Lecture 19 Wicks_theorem

We have found it necessary to introduce 2 ordering operations:

1) Normal ordering – to make the vacuum have zero energy
2) Chronological product – to be able to make the S-matrix expansion usable.

I introduced "P" for the Hamiltonian in the interaction representation, but I need to modify it slightly for individual field operators.

Write an arbitrary field operator

including annihilation

$$\alpha(x) \equiv A(x) + C(x) \qquad \beta(x) = A'(x) + C'(x)$$

including creation.

$$\bar{\alpha}(x) \equiv \bar{A}(x) + \bar{C}(x)$$

The T operation, or Wick Time Ordering Operator is defined

$$T[\alpha(x)\beta(y)] \equiv \alpha(x)\beta(y) \qquad x_0 > y_0$$
$$= P_{\alpha\beta}\, \beta(y)\alpha(x) \qquad x_0 < y_0$$

where $P_{\alpha\beta} = -1$ if fermions
$= +1$ if bosons.

This is for pairs. In general, get a factor $(-1)^P$ where $P = \#$ permutations of fermion operators

This has no real practical effect since fermion operators always appear in pairs!

Normal ordered: ($C \to L$, $A \to R$)

① $\alpha(x)\beta(y) = A(x)A'(y) + \underline{A(x)C'(y)} + C(x)A'(y) + C(x)C'(y)$

$:\alpha(x)\beta(y): = A(x)A'(y) + P_{\alpha\beta}\, C'(y)A(x) + C(x)A'(y) + C(x)C'(y)$

(use prime to indicate "from β")

Add and subtract $C'(y)A(x)$

$\cancel{\alpha(x)\beta(y) = A(x)A'(y) + A(x)C'(y) + C(x)A'(y) + C(x)C'(y) + C'(y)A(x) - C'(y)A(x)}$

* if fermions:

$:\alpha(x)\beta(y): = A(x)A'(y) - C'(y)A(x) + C(x)A'(y) + C(x)C'(y)$
$= AA' - C'A + CA' + CC'$

so ① $\alpha(x)\beta(y) = :\alpha(x)\beta(y): + C'(y)A(x) + A(x)C'(y)$

$\underline{AA'} + AC' + \underline{CA'} + \underline{CC'} = \underline{AA'} - C'A + \underline{CA'} + \underline{CC'} \quad (+ C'A + AC')$

$= :\alpha(x)\beta(y): + \{A(x), C'(y)\}$ Ⓕ

* if bosons:

#'s

$\alpha(x)\beta(y) = :\alpha(x)\beta(y): + [A(x), C'(y)]$ Ⓑ

$= AA' + C'A + CA' + CC'$

take Vacuum Expectation Value, VEV. –

$\langle 0|①|0\rangle$ $$\langle 0|\alpha(x)\beta(y)|0\rangle = \cancel{\langle 0|A(x)A'(y)|0\rangle}^{\,0} + \langle 0|A(x)C'(y)|0\rangle + \cancel{\langle 0|C(x)A'(y)|0\rangle}^{\,0} + \cancel{\langle 0|C(x)C'(y)|0\rangle}^{\,0}$$

$$= \langle 0|A(x)C'(y)|0\rangle$$

add zero $= -\langle 0|C'A|0\rangle + \langle 0|C'A|0\rangle$

rewrite for fermions $\quad$ $= 0$ explicitly

$$\langle 0|A(x)C'(y)|0\rangle = -\langle 0|C'(y)A(x)|0\rangle + \langle 0|C'(y)A(x)|0\rangle \ (=0)$$
$$+ \langle 0|A(x)C'(y)|0\rangle$$
$$= 0 + \langle 0|\{A(x), C'(y)\}|0\rangle$$
$$= \{A(x), C'(y)\}$$

or for bosons

$$= 0 + \langle 0|[A(x), C'(y)]|0\rangle$$
$$= [A(x), C'(y)]$$

Then our fermion statement Ⓕ can be written

$$= :\alpha\beta: + \langle 0|\{A, C'\}|0\rangle = :\alpha\beta: + \{A, C'\}$$

$$\alpha(x)\beta(y) = :\alpha(x)\beta(y): + \langle 0|A(x)C'(y)|0\rangle$$

But $\langle 0|:\text{anything}:|0\rangle = 0$, so

so $$\langle 0|\alpha(x)\beta(y)|0\rangle = \{A(x), C'(y)\}$$

and

Ⓑ $$\alpha(x)\beta(y) = :\alpha(x)\beta(y): + \langle 0|A(x)C'(y)|0\rangle$$ same

So $$\langle 0|\alpha(x)\beta(y)|0\rangle = [A(x), C'(y)]$$

Suppose we changed the normal product ordering — the other time relation:

$$:\beta(y)\alpha(x): = A'(y)A(x) + C'(y)A(x) + :A'(y)C(x): + C'(y)C(x)$$
$$= A'(y)A(x) + \underline{C'(y)A(x)} + \underline{P_{\alpha\beta}C(x)A'(y)} + C'(y)C(x)$$

permute \underline{some} back

$$= P_{\alpha\beta}A(x)A'(y) + \underline{C'(y)A(x)} + \underline{P_{\alpha\beta}C(x)A'(y)}$$
$$+ P_{\alpha\beta}C(x)C'(y)$$

(E) $\quad :\beta\alpha: = -AA' + C'A - CA' - CC'$
$\quad = - :\alpha\beta:$

(B) $\quad :\beta\alpha: = AA' + C'A + CA' + CC'$
$\quad = + :\alpha\beta:$

so

$$:\beta\alpha: = P_{\alpha\beta} :\alpha\beta:$$

Using the definition of the T product

BUT

either $\quad T(:\alpha\beta:) = :\alpha\beta: \quad$ OR $\quad = P_{\alpha\beta} :\beta\alpha: = P_{\alpha\beta}P_{\alpha\beta} :\alpha\beta:$

so as I said

$$T(:\alpha\beta:) = :\alpha\beta: \quad \text{-- statistics independent}$$

So,

$$T(\alpha\beta) = T(:\alpha\beta:) + T(\langle 0|\alpha\beta|0\rangle) \quad \text{a number}$$
$$= :\alpha\beta: + T(\langle 0|\alpha\beta|0\rangle)$$

already

Remember.. the whole point of $: \, :$ is that

$$\langle 0 | : \alpha\beta : | 0 \rangle = 0$$

Then

$$\langle 0 | T(\alpha\beta) | 0 \rangle = T(\langle 0 | \alpha\beta | 0 \rangle)$$

$\equiv \underbracket{\alpha\beta}$ called a "contraction"

— not an operator.

so $T(\alpha\beta) = :\alpha\beta: + \underbracket{\alpha\beta}$

Products of 3 can be done .. $\alpha(x)\beta(y)\gamma(z)$

$T(\alpha\beta\gamma)$ specific case x_0 & $y_0 > z_0$

$$T(\alpha\beta\gamma) = T(\alpha\beta)\gamma$$

$$= :\alpha\beta:\gamma + T\langle 0|\alpha\beta|0\rangle\gamma$$

$$= :\alpha\beta:\gamma + \underbracket{\alpha\beta}\,\gamma$$

⋮ lots of algebra.

$$= :\alpha\beta\gamma: + :\underbracket{\alpha\beta}\gamma: + :\underbracket{\alpha\beta\gamma}: + :\alpha\underbracket{\beta\gamma}:$$

all symmetric permutations of the $\alpha\beta\gamma$ combinations — independent of the specific x_0 & $y_0 > z_0$ conditions.

→ true for all time sequencing

Using the definition of the T. product... for any pair of operators,

$T(:\alpha\beta:) \to\ :\alpha\beta:$

$\to P_{\alpha\beta} :\beta\alpha: \ = P_{\alpha\beta} P_{\alpha\beta} :\alpha\beta:$

statistics-independent

$$T(:\alpha\beta:) = \ :\alpha\beta:$$

so, T-ordering own statistics-independent statement... B_1 or B_2

$$T(\alpha\beta) = T(:\alpha\beta:) + T(\langle 0|\alpha\beta|0\rangle) \leftarrow \text{number}$$

$$T(\alpha\beta) = \ :\alpha\beta: + T(\langle 0|\alpha\beta|0\rangle)$$

Remember, the whole point of : : is that

$\langle 0|:\alpha\beta:|0\rangle = 0$, no creation $\langle 0| \to$ $\leftarrow |0\rangle$

taking VEV:

$$\langle 0|T(\alpha\beta)|0\rangle = T\langle 0|\alpha\beta|0\rangle$$

$$\equiv \underbrace{\alpha\beta}$$ "contraction"

it's not an operator.

STOP

3 operators can be done...

$T(\alpha\beta\gamma)$ -- consider special case $\alpha(x)\beta(y)\gamma(z)$

where x_0 and $y_0 > z_0$. $\leftarrow$ specific case

$$T(\alpha\beta\gamma) = T(\alpha\beta)\gamma$$
$$= \ :\alpha\beta:\gamma + T\langle 0|\alpha\beta|0\rangle\gamma$$
$$= \ :\alpha\beta:\gamma + \underbrace{\alpha\beta}\gamma$$

$$T(\alpha\beta\gamma) = \ :\alpha\beta:\gamma + \underbrace{\alpha\beta}:\gamma: = \ \overset{①}{:\alpha\beta:\gamma} + \overset{②}{:\alpha\beta\gamma:}$$

look at this

(1.1) (1.2)

(1) $:\alpha\beta:\gamma = \underline{:\alpha\beta: C''(z)} + \underline{\underline{:\alpha\beta: A''(z)}}$

$\underbrace{}$ already normal ordered $\overset{\text{so}}{=}$ $:\alpha\beta A''(z):$

~~look at this~~

(1.2) $C(x)\,C'(y)\,A''(z) + C(x)\,A'(y)\,A''(z)$
$+ P_{\alpha\beta}\, C'(y)\,A(x)\,A''(z) + A(x)\,A''(y)\,A''(z)$

look at this

(1.1) $\underline{:\alpha\beta: C''(z)} = \left[C(x)\,C'(y) + C(x)\,A'(y) + P_{\alpha\beta}\, C'(y)\,A(x) + A(x)\,A'(y) \right] C''(z)\,)$

1.1.1 1.1.2

$= C(x)\,C'(y)\,C''(z) + C(x)\,A'(y)\,C''(z)$
$+ P_{\alpha\beta}\, C'(y)\,A(x)\,C''(z) + A(x)\,A'(y)\,C''(z)$

1.1.3 1.1.4

look at this

later, look at this

1.1.2

$\underline{C(x)\,A'(y)\,C''(z)} = C(x)\,A'(y)\,C''(z) + C(x)\,C''(z)\,A'(y) - C(x)\,C''(z)\,A'(y)$

$= C(x)\Big(A'(y)\,C''(z) + C''(z)\,A'(y) - C''(z)\,A'(y) \Big)$ 1.1.2*

$= C(x)\Big(\{A'(y), C''(z)\} - C''(z)\,A'(y) \Big)$

OR

$= C(x)\Big([A'(y), C''(y)] + C''(z)\,A'(y) \Big)$

in any case, remember that

$\langle 0|\beta\gamma|0\rangle = [A'(y), C''(z)]$

OR $\langle 0|\beta\gamma|0\rangle = \{A'(y), C''(z)\}$

Remember we constructed a situation which <u>was</u> time-ordered in $\beta\gamma$ ... so,

$$\cancel{\langle 0|\beta\gamma|0\rangle = \langle 0|T(\beta\gamma)|0\rangle = \underset{\sqcup}{\beta\gamma}}$$

$$T(\beta\gamma) = :\beta\gamma: + T\langle 0|\beta\gamma|0\rangle$$
$$\parallel \qquad\qquad\qquad \parallel$$
$$\beta\gamma = :\beta\gamma: + \underset{\sqcup}{\beta\gamma}$$

$$\cancel{\langle 0|\beta\gamma|0\rangle = 0 + \underset{\sqcup}{\beta\gamma}}$$

directly

and

$$\beta\gamma = \cancel{A'A''} + A'C'' + \cancel{C'A''} + \cancel{C'C''} = \cancel{A'A''} + P_{\beta\gamma}C''A' + \cancel{C'A''} + \cancel{C'C''} + \underset{\sqcup}{\beta\gamma}$$

$$A'(y)C''(z) = P_{\beta\gamma}C''(z)A'(y) + \underset{\sqcup}{\beta\gamma}$$

so,

1.1.2
$$C(x)A'(y)C''(z) = C(x)\left(P_{\beta\gamma}C''(z)A'(y) + \underset{\sqcup}{\beta\gamma}\right)$$

The "later" term:

same wrt x-z

1.1.3
$$P_{\alpha\beta}C'(y)A(x)C''(z) = P_{\alpha\beta}C'(y)\left(P_{\alpha\gamma}C''(z)A(x) + \underset{\sqcup}{\alpha\gamma}\right)$$

and last term:

1.1.4
$$A(x)A'(y)C''(z) = A(x)\left(P_{\beta\gamma}C''(z)A'(y) + \underset{\sqcup}{\beta\gamma}\right)$$

do it again $A(x) \longrightarrow$

1.1.4
$$A(x)A'(y)C''(z) = P_{\beta\gamma}\left(P_{\alpha\gamma}C''(z)A(x)A'(y) + \underset{\sqcup}{\alpha\gamma}A'(y)\right) + A(x)\underset{\sqcup}{\beta\gamma}$$

So, we get:

$$T(\alpha\beta\gamma) = \underset{①}{:\alpha\beta:\gamma} + \underset{②}{\underbracket{\alpha\beta}\gamma}$$

$$= \underset{(1.1)}{:\alpha\beta: C''(z)} + C(x)C'(y)A''(z) + C(x)A'(y)A''(z)$$

$$+ \underset{(1.2)}{P_{\alpha\beta}\, C'(y)A(x)A''(z) + A(x)A'(y)A''(z)}$$

$$= C(x)C'(y)C''(z) \quad 1.1.1$$

$$+ P_{\beta\gamma} C(x)C''(z)A'(y) + C(x)\underbracket{\beta\gamma} \quad — \quad (1.1.2)$$

$$+ P_{\alpha\beta}P_{\alpha\gamma}\, C'(y)C''(z)A(x) + P_{\alpha\beta}\, C(y)\underbracket{\alpha\gamma} \quad (1.1.3)$$

$$+ P_{\beta\gamma}P_{\alpha\gamma}\, C''(z)A(x)A'(y) + P_{\beta\gamma}\underbracket{\alpha\gamma}A'(y) + A(x)\underbracket{\beta\gamma} \quad (1.1.4)$$

$$= + C(x)C'(y)A''(z) + C(x)A(y)A(z) + P_{\alpha\beta}\, C(y)A(x)A''(z)$$

$$+ A(x)A'(y)A''(z) + \quad (1.2)$$

$$+ \underbracket{\alpha\beta}\gamma \quad ②$$

— terms are $:\alpha\beta\gamma:$

═ terms are $C(x)\underbracket{\beta\gamma} + A(x)\underbracket{\beta\gamma}$

≡ terms are $P_{\alpha\beta}\, C'(y)\underbracket{\alpha\gamma} + P_{\beta\gamma}\underbracket{\alpha\gamma}A'(y) = :\underbracket{\alpha\beta\gamma}:$

So,

$$T(\alpha\beta\gamma) = :\alpha\beta\gamma: + :\underbracket{\alpha\beta}\gamma: + :\underbracket{\alpha\beta\gamma}: + :\alpha\underbracket{\beta\gamma}:$$

all are symmetric permutations of the $\alpha\beta\gamma$ contractions

This is a general theorem, important for S matrix physics.

Wick's Theorem

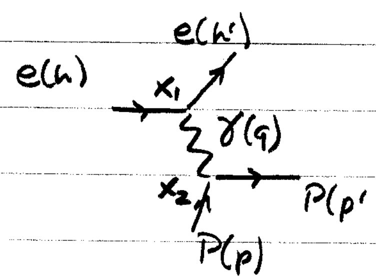
$$T(ABC\cdots XYZ) = \,:ABC\cdots XYZ:$$
$$+ :\underbrace{AB}C\cdots XYZ: + :\underbrace{A}BC\cdots \underbrace{X}YZ: \; + \text{ etc}$$

+ ... all possible pairs + all double combinations + all additional pair combinations.

until everything is contracted

WHY? YOU MIGHT ASK...

Imagine we're considering the process $e(k)P(p) \to e(k')P(p')$



$|i\rangle = |e(k)\,P(p)\rangle$

$|f\rangle = |e(k')\,P(p')\rangle$

$$\mathcal{L}_I(x) = -e\overline{\Psi}(x)\gamma^\mu \Psi(x) A_\mu(x)$$ and from the

S matrix expansion we're obligated to evaluate

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2 \cdots \int_{-\infty}^{\infty} d^4x_n \, P\left[\mathcal{L}_I(x_1)\, \mathcal{L}_I(x_2) \cdots \mathcal{L}_I(x_n) \right]$$

for our problem

$$P\left[\mathcal{L}_I(x_1)\,\mathcal{L}_I(x_2)\right] = \begin{cases} \mathcal{L}_I(x_1)\,\mathcal{L}_I(x_2) & x_{10} > x_{20} \\ \mathcal{L}_I(x_2)\,\mathcal{L}_I(x_1) & x_{20} > x_{10} \end{cases}$$

it's conditional.

One would just do it.. keep track of all time ordering and explicitly calculate (that's what Sakurai does).

However, Wick's Theorem greatly simplifies things as it can very quickly become unreasonably complicated otherwise.

It's a bit of magic.. the P product and the T product are the same for bosons and for pairs of fermion operators.

So, the conditional, complicated P expansion is simplified to a sum of terms.

Here's how it simplifies...

consider $\alpha(x)\beta(y) = \,:\alpha(x)\beta(y): + \langle 0|A(x)C'(y)|0\rangle$

$= \,:\alpha(x)\beta(y): + \{A(x), C'(y)\}$

a general fermion statement

• Suppose $\beta(y) \equiv \alpha(y)$ same field op as $\alpha(x)$, at y instead of x

$\alpha(x)\alpha(y) = \,:\alpha(x)\alpha(y): + \{A(x), C'(y)\}$

$\downarrow$ corresponds to

$\{a, b^{\dagger}\} = 0$ so,

$\alpha(x)\alpha(y) = \,:\alpha(x)\alpha(y):$

so $\langle 0|\alpha(x)\alpha(y)|0\rangle = \langle 0|:\alpha(x)\alpha(y):|0\rangle = 0$ by definition.

then $T\langle 0|\alpha(x)\alpha(y)|0\rangle = \underbrace{\alpha\alpha} = 0$

• Suppose $\beta(y) \equiv \bar{\alpha}(y)$

$\alpha(x)\bar{\alpha}(y) = \,:\alpha(x)\bar{\alpha}(y): + \{A(x), \bar{C}'(y)\}$

$\downarrow$ corresponds to

$\{a, a^{\dagger}\} \neq 0$

so $\underbrace{\alpha\bar{\alpha}} \neq 0$

similarly $\underbrace{\bar{\alpha}\bar{\alpha}} = 0$, $\underbrace{\alpha\phi} = 0$

$\longrightarrow$ this will reduce the number of terms by a considerable amount.

So, inside the Smatrix we had for $S^{(2)}$

$$P\left[\ \bar{\psi}(x_1)\gamma^\mu\psi(x_1)A_\mu(x_1)\,\bar{\psi}(x_2)\gamma^\nu\psi(x_2)A_\nu(x_2)\right]$$

which is $= \quad T\left[\qquad\qquad\qquad\right]$

which, from Wick's Theorem becomes,

$$= \gamma^\mu_{ij}\gamma^\nu_{lm}\,T\left[\ \underbrace{\bar{\psi}_i(x_1)\psi_j(x_1)A_\mu(x_1)\,\bar{\psi}_l(x_2)\psi_m(x_2)A_\nu(x_2)}_{6\text{ operators}}\right]$$

matrix indices, so can move them around.

$$= \gamma^\mu_{ij}\gamma^\nu_{lm}\sum_W \Theta^{(W)}_{\mu\nu}$$

the sum of operator combinations in the Wick expansion — $:\bar{\psi}\psi A\bar{\psi}\psi A:$ + all contractions. >50 terms.

BUT, from above since

$$\underbrace{\psi\psi} = \underbrace{\psi A} = \underbrace{\bar{\psi}A} = 0$$

... 8 terms are left which are non-zero