The cross section at this point is still a function of the photon polarizations, → want $d\sigma/d\Omega'(\gamma')$

$$d\sigma(\lambda,\lambda') = \frac{1}{\underset{\underset{|0-\vec{\beta}|=1}{\shortparallel}}{|\vec{v}_e-\vec{v}_\gamma|}} (2\pi)^4\delta^4(p+k-p'-k') \frac{\overline{\sum_i}\sum_f |T|^2}{\underset{\underset{2m}{\shortparallel}}{2E\,2\omega}} \frac{d^3p'}{(2\pi)^3 2E'} \frac{d^3k'}{(2\pi)^3 2\omega'}$$

$$= \frac{(2\pi)^4\delta(p+k-p'-k')}{16mE'\omega\omega'(2\pi)^6} \overline{\sum_i}\sum_f |T|^2 d^3p'\, d^3k'$$

We're calculating in terms of outgoing photon, so integrate outgoing electron away.

$$\delta^4(p+k-p'-k') = \underbrace{\delta^3(\overset{0}{\cancel{\vec{p}}}+\vec{k}-\vec{p}\,'-\vec{k}')}_{\text{integrate } d^3p'}\delta(m+\omega-E'-\omega')$$

which enforces $\vec{p}\,' = \vec{k}' - \vec{k}$

$$d\sigma(\lambda\lambda') = \frac{\delta(m+\omega-E'-\omega')\overline{\sum_i}\sum_f |T|^2 \omega'^2 d\omega' d\Omega'}{16mE'\omega\omega'(2\pi)^2}$$

$$\frac{d\sigma(\lambda\lambda')}{d\Omega} = \int \frac{1}{16mE'}\left(\frac{\omega'}{\omega}\right)\frac{1}{2\pi}\overline{\sum_i}\sum_f |T|^2 \delta(m+\omega-E'-\omega')d\omega'$$

Kinematics thinking required to unravel the correlations in the δ function. $\delta(m+\omega - E' - \omega')$

$\omega' \leftarrow f(\omega)$

$E' \leftarrow f(\vec{p}') \to f(\omega')$

$\to$ treat as a function of ω' //

$p^{\mu\prime} = p^\mu + k^\mu - k'^\mu$

$p'^2 = p^2 + k^2 + k'^2 + 2p\cdot k - 2p\cdot k' - 2k\cdot k'$

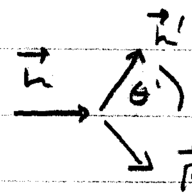
$m^2 = m^2 + 0 + 0 + 2m\omega - 2m\omega' - 2k\cdot k'$

$2m\omega' = 2m\omega - 2k\cdot k'$ ✓

so

$$k\cdot k' = m(\omega - \omega') \qquad ①$$

but



$\vec{p}' = \vec{k} - \vec{k}'$ from δ functions ✓

$$k\cdot k' = \omega\omega' - \vec{k}\cdot\vec{k}'$$
$$= \omega\omega'(1-\cos\theta')$$

from ① $\quad m(\omega-\omega') = \omega\omega'(1-\cos\theta')$

$$\omega' = \frac{m\omega}{\omega - \omega\cos\theta' + m} \qquad \star$$

work on p':

$E'^2 = \vec{p}'\cdot\vec{p}' + m^2 \qquad$ but $\vec{p}' = \vec{k} - \vec{k}'$

$\vec{p}'\cdot\vec{p}' = \vec{k}\cdot\vec{k} + \vec{k}'\cdot\vec{k}' - 2\vec{k}\cdot\vec{k}'$

$= \omega^2 + \omega'^2 - 2\omega\omega'\cos\theta'$

$E'^2 = \omega^2 + \omega'^2 - 2\omega\omega'\cos\theta' + m^2 \quad = \underline{\underline{f(\omega')}}$

so, $\quad \delta[f(\omega')]$

$$f(\omega') = \omega' + E' - \omega - m$$

the root value is $\omega'_r \equiv \omega + m - E'$ energy conservation

$$\frac{df}{d\omega'} = 1 + \underset{\overset{\shortparallel}{\frac{1}{2}(2\omega' - 2\omega\cos\theta')\frac{1}{E'}}}{\frac{dE'}{d\omega'}}$$

so

$$\frac{df}{d\omega'} = 1 + \frac{1}{2}\frac{(2\omega' - 2\omega\cos\theta')}{E'(\omega')}$$

$$= 1 + \frac{\omega' - \omega\cos\theta'}{E'(\omega')}$$

$$= \frac{E'(\omega') - \omega' - \omega\cos\theta'}{E'(\omega')}$$

$$\boxed{\delta[f(x)] = \left|\frac{\partial f}{\partial x}\right|^{-1}_{x=x_r} \delta(x - x_r) \quad \text{where } f(x_r) = 0}$$

$$\left|df/d\omega'\right|_{\omega'=\omega'_r} = \left.\frac{\cancel{E'} + \omega + m - \cancel{E'} - \omega\cos\theta'}{E'}\right|_{\omega'=\omega'_r}$$

so,

$$\delta[f(\omega')] = \frac{\delta(\omega' - \omega'_r)\,E'(\omega'_r)}{m + \omega - \omega\cos\theta'}$$

and

$$\frac{d\sigma}{d\Omega'} = \frac{1}{64\pi^2}\frac{1}{m\omega}\int d\omega'\,\frac{\omega'}{\cancel{E'}}\,\bar{\Sigma}\Sigma|T|^2\left.\frac{\delta(\omega - \omega_r)\cancel{E'}}{m + \omega - \omega\cos\theta'}\right|_{\omega'=\omega'_r}$$

remember from energy conservation we found

$$\omega' = \frac{m\omega}{\omega - \omega\cos\theta' + m} = \omega'_r \quad \star$$

aside:

for $\quad \omega = \frac{2\pi}{\lambda} \quad \Rightarrow \quad \frac{2\pi}{\lambda'} = \omega' = \frac{m\omega}{\omega - \omega\cos\theta' + m}$

$$\frac{2\pi}{\lambda'} = \frac{2\pi/\lambda \, m}{2\pi/\lambda - 2\pi/\lambda \cos\theta' + m}$$

$$\frac{\lambda'}{2\pi} = \frac{2\pi/\lambda - 2\pi/\lambda \cos\theta' + m}{2\pi/\lambda \, m} = \frac{1 - \cos\theta' + m\lambda/2\pi}{m}$$

$$\lambda' = \lambda + \frac{2\pi}{m}(1 - \cos\theta')$$

which is comptons formula.

so, we can write $\quad \omega + m - \omega\cos\theta' = m\omega/\omega'_r$

$$\frac{d\sigma}{d\Omega'} = \frac{1}{64\pi^2}\frac{1}{m\omega}\int d\omega'\, \underline{\omega'}\, \delta(\omega'-\omega'_r)\, \frac{\omega'_r}{m\omega}\, \bar{\Sigma}\Sigma|T|^2$$

$$= \frac{1}{64\pi^2}\frac{\omega_r'^2}{m^2\omega^2}\, \bar{\Sigma}\Sigma|T|^2\Big|_{\omega'=\omega'_r}$$

maybe

actually writing the matrix element,

$$\frac{d\sigma(\lambda\lambda')}{d\Omega'} = \frac{e^4\omega_r'^2}{64\pi^2 m^2\omega^2}\left[\frac{(\omega-\omega'_r)^2}{\omega\omega'_r} + 4(\varepsilon\cdot\varepsilon')^2\right]$$

for unpolarized photons and no detected polarizations,

$$\frac{d\sigma}{d\Omega'} = \bar{\sum_\lambda}\sum_{\lambda'}\frac{d\sigma}{d\Omega}(\lambda\lambda')$$

$$= \frac{1}{2}\sum_\lambda\sum_{\lambda'}\frac{d\sigma}{d\Omega'}(\lambda\lambda') \qquad \text{look at } 2^{\text{nd}} \text{ term}$$

$$(\varepsilon(k)\cdot\varepsilon(k'))^2 \Sigma = \sum_\lambda \epsilon_{(\lambda)\mu}(k)\,\epsilon_{(\lambda)\nu}(k)\sum_{\lambda'}\epsilon_{(\lambda')}{}^{\mu}(k')\,\epsilon_{(\lambda')}{}^{\nu}(k')$$

$$\parallel$$

$$\varepsilon_\mu \varepsilon^{\mu\prime}\varepsilon_\nu\varepsilon^{\nu\prime}$$

matrix indices explicitly shown--
terms rearranged

remember

$$\sum \epsilon_{(\lambda)\mu}(k)\,\epsilon_{(\lambda)\nu}(k) = -g_{\mu\nu} - \frac{(k_\mu k_\nu - (k\cdot\eta)(k_\mu\eta_\nu + \eta_\mu k_\nu) + k^2\eta_\mu\eta_\nu)}{(k\cdot\eta)^2 - k^2}$$

$\omega^2 \qquad \phi \qquad \phi$

remember $\quad \eta_\mu \equiv (1, \vec{0})$
(p 248)

for real photons, $(\epsilon\cdot\epsilon')^2 = (-\vec{\epsilon}\cdot\vec{\epsilon}\,')^2$

so

$$\sum_\lambda \epsilon_{(\lambda)i}(k)\,\epsilon_{(\lambda)j}(k) = -g_{ij} - \frac{k_i k_j}{\omega^2} = \delta_{ij} - \frac{k_i k_j}{\omega^2}$$

and

$$\frac{1}{2}\sum_\lambda\sum_{\lambda'}(\epsilon\cdot\epsilon')^2 = \frac{1}{2}\left(\delta_{ij} - \frac{k_i k_j}{\omega^2}\right)\left(\delta_{ij} - \frac{k_i' k_j'}{\omega'^2}\right)$$

$$= \frac{1}{2}\left(\delta_{ij}\delta_{ij} - \frac{k'\cdot k'}{\omega'^2} - \frac{k\cdot k}{\omega^2} + \frac{(\vec{k}\cdot\vec{k}')^2}{\omega^2\omega'^2}\right)$$

$$= \frac{1}{2}\left(3 - 1 - 1 + \cos^2\theta'\right)$$

$$= \frac{1}{2}\left(1 + \cos^2\theta'\right)$$

so,

$$\frac{d\sigma}{d\Omega'} = \frac{e^4}{64\pi^2}\frac{\omega_r'^2}{m^2\omega^2}\left[\frac{2(\omega-\omega_r')^2}{\omega\omega_r'} + 2\left(1+\cos^2\theta'\right)\right]$$

$$[\;] = \left[\frac{2(\omega^2+\omega_r'^2)}{\omega\omega_r'} \underbrace{- \frac{4\omega\omega_r'}{\omega\omega_r'} + 2}_{-4+2=-2} + 2\cos^2\theta'\right]$$

$$= 2\left[\frac{\omega^2+\omega_r'^2}{\omega\omega_r'} - 1 + \cos^2\theta'\right]$$

$$= 2\left[\frac{\omega^2+\omega_r'^2}{\omega\omega_r'} - \sin^2\theta'\right]$$

and

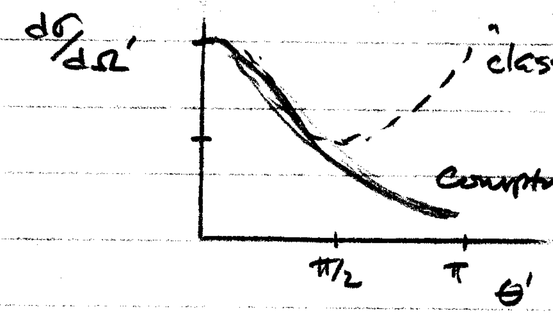
$$\frac{d\sigma}{d\Omega'} = \frac{e^4}{32\pi^2}\frac{\omega_r'^2}{m^2\omega_r^2}\left[\frac{\omega}{\omega_r'} + \frac{\omega_r'}{\omega} - \sin^2\theta'\right]$$ Compton Cross Section.

or Klein-Nishina 1929

where $$\omega_r' \equiv \frac{m\omega}{m+\omega-\omega\cos\theta'}$$

A "classical" limit is where $\omega\hbar \ll mc^2$ & $\omega \sim \omega'$. Then,

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^4}\frac{1}{m^2}\left(1+\cos^2\theta'\right)$$



→ confirmed in experiments in early '30s, I think

The total classical cross section: $$\sigma = \int \frac{d\sigma}{d\Omega'}\,d\Omega'$$

$$= \frac{e^4}{2\pi}\frac{1}{3m^2}$$

recall the classical electron radius is

$$r_0 \equiv \frac{e^2}{4\pi m}.$$

so $$\sigma = \frac{8\pi r_0^2}{3}$$ the Thomson cross section. relevant for xrays.

Generally, $\omega > \omega' > mc^2$. If $\omega >> mc^2$, from the Compton formula

$$\omega' = \frac{\omega m}{m + \omega(1-\cos\theta')} \approx \frac{m}{1-\cos\theta'} = \frac{m}{2\sin^2\theta/2}$$

and in this HR limit $\qquad \theta^2 >> \frac{2m}{\omega}$

$$\frac{d\sigma}{d\Omega} \simeq \frac{e^4}{32\pi^2 m^2 \omega}\,\frac{m}{2\sin^2\theta/2} \qquad \alpha = \frac{e^2}{4\pi}$$

$$= \frac{r_0^2\, m}{4\sin^2\theta/2\;\omega} = \frac{\alpha^2}{4m\omega\,\sin^2\theta/2}$$

The total cross section will involve $\iint \frac{d(\cos\theta)\,d\varphi}{1-\cos\theta}$.

The integration limits are $\int_{-1}^{1-m/\omega} \frac{dx}{1-x} = -\ln(1-x)\Big|_{-1}^{1-m/\omega}$

$$= -\ln(m/\omega) + \ln(2)$$
$$= \ln(2\omega/m)$$

so

$$\sigma_{HR} \simeq \frac{\alpha^2 \pi}{\omega m} \ln\left(\frac{2\omega}{m}\right)$$