Lecture 15 Interaction Picture

In the regular approach to perturbation theory above, the expansion of the perturbed states in terms of the unperturbed states was crucial. This was working in the Schrödinger picture which requires the inclusion of the unperturbed, uninteresting, stuff.

There is a simpler picture, the Interaction Representation (or Dirac)

Consider a time-translation operator generated by only H_0 in $H = H_0 + H_I \ldots$

recall $\quad |\hat{\phi}(t)\rangle_H = T^{-1}(t\,t_0)\,|\phi(t)\rangle_S$

then, now define

$$|\tilde{\phi}(t)\rangle_I = R^{-1}(t\,t_0)\,|\phi(t)\rangle_S$$
$$= R^{-1}(t\,t_0)\,T(t\,t_0)\,|\hat{\phi}\rangle_H$$

where:

T generated by H $\qquad T = e^{-iHt}$ $\qquad H_{0_S} = H_{0_H}$

R generated by H_0 $\qquad R = e^{-iH_{0_S}t}$

That means $\qquad i\dfrac{\partial R}{\partial t} = H_{0_S} R$

which is like $\qquad i\dfrac{\partial T}{\partial t} = (H_0 + H_I)\,T$

Substituting $|\phi(t)\rangle_S = R(t\,t_0)\,|\tilde{\phi}(t)\rangle_I$ into Schrödinger equation.

$$i\frac{\partial}{\partial t}|\phi(t)\rangle_s = (H_0 + H_I)_s|\phi(t)\rangle_s$$ get

$$i\frac{\partial}{\partial t}R|\tilde{\phi}(t)\rangle_I = (H_0 + H_I)R|\tilde{\phi}(t)\rangle_I$$

$$i\underbrace{\left(\frac{\partial R}{\partial t}\right)}_{= \cancel{H_{0_s}R}}|\hat{\phi}(t)\rangle_I + iR\frac{\partial}{\partial t}|\hat{\phi}(t)\rangle_I = \cancel{H_{0_s}R|\hat{\phi}(t)\rangle_I} + H_{I_s}R|\hat{\phi}(t)\rangle_I$$

so

$$iR\frac{\partial}{\partial t}|\hat{\phi}(t)\rangle_I = H_I R|\hat{\phi}(t)\rangle_I$$

operate $R^+ \longrightarrow$

$$i\frac{\partial}{\partial t}|\hat{\phi}\rangle_I = R^+(t t_0)\, H_I\, R(t t_0)|\hat{\phi}(t)\rangle_I$$

we can therefore designate any operator as in the Interaction Representation

$$\hat{O}(t) = R^+(t t_0)\, O\, R(t t_0)$$

$$= e^{iH_0(t-t_0)}\, O\, e^{-iH_0(t-t_0)}$$

so,

$$i\frac{\partial}{\partial t}|\hat{\phi}(t)\rangle_I = \hat{H}_I|\hat{\phi}(t)\rangle_I$$ not the Schrödinger equation

and we see that the time evolution of $|\hat{\Phi}\rangle_I$ is driven by $\hat{H}_I$

Meanwhile, since $\tilde{O} = R^{-1} O R$

$$\frac{d\tilde{O}}{dt} = \underbrace{\frac{\partial R^{-1}}{\partial t}}_{iR^{-1}H_0} O R + R^{-1} O \underbrace{\frac{\partial R}{\partial t}}_{-iH_0 R}$$

$$= i\left(R^{-1} H_0 O R - R^{-1} O H_0 R\right)$$

$$= i\left(R^{-1} H_0 R R^{-1} O R - R^{-1} O R R^{-1} H_0 R\right)$$

$$= i\left(\hat{H}_0 \tilde{O} - \tilde{O} \hat{H}_0\right)$$

and

$$\frac{d\hat{O}}{dt} = i\left[\hat{H}_0, \hat{O}\right]$$

<u>operator's</u> time evolution governed by $\hat{H}_0$

This is crucial for field theory $\Rightarrow$ $\psi(x)$, $\phi(x)$, etc can continue to have $e^{i\omega t}$ dependence

Let's consider time <u>evolution</u> within the interaction representation. That is, just as we had

$$|\phi(t)\rangle_s = T(t\,t_0)\,|\phi(t_0)\rangle_s$$

a new time evolution operator for the interaction rep

we now state,

$$|\hat{\phi}(t)\rangle = U(t\,t_0)\,|\hat{\phi}(t_0)\rangle$$

we know U, we know everything!

where

$$\frac{\partial}{\partial t}|\hat{\phi}(t)\rangle = -i\hat{H}_I\,|\hat{\phi}(t)\rangle$$

so,

$$\frac{\partial}{\partial t'} U(t'\,t)\,|\hat{\phi}(t)\rangle = -i\hat{H}_I\, U(t'\,t)\,|\hat{\phi}(t)\rangle$$

or, as an operator equation (switching dummy times again)

$$\frac{\partial}{\partial t} U(t\,t_0) = -i\hat{H}_I\, U(t\,t_0)$$

This can be formally written as an integral-operator equation.

$$U(t\,t_0) = 1 - i\int_{t_0}^{t} dt'\, \tilde{H}_I(t')\, U(t'\,t_0)$$

with the boundary condition $U(t\,t) \equiv \mathbb{1}$.

To solve this, appeal to an iterative approach.

$$U(t\,t_0) = 1 - i\int_{t_0}^{t} dt_1\, \tilde{H}_I(t_1)\, U(t_1\, t_0)$$

$$= 1 - i\int_{t_0}^{t} dt_1\, \tilde{H}_I(t_1)\left[1 - i\int_{t_0}^{t_1} dt_2\, \tilde{H}_I(t_2)\, U(t_2\, t_0)\right]$$

(forget tilde .. all Interaction Rep).-

$$U(t\,t_0) = 1 - i\int_{t_0}^{t} dt_1\, H_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1\, H_I(t_1)\int_{t_0}^{t_1} dt_2\, H_I(t_2)\, U(t_2\, t_0)$$

$$t_0 \le t_1 \le t \qquad t_0 \le t_1 \le t \qquad t_0 \le t_2 \le t_1$$

which can be continued indefinitely.

$$U(t\,t_0) = 1 - i\int_{t_0}^{t} dt_1\, H_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\, H_I(t_1)\, H_I(t_2)$$

$$+ \cdots + (-i)^n\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, H_I(t_1)\, H_I(t_2) \cdots H_I(t_n)$$

Associate

$$U(t,t_0) = U^{(0)}(t,t_0) + U^{(1)}(t,t_0) + \cdots = \sum_{i=0}^{\infty} U^{(i)}(t,t_0)$$

$$U^{(0)} = \mathbb{1}$$

$$U^{(1)}(t,t_0) = -i\int_{t_0}^{t} dt_1\, H_I(t_1)$$

etc.

Remember how time evolution proceeds for $\hat{O}$

$$H_I(t') = e^{iH_0 t} H_I e^{-iH_0 t}$$

what about
$$[H_I(t), H_I(t')] = \left(e^{iH_0 t} H_I e^{-iH_0 t} e^{iH_0 t'} H_I e^{-iH_0 t'} - \cdots\right)$$
$$\neq 0 \quad \text{unless} \quad [H_0, H_I] = 0$$

which may not necessarily happen. So, the Hamiltonians at different times may not commute $\Rightarrow$ the ordering in time has to be preserved: $HH' \neq H'H$

Go back to the iterated sum,

$$1 - \int_{t_0}^{t} \text{——} + \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H(t_1)\, H(t_2)$$

$t_0 < t_1 < t$

$t_0 \leq t_1 \leq t$

$t_0 \leq t_2 \leq t_1$

note the position of the Hamiltonian operators.

later > earlier

earlier

later

$$H_I(t_1)\,H_I(t_2)$$
later earlier

Left to Right. — decreasing time order

the n^{th} order term

$$U^{(n)}(t,t_0) = (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \cdots \int_{t_0}^{t_{n-1}} dt_n \; H_I(t_1)\,H_I(t_2) \cdots H(t_n)$$

$$= (-i)^n \int_{t_0}^{t} dt_1 \int d\vec{x} \int_{t_0}^{t_1} dt_2 \int d\vec{x} \cdots \int_{t_0}^{t_{n-1}} dt_n \int d\vec{x} \; \underbrace{\mathcal{H}_I(t_1)\,\mathcal{H}_I(t_2) \cdots \mathcal{H}_I(t_n)}_{\text{still ordered}}$$

There is a trick to change this situation having a unique time le [illegible] integration limit. — a new ordering operation

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$$P\left[\theta(t_1)\,\theta(t_2) - \theta(t_n)\right] \equiv$$ product of operators arranged L to R in decreasing time order w/out regard to commutation relations

"Chronological Ordering"

For example

L > E

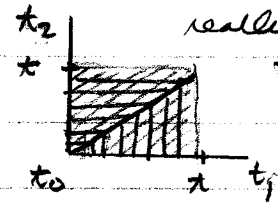
$$P[\theta(x)\,\theta(x')] = \begin{cases} \theta(x)\,\theta(x') & \text{if } t > t' \\ \theta(x')\,\theta(x) & \text{if } t' > t \end{cases}$$

$$P[\theta(x)\,\theta(x')\,\theta(x'')] = \theta(x)\,\theta(x')\,\theta(x'') \quad \text{if } t > t' > t''$$

etc

Look at the order 2 integral, $U^{(2)}$

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1) H_I(t_2) \xrightarrow[\text{whole region}]{}$$



really an integration from t_0 to t

$2^{nd}\int$ says: for given t_1, integrate t_2 from $t_0 \to t_1$

so $t_2 < t_1$

In 1949 Dyson realized that this is the same as

common

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\; \theta(t_1 - t_2)\, H_I(t_1)\, H_I(t_2)$$

$$\theta(t) = +1 \quad \text{for } t>0$$
$$= 0 \quad t<0$$

or
$$\theta(t_1 - t_2) = +1 \quad \text{for } t_1 - t_2 > 0$$
$$t_1 > t_2$$
$$= 0 \quad \text{for } t_1 - t_2 < 0$$
$$t_1 < t_2$$

So he invented an operation — the Chronological Product, which does the following,

$$P[H_I(t_1) \cdots H_I(t_n)] = \text{permutations}\Big[\theta(t_{\alpha_1} - t_{\alpha_2}) \cdots \theta(t_{\alpha_n} - t_{\alpha_{n-1}}) \cdot H_I(t_{\alpha_1}) \cdots H_I(t_{\alpha_n})\Big]^{*}$$

$\rightarrow$ looks at all ways of writing the ordered H_I's and leaves behind as non-zero only the one that has

L $\rightarrow$ R in decreasing time order.

(* we'll change this one more time for statistics later)

$L > E$

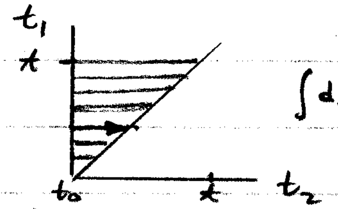
so $\quad P[\theta(x)\theta(x')] = \theta(x)\theta(x')$ if $t > t'$

$\qquad\qquad\qquad\qquad\quad \theta(x')\theta(x)$ if $t' > t$

Consider, then,

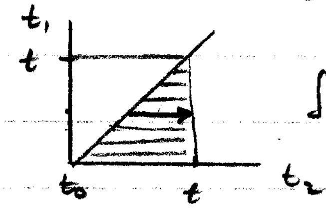
$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \, P[H_I(t_1) H_I(t_2)] =$$

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, H_I(t_1) H_I(t_2)$$



$t_2 < t_1$

$$+ \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 \, H_I(t_2) H_I(t_1)$$



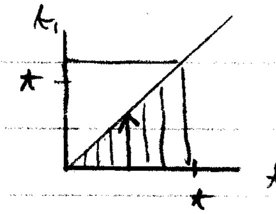
$t_2 < t$

$t_2 > t_1$

Look at 2nd term on RHS...

change the order of integration

$$\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \, H_I(t_2) H_I(t_1)$$



Same area.

$t_1 < t_2$ still

Now change names of labels,

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \, H_I(t_1) H_I(t_2) \qquad t_2 < t_1$$

$\rightarrow$ this is the same as the 1st term on RHS

SO

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, P\left[H_I(t_1) H_I(t_2) \right] = \underbrace{2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1) H_I(t_2)}_{U^{(2)}}$$

$t_1 > t_2$ period.

and so

$$U(t\,t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n\, P\left[H_I(t_1) \cdots H_I(t_n) \right]$$

$$= \sum \frac{(-i)^n}{n!} \int d^4x_1 \int d^4x_2 \cdots \int d^4x_n\, P\left[\mathcal{H}_I(x_1) \mathcal{H}_I(x_2) \cdots \mathcal{H}(x_n) \right]$$

The physical interest is all times: $t_0 \to -\infty$, $t \to \infty$

and the time evolution operator which is introduced to do that is

$$S \equiv \lim_{\substack{t_0 \to -\infty \\ t \to \infty}} U(t\,t_0)$$ called the "S Matrix"

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2 \cdots \int_{-\infty}^{\infty} d^4x_n\, P\left[\mathcal{H}_I(x_1) \cdots \mathcal{H}_I(x_n) \right]$$

The physical significance is the following:

If a system is in state $|\Psi(t_0)\rangle$ at $t = t_0$
the probability that it will evolve to system
$|\Psi(t')\rangle$ at $t = t'$