So,

$$* \quad J^{(2)}(e\gamma \to e\gamma) = \frac{(-ie)^2}{2} \int d^4x_1 \int d^4x_2 \langle e\gamma | \gamma^\mu_{ij} \gamma^\nu_{\ell m} \{ \bar{\psi}_i^-(x_1) \psi_m^+(x_2) A_\mu(x_1) A_\nu(x_2)$$

$$* \quad \times \langle 0 | T[\psi_j(x_1) \bar{\psi}_\ell(x_2)] | 0 \rangle + \bar{\psi}_\ell^-(x_2) \psi_j^+(x_1) A_\mu^+(x_1) A_\nu^-(x_2)$$

$$\times \langle 0 | T[\bar{\psi}_m(x_1) \psi_i(x_2)] | 0 \rangle \} | e\gamma \rangle$$

We need to deal with the contracted terms.

eg, what is $\langle 0 | T[\psi_j(x_1) \bar{\psi}_\ell(x_2)] | 0 \rangle$

We know it's

$$T[\psi_j(x_1) \bar{\psi}_\ell(x_2)] = \psi_j(x_1) \bar{\psi}_\ell(x_2)\, \theta(x_{10} - x_{20}) - \bar{\psi}_\ell(x_2) \psi_j(x_1)\, \theta(x_{20} - x_{10})$$

But, let's see it happen naturally. —

look at 1st term $x_{10} > x_{20}$

$$\langle 0 | T(\psi_j(x_1) \bar{\psi}_\ell(x_2)) | 0 \rangle = \langle 0 | T(\underbrace{\psi_j^+ \bar{\psi}_\ell^+ + \psi_j^+ \bar{\psi}_\ell^- + \psi_j^- \bar{\psi}_\ell^+ + \psi_j^- \bar{\psi}_\ell^-}_{\text{already time ordered}}) | 0 \rangle$$

$$\propto \langle 0 | \underset{0}{\underset{\parallel}{ab}} + aa^+ + \underset{0}{\underset{\parallel}{b^+b}} + \underset{0}{\underset{\parallel}{b^+a^+}} | 0 \rangle$$

$$= \langle 0 | \psi_j^+(x_1) \bar{\psi}_\ell^-(x_2) | 0 \rangle \qquad x_{10} > x_{20}$$

$x_{20} > x_{10}$

$$\langle 0 | T(\psi_j(x_1)\bar{\psi}_\ell(x_2))|0\rangle = \langle 0|T(\psi_j^+\bar{\psi}_\ell^+ + \psi_j^+\bar{\psi}_\ell^- + \psi_j^-\bar{\psi}_\ell^+ + \psi_j^-\bar{\psi}_\ell^-)|0\rangle$$

$$= -\langle 0| \bar{\psi}_\ell^+\psi_j^+ + \bar{\psi}_\ell^-\psi_j^+ + \bar{\psi}_\ell^+\psi_j^- + \bar{\psi}_\ell^-\psi_j^- |0\rangle$$

$$\propto \langle 0| \underset{\substack{\parallel\\0}}{ba} + \underset{\substack{\parallel\\0}}{a^+a} + \underset{\substack{\neq\\0}}{bb^+} + \underset{\substack{\parallel\\0}}{a^+b^+} |0\rangle$$

$$= -\langle 0| \bar{\psi}_\ell^+(x_2)\psi_j^-(\underline{x_1})|0\rangle \qquad x_{20} > x_{10}$$

Look explicitly at the first one $\langle 0|\psi_j^+(x_1)\bar{\psi}_\ell^-(x_2)|0\rangle$

$$\psi_j^{(+)}(x_1) = \sum_{i=1}^{2}\int dK\; a^{(i)}(k)u_j^{(i)}(k)\,e^{-ik\cdot x_1}$$

$$\bar{\psi}_\ell^-(x_2) = \sum_{m=1}^{2}\int dK'\; \bar{u}_\ell^{(m)}(k')\,a^{+(m)}(k')\,e^{ik'\cdot x_2}$$

$$\langle 0|\psi_j^+(x_1)\bar{\psi}_\ell^-(x_2)|0\rangle = \sum_i\sum_m \int\frac{d^3k}{(2\pi)^3 2E}\int\frac{d^3k'}{(2\pi)^3 2E'}\; u_j^{(i)}(k)\bar{u}_\ell^{(m)}(k')$$
$$\times\, e^{-i(k\cdot x_1 - k'\cdot x_2)}\langle 0|a^{(i)}(k)a^{+(m)}(k')|0\rangle$$

$\downarrow$ use anticommutator by adding

$\langle 0|a^+a|0\rangle \quad (=0)$

$(2\pi)^3 2E\,\delta(\vec{k}-\vec{k}')\,\delta_{im}$

do $\int d^3k'$ integration

$$= \sum_i \int dK \quad u_j^{(i)}(k)\, \bar{u}_\ell^{(i)}(k)\, e^{-ik\cdot(x_1-x_2)}$$

use completeness .. to project out the + energy particles

$$= \int dK \ (\not{k}+m)_{j\ell}\, e^{-ik\cdot(x_1-x_2)} \qquad x_{10} > x_{20}$$

The second one

$$\langle 0|\, \bar{\Psi}_\ell^{+}(x_2)\, \Psi_j^{-}(x_1)\,|0\rangle = \sum_i \sum_m \int dK \int dK' \ \bar{v}_\ell^{(i)}(k')\, v_j^{(m)}(k)\, e^{-i(k\cdot x_2 - k'\cdot x_1)}$$

$$\times \langle 0|\, b^{(i)}(k)\, b^{+(m)}(k')\,|0\rangle$$

since $\displaystyle \Psi_j^{-}(x_1) = \sum_{i=1}^{2} \int dK'\ b^{+(i)}(k')\, v_j(k')\, e^{ik'\cdot x_1}$

$$\bar{\Psi}_\ell^{+}(x_2) = \sum_{m=1}^{2} \int dK \ \bar{v}_\ell^{(m)}(k)\, b^{(m)}(k)\, e^{-ik\cdot x_2}$$

In the same fashion

$$\langle 0|\, \bar{\Psi}_\ell^{+}(x_2)\, \Psi_j^{-}(x_1)\,|0\rangle = \sum_i \int dK \ v_j^{(i)}(k)\, \bar{v}_\ell^{(i)}(k)\, e^{ik\cdot(x_1-x_2)}$$

$$= \int dK \ (\not{k}-m)_{j\ell}\, e^{ik\cdot(x_1-x_2)}$$

So, going all the way back .. remembering the negative sign,

$\langle 0|T(\psi_j(x_1)\bar{\psi}_\ell(x_2))|0\rangle$

$$= \int dK\, (\not{k}+m)_{j\ell}\, e^{-ik\cdot(x_1-x_2)}\, \theta(x_{10}-x_{20})$$

$$- \int dK\, (\not{k}-m)_{j\ell}\, e^{ik\cdot(x_1-x_2)}\, \theta(x_{20}-x_{10})$$

Look at the second term...

$$-\int dK\, (\gamma^0 k_0 - \vec{\gamma}\cdot\vec{k} - m)_{j\ell}\, e^{ik\cdot(x_1-x_2)}$$

note $\quad ik\cdot(x_1-x_2) = iE(x_{10}-x_{20}) - \vec{k}\cdot(\vec{x}_1-\vec{x}_2)$

$$= (-i)(-E)(x_{10}-x_{20}) - \vec{k}\cdot(\vec{x}_1-\vec{x}_2)$$

change $\quad \vec{k} \to -\vec{k} \equiv \vec{p}$

$$-\int dP\, [(-\gamma^0)(-E) + \vec{\gamma}\cdot\vec{p} - m]_{j\ell}\, e^{-i(-E)(x_{10}-x_{20})}\, e^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)}$$

if $\quad p^\mu = (-E, \vec{p}) \quad$ then

$$\int dP[\underbrace{\gamma^0 p_0 - \vec{\gamma}\cdot\vec{p} + m}_{(\not{p}+m)}]_{j\ell}\, \underbrace{e^{-ip^0\cdot(x_{10}-x_{20})}\, e^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)}}_{e^{-ip\cdot(x_1-x_2)}}$$

Of course p^μ is a dummy, so can call it k^μ with understanding that in 2nd term $k^0 = -E$

$$\langle 0|T(\psi_j(x_1)\bar{\psi}_\ell(x_2))|0\rangle = \int dK\, (\not{k}+m)_{j\ell}\, e^{-ik\cdot(x_1-x_2)}$$

for $\quad x_{10} > x_{20}$ then $k^0 = E$; for $x_{20} > x_{10}$ then $k^0 = -E$

where we are...

I showed the non-zero terms in the Wick expansion for $\bar{\Psi}(x)\gamma^\mu\Psi(x)A_\mu(x)$ to second order to include a variety of stand-alone, familiar processes (Compton scattering, Möller Scattering, and Bhabha scattering) as well as some less familiar, "add-on" processes (...
(electron/positron self energy, photon vacuum polarization) and some non-interesting, unmeasurable processes (un-scattering & vacuum fluctuation).

We are working specifically on $e\gamma \to e\gamma$

$$* \quad J^{(2)}(e\gamma\to e\gamma) = \frac{(-ie)^2}{2}\int d^4x_1 \int d^4x_2 \, \langle e\gamma | \gamma^\mu_{ij}\gamma^\nu_{lm}$$

$$\{ \bar{\Psi}_i(x_1)\Psi_m^+(x_2) A_\mu(x_1) A_\nu(x_2) \langle 0|T[\Psi_j(x_1)\bar{\Psi}_l(x_2)]|0\rangle$$

$$* \quad - \bar{\Psi}_l(x_2)\Psi_j^+(x_1) A_\mu(x_1) A_\nu(x_2) \langle 0|T[\bar{\Psi}_i(x_1)\Psi_m(x_2)]|0\rangle \} |e\gamma\rangle$$

we're concentrating on the contracted term, which I showed to be

$$\langle 0|T(\bar{\Psi}(x_1)\Psi(x_2))|0\rangle = \int dK\,(\not{k}+m)\,e^{-ik\cdot(x_1-x_2)}$$

with $\quad k^0 = E \quad x_{10} > x_{20}$

$\quad\quad\;\; k^0 = -E \quad x_{20} > x_{10}$

Look at the special integral

$$I = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d^4k \left(\frac{i}{\not{k} - m} \right)_{j\ell} e^{-ik(x_1 - x_2)}$$

no constraint on k^μ

multiply by $\frac{\not{k}+m}{\not{k}+m}$

denominator –

$$(\not{k}+m)(\not{k}-m) = \gamma^\nu k_\nu \gamma^\mu k_\mu - \gamma^\nu k_\nu m + m\gamma^\mu k_\mu - m^2$$
$$= \gamma^\nu k_\nu \gamma^\mu k_\mu - m^2$$

note $$A_\nu B_\mu \gamma^\mu \gamma^\nu = B_\mu A_\nu \left(-\gamma^\mu \gamma^\nu + 2g^{\mu\nu} \right)$$
$$= -B_\mu A_\nu \gamma^\mu \gamma^\nu + 2A\cdot B$$

or

$$\not{A}\not{B} = -\not{B}\not{A} + 2A\cdot B$$
$$\{\not{A}, \not{B}\} = 2A\cdot B$$

so

$$\{\not{k}, \not{k}\} = 2k\cdot k$$
$$\not{k}\not{k} = k\cdot k = k^2$$

and denominator becomes.

$$k^2 - m^2 = k_0 k_0 - \vec{k}\cdot\vec{k} - m^2$$
$$= \left(k_0 - \sqrt{\vec{k}\cdot\vec{k} + m^2}\right)\left(k_0 + \sqrt{\vec{k}\cdot\vec{k} + m^2}\right)$$

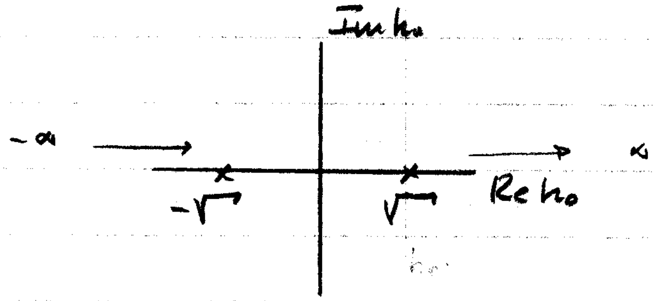
and I get

$$I = \frac{i}{(2\pi)^4} \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} dk^0 \left[\frac{(\not{k}+m)\, e^{-ik\cdot(x_1-x_2)}}{\left(k_0-\sqrt{\vec{k}\cdot\vec{k}+m^2}\right)\left(k_0+\sqrt{\vec{k}\cdot\vec{k}+m^2}\right)} \right]$$

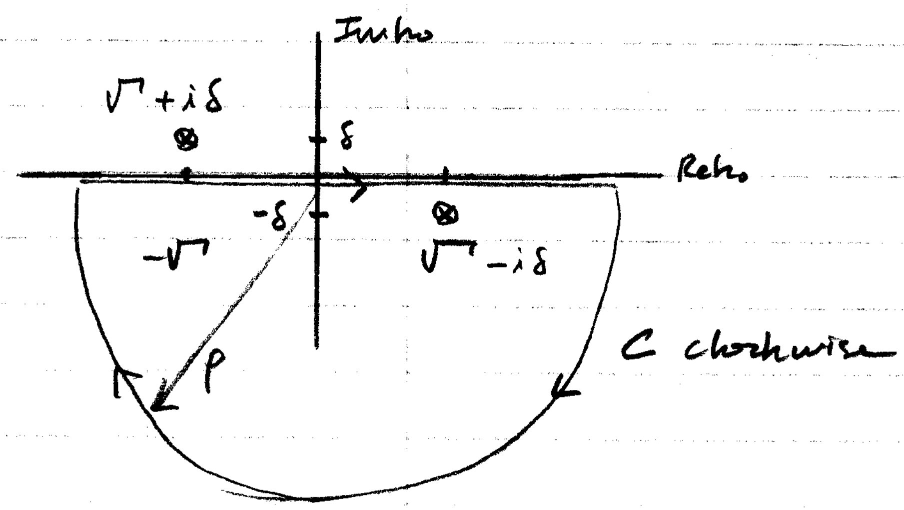
which has poles at $k_0 = \pm\sqrt{\vec{k}\cdot\vec{k}+m^2}$

So, $$I_0 = \int_{-\infty}^{\infty} dk_0 \, \frac{(\not{k}+m)\, e^{-ik^0(x_{10}-x_{20})}}{\left(k_0-\sqrt{}\right)\left(k_0+\sqrt{}\right)}$$

Map this onto the complex k_0 plane.



avoid the poles in the standard way by shifting them and use Cauchy's Theorem



As $\rho \to$ large and $h^0 = -i|h^0|$, the exponential becomes,

$$e^{-ih^0(x_{10}-x_{20})} \longrightarrow e^{-i(-i|h^0|)(x_{10}-x_{20})}$$

$$= e^{-|h^0|\underbrace{(x_{10}-x_{20})}_{>0}}$$

so as $\rho \to \infty$ $|h^0| \to \infty$ and exponential damps.

Then,

$$\oint_C dh_0 (\quad) = -2\pi i\,(\Sigma \overset{R}{\text{residues of enclosed poles}})$$

$$= \int_{-\infty}^{\infty} dh_0 \,(\text{real, straight line})$$

$$+ \int_{-\infty}^{\infty} \cancel{(\text{curve})} \to 0$$

$$-2\pi i R = \int_{-\infty}^{\infty} (\,) dh_0 = I_0$$

The pole is simple, so the residue is

$$\frac{(\not{k}+m)\, e^{-ih^0(x_{10}-x_{20})}}{\cancel{(h^0-\sqrt{}+i\delta)}(h^0+\sqrt{}-i\delta)} \cancel{(h^0-k_p^0)} \Bigg|_{h^0=k_p=\sqrt{}-i\delta}$$

$$I_0 = -2\pi i \frac{(\not{K}_p + m)\, e^{-i[\sqrt{\ } - i\delta](x_{10}-x_{20})}}{(\sqrt{\ } - i\delta + \sqrt{\ } - i\delta)}$$

$\delta \to 0$

$$I_0 = -2\pi i \frac{(\not{K} + m)\, e^{-i\sqrt{\vec{k}\cdot\vec{k}+m^2}\,(x_{10}-x_{20})}}{2\sqrt{\vec{k}\cdot\vec{k}+m^2}}$$

and

$$I = \frac{1}{\cancel{(2\pi)^4}\ (2\pi)^3} \cancel{(-2\pi i)}\, \cancel{i} \int d^3k \frac{(\not{k}+m)\, e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{2\sqrt{\vec{k}\cdot\vec{k}+m^2}}\, e^{-i\sqrt{\vec{k}\cdot\vec{k}+m^2}\,(x_{10}-x_{20})}$$

$$= \int \frac{d^3k}{(2\pi)^3 2E} (\not{k}+m)\, e^{-ik\cdot(x_1-x_2)} \qquad E = +\sqrt{\vec{k}\cdot\vec{k}+m^2}$$

$$= \int dK\, (\not{k}+m)\, e^{-ik\cdot(x_1-x_2)}$$

which is what we had for

$$\langle 0 | T(\psi(x_1)\bar{\psi}(x_2)) | 0 \rangle \qquad \text{for } x_{10} > x_{20}$$

Note if we had had <u>scalar</u> fields, one can go all the way back to the original T product and would find

$$\int dP\, e^{-ip\cdot(x_1-x_2)} = \langle 0 | T(\phi(x_1)\phi^\dagger(x_2)) | 0 \rangle \equiv i\Delta^{(+)}(x_1-x_2)$$

$x_{10} > x_{20}$

called the Positive Frequency propagator — one of a family

Likewise, $\langle 0 | T(\varphi^+(x_2)\,\phi(x_1))|0\rangle \equiv -i\Delta^{(-)}(x_1-x_2)$

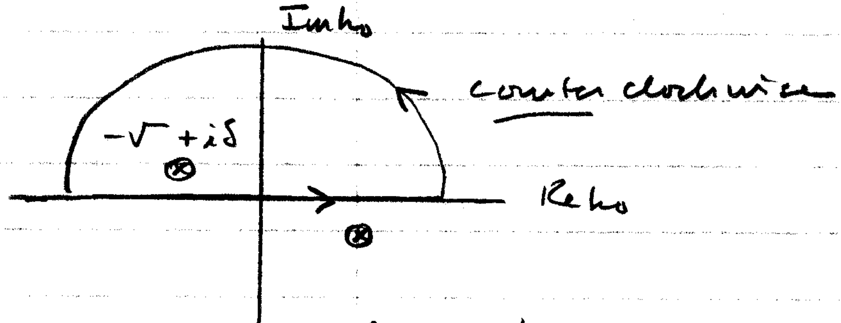
$$= \int dP\, e^{+ip\cdot(x_1-x_2)} \quad \text{for} \quad x_{20} > x_{10}$$

In general the Feynman scalar Propagator is

$$i\Delta_F(x_1-x_2) = i\Delta^{(+)}(x_1-x_2)\,\theta(x_{10}-x_{20}) - i\Delta^{(-)}(x_1-x_2)\,\theta(x_{20}-x_{10})$$

We've just seen that the Dirac propagator for $x_{20} > x_{20}$ looks like that of the scalar, but for the projection operator (as I hinted at before)

Back to Dirac fields.. what about $x_{20} > x_{10}$?



so $\quad I_0 = +2\pi i\,(\sum R)\Big|_{k^0 = -\sqrt{\ } + i\delta}$

so $\quad I = (2\pi i)\dfrac{i}{(2\pi)^4}\displaystyle\int \dfrac{d^3k}{2(-\sqrt{\vec{k}\cdot\vec{k}+m^2})}\, e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\,(\not{k}+m)\,e^{-i(-\sqrt{\vec{k}\cdot\vec{k}+m^2})(x_{10}-x_{20})}$

Call $\quad k_0 = -\sqrt{\vec{k}\cdot\vec{k}+m^2} \equiv -E$

and $\quad I = -\displaystyle\int \dfrac{d^3k}{(2\pi)^3(-2E)}\,(\not{k}+m)\,e^{-ik\cdot(x_1-x_2)}$

which is what we need for $\langle 0|T(\psi(x_1)\bar{\psi}(x_2))|0\rangle$
when $x_{20} > x_{10}$

So, we have a single way of writing $\psi(x_1)\bar{\psi}(x_2)$ (contracted)

regardless of the time ordering.

$\langle 0|T(\psi_j(x_1)\bar{\psi}_\ell(x_2))|0\rangle$

$$= \frac{i}{(2\pi)^4}\int \frac{d^4k\ (\not{k}+m)_{j\ell}\, e^{-ik\cdot(x_1-x_2)}}{\left[k_0-\sqrt{\vec{k}\cdot\vec{k}+m^2}+i\delta\right]\left[k_0+\sqrt{\vec{k}\cdot\vec{k}+m^2}-i\delta\right]}$$

$$\text{denominator} = k_0^2-(\vec{k}\cdot\vec{k}+m^2)+\underbrace{2i\delta\sqrt{\vec{k}\cdot\vec{k}+m^2}}_{\text{call this } i\eta}+\cancel{\delta^2}$$

$$= k\cdot k+m^2+i\eta$$

and

$$\langle 0|T(\psi_j(x_1)\bar{\psi}_\ell(x_2))|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{(\not{k}+m)_{j\ell}}{(k^2+m^2+i\eta)}e^{-ik\cdot(x_1-x_2)}$$

usually written

$$= \int \frac{d^4k}{(2\pi)^4}\left(\frac{i}{\not{k}-m}\right)e^{-ik\cdot(x_1-x_2)}$$

$$\equiv iS_F(x_1-x_2)$$

the Feynman fermion propagator

Notice

$$S_{F_{jl}}(x_1-x_2) = (i\not\partial+m)_{jl}\,\Delta^{(+)}(x_1-x_2)\,\theta(x_{10}-x_{20})$$

$$- (i\not\partial+m)_{jl}\,\Delta^{(-)}(x_1-x_2)\,\theta(x_{20}-x_{10})$$

Remember what these are...

look at $\quad S_F(x_1-x_2) = \frac{-i}{(2\pi)^4}\int d^4k \,\frac{(\not k + m)\, e^{-ik\cdot(x_1-x_2)}}{\not k - m}$

operate with the Dirac Equation operator.

$$(i\not\partial_1 - m)\,S_F(x_1-x_2) = \frac{-i}{(2\pi)^4}\cdot i\int d^4k\,\frac{(i(-i\not k)-m)\,e^{-ik\cdot(x_1-x_2)}}{\not k - m}$$

$$= \frac{1}{(2\pi)^4}\int d^4k\,\frac{\not k - m}{\not k - m}\,e^{-ik\cdot(x_1-x_2)}$$

$$= \frac{1}{(2\pi)^4}\int d^4k\, e^{-ik\cdot(x_1-x_2)}$$

$$= \delta(x_1-x_2)$$

proving that $S_F(x_1-x_2)$ is the Green's Function for the Dirac Equation.

We will need the same thing for the photon. Again

$\langle 0 | T(A_\mu(x_1) A_\nu(x_2)) | 0 \rangle =$

$\langle 0 | A^+_\mu(x_1) A^-_\nu(x_2) | 0 \rangle \theta(x_{10} - x_{20}) + \langle 0 | A^-_\nu(x_2) A^+_\mu(x_1) | 0 \rangle \theta(x_{20} - x_{10})$

proceed the same way.

first, for $x_{10} > x_{20}$

$$\langle 0 | T(\quad) | 0 \rangle = \int dK \int dK' \sum_{\lambda=0}^{3} \sum_{\lambda'=0}^{3} \epsilon_{\mu(\lambda)}(k) \epsilon_{\nu(\lambda')}(k') \cdot$$
$$e^{-ik\cdot x_1} e^{ik'\cdot x_2} \langle 0 | a_{(\lambda)}(k) a^+_{(\lambda')}(k') | 0 \rangle$$

Remember, for photons the commutation relations are a little odd

$$[a_{(\lambda)}(k), a^+_{(\lambda')}(k')] = -g_{\lambda\lambda'} 2k_0 (2\pi)^3 \delta(\vec{k} - \vec{k}')$$

$$iD_{F\mu\nu}(x_1 - x_2) = \int dK \sum_{\lambda=0}^{3} (-g_{\lambda\lambda'}) \epsilon_{\mu(\lambda)}(k) \epsilon_{\nu(\lambda)}(k) e^{-ik\cdot(x_1 - x_2)}$$

Remember the completeness relation

$$\sum_{\lambda=0}^{3} g_{\lambda\lambda} \epsilon_{\mu(\lambda)}(k) \epsilon_{\nu(\lambda)}(k) = g_{\mu\nu} \qquad (\text{I wrote } \textstyle\sum \rho_\lambda \epsilon_\lambda \epsilon_\lambda = -g_{\mu\nu}$$
same.)

With the same logic as before, we get a similar expansion for $x_{20} > x_{10}$.

For the photon, we consider an integral like

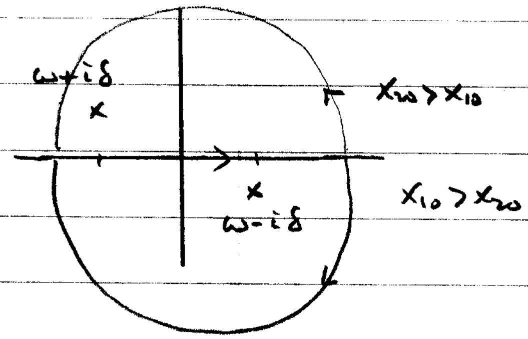
$$\frac{i}{(2\pi)^4} \int \frac{d^4k}{k^\mu k_\mu} e^{-ik\cdot(x_1 - x_2)}$$

$$k^2 = k_0^2 - \vec{k}\cdot\vec{k} = (k_0 - \sqrt{\vec{k}\cdot\vec{k}})(k_0 + \sqrt{\vec{k}\cdot\vec{k}})$$

which leads to $$I_0 = \int_{-\infty}^{\infty} dk_0 \, \frac{e^{-ik_0(x_{10}-x_{20})}}{(k_0+\omega)(k_0-\omega)}$$

where $\omega \equiv \sqrt{\vec{k}\cdot\vec{k}}$

Again,



and we get

$$D_{F\mu\nu}(x_1-x_2) = \int \frac{d^4k}{(2\pi)^4} \, e^{-ik\cdot(x_1-x_2)} \, D_{F\mu\nu}(k)$$

where $$D_{F\mu\nu}(k) = \frac{-g_{\mu\nu}}{k^2+i\epsilon}$$ like before

This is the momentum space Feynman Propagator.

Note, this is in a particular gauge – The Feynman Gauge. We can generalize as before with the gauge fixing parameter, in which we would have

$$D_{F\mu\nu}(k) = \frac{-g_{\mu\nu}}{k^2+i\epsilon} + \frac{\xi-1}{\xi} \frac{k_\mu k_\nu}{(k^2+i\epsilon)^2}$$

Feynman $\xi = 1$

Landau $\xi = \infty$ (transverse in 4d, ie $k^\mu D_{F\mu\nu} = 0$)

This is all covariant -- we could have done it for only the physical, transverse states -- that's often done. However, it is instructive to break down the components of the polarization sum.

Define the unit vector:

$$\eta_\mu \equiv (1, \vec{0}) \qquad \text{timelike and } \perp \text{ to } \epsilon^\mu_{(1,2)}$$

and define another unit vector:

$$\pi^\mu \equiv \frac{k^\mu - (k\cdot\eta)\eta^\mu}{\sqrt{(k\cdot\eta)^2 - k^2}} \qquad \text{spacelike and orthogonal: } \pi\cdot\epsilon = 0, \quad \pi\cdot\eta = 0$$

So, this provides a basis $(\eta_\mu, \pi^\mu, \epsilon^\mu_{(1,2)})$ for which we can write a completeness relation

$$\eta^\mu\eta^\nu - \sum_{i=1}^{2}\epsilon^\mu_{(i)}\epsilon^\nu_{(i)} - \pi^\mu\pi^\nu = g^{\mu\nu}$$

$$\vdots$$

so,

$$iD_F^{\mu\nu}(k) = \frac{\sum_{i=1}^{2}\epsilon^\mu_{(i)}\epsilon^\nu_{(i)}}{k^2+i\epsilon} \qquad \text{transverse piece.}$$

$$+ \cancel{\frac{1}{k^2+i\epsilon}}\;\frac{\cancel{k^2}\eta^\mu\eta^\nu}{\sqrt{(k\cdot\eta)^2-k^2}} \qquad \text{Coulomb piece}$$

$$+ \frac{1}{k^2+i\epsilon}\;\frac{k^\mu k^\nu - (k^\mu\eta^\nu + k^\nu\eta^\mu)(k\cdot\eta)}{\sqrt{(k\cdot\eta)^2-k^2}} \qquad \text{"residual piece"}$$

Notice, the Coulomb piece is a combination of the scalar and longitudinal pieces -- then in the frame in which $\eta = (1,0,0,0)$

$$D_F^{\mu\nu}(k)_{\text{Coulomb}} = \frac{g^{\mu 0} g^{\nu 0}}{|\vec{k}|^2}$$

· couples only to $\mu,\nu = 0$ photon "leg"

→ the static Coulomb potential emerges and is, as advertised, a mixture of the two separately unphysical pieces of the polarization. → our $\sum_{\lambda=0}^{3}$ sum includes the static potential.

The "residual piece is not observable due to current conservation. That is, in momentum space

$$k^\mu j_\mu(k) = 0$$

so any coupling of the $D_F^{\mu\nu}$ propagator with a conserved EM current leads to the k^μ parts of $D^{\mu\nu}_{F\,\text{residual}}$ to give no real contribution.