}\right)}{(N_{inc}/\Delta A)}$$

since the N's accumulate, calculate rates by dividing by Δt

$$= \frac{(N_s/\Delta t)/\Delta\Omega}{\underbrace{N_{in}/\Delta A/\Delta t}_{j_{in}}} \qquad \Delta\Omega = \frac{\Delta A}{r^2}, \quad \underbrace{(N_{sc}/\Delta t/\Delta A)}_{j_{sc}}\cdot r^2$$

$$= \frac{j_{sc}\cdot r^2}{j_{in}}$$

so,

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 \frac{v_{out}}{v_{in}}$$

$f(\theta,\phi)$ is the scattering amplitude — a measurable.

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Suppose we've got the free particle Schrödinger equation

$$H\psi(r) = E\psi(r) \quad \longrightarrow \quad \frac{\nabla^2}{2m}\psi = \frac{k^2}{2m}\psi$$

for a spherical Hamiltonian,

$$H = -\frac{1}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\vec{L}^2}{r^2}\right]$$

since $\vec{L}^2 Y_\ell^m(\theta,\varphi) = \ell(\ell+1)Y_\ell^m(\theta,\varphi)$

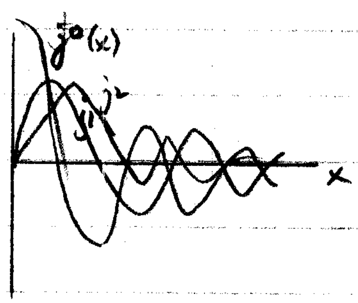
$L_z Y_\ell^m(\theta,\varphi) = m\,Y_\ell^m(\theta,\varphi)$

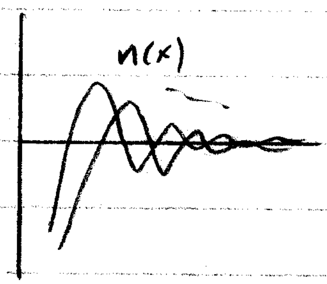
we can write $\psi(r) = R_{k,\ell}(r)Y_\ell^m(\theta,\varphi)$

Substituting,

$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right]\right\}R_{k,\ell}(r) = 0$$

The general solution to this gives spherical Bessel and Neumann functions. However, specifying finiteness at the





origin means only $j_\ell(kr)$ contribute

So, $\psi(r) \sim j_\ell(kr) Y_\ell^m(\theta,\varphi)$, we have 2 bases ($e^{ikz}$ and $j(kr)$)

and can expand one in terms of the other,

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(k)\, j_\ell(kr)\, Y_\ell^m(\theta,\varphi)$$

Presume incoming wave along z so things are independent of φ, so really $m=0$

so, $\vec{k}\cdot\vec{r} = kz$ for the incoming waves ...

$$e^{ikz} = \sum_{\ell=0}^{\infty} (2\ell+1)\, i^\ell j_\ell(kr)\, P_\ell(\cos\theta)$$ Bauer's Formula.

Far from the scattering center, specifically $kr \gg \ell$

$$j_\ell(kr) \longrightarrow \frac{\sin(kr - \ell\pi/2)}{kr}$$ so,

$$e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, P_\ell(\cos\theta)\, \frac{\sin(kr - \ell\pi/2)}{kr}$$

whatever happens, this must be the solution far from source.

Now presume some spherically symmetric potential

$$H\psi(r) + V(r)\psi = E\psi(r)$$

$V(r) \to 0$ as $r \to \infty$
~~no free particle eigst~~

this adds $2mV(r)$ to radial equation and now both Bessel-like functions must be considered.

$$R_{kl}(r) \sim a_l j_l(kr) + b_l n_l(kr)$$
$$\xrightarrow[r\to\infty]{} \frac{1}{kr}\left[a_l \sin\left(kr - \frac{l\pi}{2}\right) - b_l \cos\left(kr - \frac{l\pi}{2}\right)\right]$$
$$\sim c_l \frac{\sin\left(kr - \frac{l\pi}{2} + \delta_l\right)}{kr}$$

the presence of the scattering potential shifts the <u>phase</u> from the un scattered wave.

Remembering $\quad \psi_{sc} = f(\theta,\varphi)\frac{e^{ikr}}{r} \quad$ we have

2 forms for the total wave function after the scattering.

$$\psi_{out} = \sum_{l=0}^{\infty} c_l P_l(\cos\theta) \frac{\sin\left(kr - \frac{l\pi}{2} + \delta_l\right)}{kr}$$

change of <u>phase</u> → elastic

and

change of <u>amplitude</u> & phase → inelastic

$$\psi_{out} = \sum_{l=0}^{\infty} i^l(2l+1) P_l(\cos\theta) \frac{\sin\left(kr - \frac{l\pi}{2}\right)}{kr} + f(\theta,\varphi)\frac{e^{ikr}}{r}$$

comparing allows us to identify

$$c_l = i^l(2l+1)e^{i\delta_l} \quad \text{so}$$

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)$$

so, $$\sigma_{total} = \int \frac{d\sigma}{d\Omega}\, d\Omega$$

$$= \int |f(\theta,\varphi)|^2 d\Omega$$ for elastic scattering

using orthogonality for P_ℓ

$$\sigma_{tot} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)\sin^2\delta_\ell$$

Notice,

$$f(0,0) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell$$

$$= \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\cos\delta_\ell\sin\delta_\ell + i\frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$

and we find,

$$\sigma_{tot} = \frac{4\pi}{k}\,\mathrm{Im}\left[f(0)\right]$$

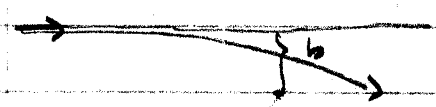
called the Optical Theorem.

What's going on here is

$$j_{out} \sim \Big\{ \underbrace{(\psi_{in}^* + \psi_{sc}^*)\vec{\nabla}(\psi_{in} + \psi_{out})}_{} - (\quad)\vec{\nabla}(\quad) \Big\}$$

leads to interference → that's what the phase shift is — destructive interference between scattered wave and incident, transmitted wave

In forward directions, the depletion goes everywhere else, and amount is "black" portion directly forward of target.



$p = mv$

$L = mvb$

$$mvb = [\ell(\ell+1)]^{\frac{1}{2}}\hbar$$

which limits # partial waves in $\sigma_{tot.}$

If potential range is R -- b must be less than R,

$$[\ell(\ell+1)]^{\frac{1}{2}} < kR$$

so, S wave scattering is a low energy scattering --

Suppose the potential is a hard sphere:

$$V(r) = \infty \qquad r < R$$
$$= 0 \qquad r > R$$

Since ∞, the wavefunction will not penetrate the $r < R$ region and outside it will be the free solution.
Now, the Neumann functions can not be eliminated from the free particle solutions

so, $R'_{kl} \propto A_l j_l(kr) + B_l n_l(kr) \qquad r > R$

comparing this solution with asymptotic —

$$\sin\left(kr - \frac{l\pi}{2} + \delta_l\right) = A_l \sin\left(kr - \frac{l\pi}{2}\right) - B_l \cos\left(kr - \frac{l\pi}{2}\right)$$

$A_l = \cos\delta_l \qquad B_l = \sin\delta_l$ so, everywhere

$$R'_{l,k} = j_l(kr)\cos\delta_l - n_l(kr)\sin\delta_l \qquad r > R$$

So, at the edge $\quad R'_{l,k}(R) = 0 = j_l(kR)\cos\delta_l - n_l(kR)\sin\delta_l$

$$\tan\delta_l = \frac{j_l(kR)}{n_l(kR)} \qquad \text{consider } l=0$$

$$\tan\delta_0 = \frac{\sin kR/kR}{-\cos kR/kR} = -\tan kR \Rightarrow \delta_0 = -kR$$

— generally the phase shift for a repulsive potential is negative

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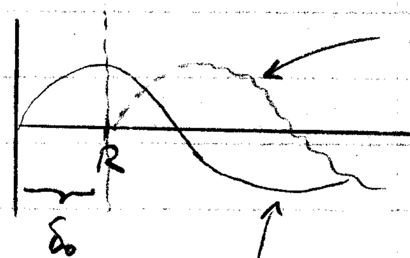
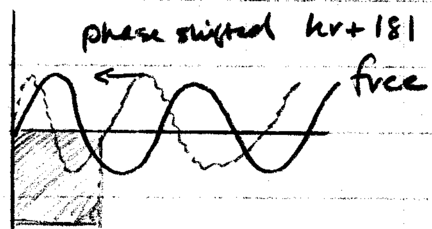
so, the radial wave function

$$R_{k0}(r) \sim \frac{\sin kr}{kr}\cos\delta_0 + \frac{\cos kr}{kr}\sin\delta_0$$

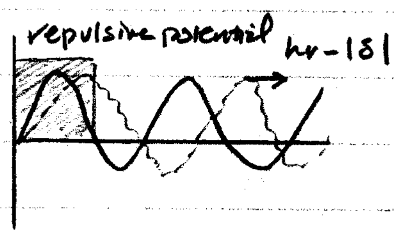
$$= \frac{1}{kr}\sin(kr+\delta_0)$$

$l=0$





positive phase shift



negative phase shift

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This is used to understand the potential

Suppose from

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \qquad f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)$$

consider only S and P waves.

$$f(\theta) = \frac{1}{k}\left(e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin\delta_1\cos\theta \right)$$

so

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left(\sin^2\delta_0 + 9\sin^2\delta_1\cos^2\theta + 6\sin\delta_0\sin\delta_1\cos\theta \right)$$

$$= \frac{1}{k^2}\left(a_0 + a_2\cos^2\theta + a_1\cos\theta \right)$$

So, measurement of the differential cross section leads to a distribution in θ that can be fit for partial wave components and phase shifts