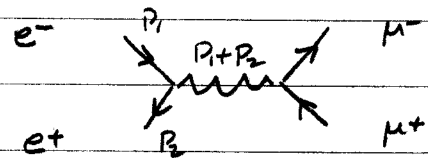
Consider the benchmark reaction

$$e^-(p_1) + e^+(p_2) \longrightarrow \mu^+(p_3) + \mu^-(p_4)$$

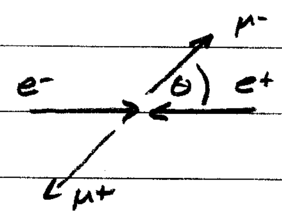
There is a single graph -- an s-channel reaction.



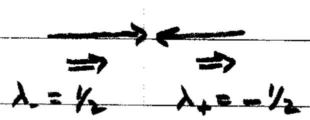
$$(-i)\bar{v}(p_2)(-ie\gamma_\mu)u(p_1)\frac{(-ig^{\mu\nu})}{(p_1+p_2)^2}\bar{u}(p_3)(-ie\gamma_\nu)v(p_4)$$

Of course we can do the complete calculation with the full T-matrix. However before doing that there is a lot to be learned from an angular momentum analysis. This can always be done -- easily in an s-channel reaction.

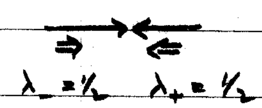
In the center of momentum frame



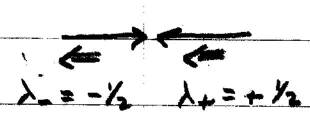
The total angular momentum state through a virtual photon is $J=1$. This happens for



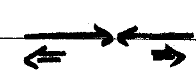
$J_z = 1$



$J_z = 0$



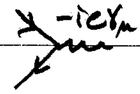
$J_z = -1$



$J_z = 0$

Our initial state will look like

$$\bar{v}(p_2)\,\gamma_\mu\, u(p_1)$$



$$= (\bar{v}_L + \bar{v}_R)\gamma_\mu(u_L + u_R) = \bar{v}_L\gamma_\mu u_L + \bar{v}_L\gamma_\mu u_R + \bar{v}_R\gamma_\mu u_L + \bar{v}_R\gamma_\mu u_R$$

look at $\quad \bar{v}_L\gamma_\mu u_L = v_L^{\dagger}\gamma_0\gamma_\mu u_L$

$$= v^{\dagger}\left(\frac{1+\gamma_5}{2}\right)\gamma_0\gamma_\mu\left(\frac{1-\gamma_5}{2}\right)u$$

$$= \frac{1}{4}v^{\dagger}\gamma_0(1-\gamma_5)(1+\gamma_5)\gamma_\mu u \quad = 0$$

also, $\quad \bar{v}_R\gamma_\mu u_R = 0$

So, we only have $J_z = \pm 1$ states which will contribute in formation of the $J=1$ circumstance.

The final state can be more complicated and the helicity and angular distribution structures are entwined.

Let $\quad \lambda = \lambda_+ - \lambda_- = \pm 1$

$\rho = \lambda_3 - \lambda_4 = 0, \pm 1 \quad$ for a general final state

We can develop the picture in an helicity basis.
The amplitude will be proportional to

$$A \propto \mathcal{M}_\lambda(e^+e^-)\,\mathcal{M}^{\rho}(34)\,D^{J}(\alpha\beta\gamma)^{\lambda}{}_{\rho}$$

In the basis in which J_3 is diagonal,

$$D^J(\alpha\beta\gamma)^{m'}{}_m = e^{-i\alpha m'} d^J(\beta)^{m'}{}_m e^{-i\gamma m}$$

For our polar geometry, $\beta = \theta$ and $\alpha = -\gamma \equiv \varphi$, so

$$A \propto \mathcal{M}_\lambda \mathcal{M}^\rho d^1(\theta)^\lambda{}_\rho e^{-i\varphi(\lambda-\rho)} \quad \text{and}$$

$$\frac{d\sigma}{d\Omega} \propto \sum_{\lambda=\pm1} \underbrace{\sum_{\rho=0,\pm1}}_{\text{in general for now...}} |d^1(\theta)^\lambda{}_\rho|^2 \sigma(e^+e^- \to 34)^\rho{}_\lambda$$

cross section for a particular helicity configuration.

$$= (d^1(\theta)^1{}_0)^2 \sigma^0{}_1 + (d^1(\theta)^{-1}{}_0)^2 \sigma^0{}_{-1}$$

$$+ (d^1(\theta)^1{}_1)^2 \sigma^1{}_1 + (d^1(\theta)^1{}_{-1})^2 \sigma^{-1}{}_1$$

$$+ (d^1(\theta)^{-1}{}_1)^2 \sigma^1{}_{-1} + (d^1(\theta)^{-1}{}_{-1})^2 \sigma^{-1}{}_{-1}$$

The "little d's" we know--

$$d^1(\theta)^1{}_1 = d^1(\theta)^{-1}{}_{-1} = \tfrac{1}{2}(1+\cos\theta)$$
$$d^1(\theta)^1{}_{-1} = -d^1(\theta)^{-1}{}_1 = \tfrac{1}{2}(1-\cos\theta)$$
$$d^1(\theta)^1{}_0 = \sqrt{1/2}\,\sin\theta.$$
$$d^1(\theta)^{-1}{}_0 = 0$$

so

$$\frac{d\sigma}{d\Omega} = (d^1(\theta)^1{}_0)^2 \sigma^0{}_1 + (d^1(\theta)^1{}_1)^2 [\sigma^1{}_1 + \sigma^{-1}{}_{-1}]$$
$$+ (d^1(\theta)^1{}_{-1})^2 [\sigma^{-1}{}_1 + \sigma^1{}_{-1}]$$

The final states can all be labeled singly, since they all come through a single photon, $\sigma^{\rho}_{\lambda} \to \sigma^{\rho}$.

So,

$$\frac{d\sigma}{d\Omega} = \left[d^1(\theta)^1{}_0\right]^2 \sigma^0 + 2\left\{\left[d^1(\theta)^1{}_1\right]^2 + \left[d^1(\theta)^1{}_{-1}\right]\right\} \sigma^T$$

where $\sigma^T \equiv \dfrac{\sigma^1 + \sigma^{-1}}{2}$ for "transverse"

(n.b. $\sigma^1 = \sigma^{-1}$ from parity conservation and σ^0 is often called σ^L for "longitudinal")

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{2}\sin^2\theta\,\sigma^L + 2\left\{\tfrac{1}{4}(1+\cos^2\theta + 2\cos\theta) + \tfrac{1}{4}(1+\cos^2\theta - 2\cos\theta)\right\}\sigma^T$$

$$= \tfrac{1}{2}\sin^2\theta\,\sigma^L + (1+\cos^2\theta)\sigma^T$$

For our particular case $\sigma^L = 0$ because we have $\mu^+\mu^-$ (distinct from e^+e^-) and vector coupling, so that we have only $J^3 = \pm 1$ final states as well. (This would be different had we had $e^+e^- \to \pi^+\pi^-$, for example.) So, $\sigma^L(e^+e^- \to f^+f^-)$ where $f \neq e$.

So, $$\frac{d\sigma}{d\Omega}(e^+e^- \to \gamma \to \mu^+\mu^-) = (1+\cos^2\theta)\sigma^T$$

This has served as a key signature for new particle production -- determining that final states are fermions.

The full calculation is straightforward. From above,

$$T = \frac{e^2}{(p_1+p_2)^2} \bar{v}(p_2)\gamma_\mu u(p_1)\bar{u}(p_3)\gamma^\mu v(p_4)$$

Assume no spin detection and unpolarized beams, so we'll $\bar{\sum_i}\sum_f = 1/4 \sum_i \sum_f$ and.

$$1/4 \sum_{e^-}\sum_{e^+}\sum_{\mu^-}\sum_{\mu^+} |T|^2 = 1/4 \frac{e^4}{(p_1+p_2)^4} \sum\sum\sum\sum j^{elec}_{\mu\nu} j^{\mu\nu}_{muon}$$

where the electron and muon tensors are

$$j^{elec}_{\mu\nu} = [\bar{v}_e(p_2)\gamma_\mu u_e(p_1)][\bar{v}_e(p_2)\gamma_\nu u_e(p_1)]^\dagger$$

$$j^{\mu\nu}_{muon} = [\bar{u}_\mu(p_3)\gamma^\mu v_\mu(p_4)][\bar{u}_\mu(p_3)\gamma^\nu v_\mu(p_4)]^\dagger$$

We'll work in the massless limit (as in the above helicity basis discussion)

$$j^{\mu\nu}_{muon} = \bar{u}(p_3)\gamma^\mu v(p_4)\, v^\dagger(p_4)\gamma^{\nu\dagger}\bar{u}^\dagger(p_3) \qquad (\mu \text{ spinors implied})$$

$$= \bar{u}(p_3)\gamma^\mu v(p_4)\,\bar{v}(p_4)\overline{\gamma^\nu}\, u(p_3)$$

bring in the helicity summations. — add matrix indices

$$= \sum_{\mu^-} u_\alpha(p_3)\bar{u}_\beta(p_3)\gamma^\mu_{\alpha\beta}\sum_{\mu^+} v_\gamma(p_4)\bar{v}_\delta(p_3)\bar{\gamma}^\nu_{\delta\alpha}$$

use Casimir's Trick..

$$= Tr[(\not{p}_3+m_3)\gamma^\mu(\not{p}_4-m_4)\bar{\gamma}^\nu]$$

$$\longrightarrow Tr(\not{p}_3\gamma^\mu\not{p}_4\gamma^\nu) \quad \text{in massless limit.}$$

Likewise $\quad \sum\sum j^{dec}_{\mu\nu} = Tr(\not{p}_1 \gamma_\mu \not{p}_2 \gamma_\nu)$

Putting it together

$$\tfrac{1}{4}\sum\sum\sum\sum |T|^2 = \frac{e^4}{4(p_1+p_2)^4}\left\{ Tr(\not{p}_1\gamma_\mu\not{p}_2\gamma_\nu)\, Tr(\not{p}_3\gamma^\mu \not{p}_4 \gamma^\nu)\right\}$$

$$\{\} = 16\left(p_{1\mu}p_{2\nu} + p_{2\mu}p_{1\nu} - g_{\mu\nu}\, p_1\cdot p_2\right)\left(p_3^\mu p_4^\nu + p_4^\mu p_3^\nu - g^{\mu\nu} p_3\cdot p_4\right)$$

doing the contractions and a little algebra,

$$\{\} = 32\left(p_1\cdot p_3\, p_2\cdot p_4 + p_1\cdot p_4\, p_2\cdot p_3\right) = 32(t^2+u^2)$$

Choose a frame: cm where $E_1 = E_2 \equiv E$

$p_1 = p_2 = E$ (massless limit)

$E_3 = E_4 \equiv E'$

$p_3 = p_4 = E'$

$$\{\} = 32\Big[(EE' - EE'\cos\theta)(EE' - EE'\cos\theta) + (EE' - EE'\underbrace{\cos(\pi-\theta)}_{=\,-\cos\theta})(EE' - EE'\cos(\pi-\theta))\Big]$$

$$= 32\, E^2E'^2\Big[\underbrace{(1-\cos\theta)^2}_{t^2} + \underbrace{(1+\cos^2\theta)^2}_{u^2}\Big]$$

$$= 64\, E^2E'^2(1+\cos^2\theta)$$

Since, $\quad E_1 + E_2 = E_3 + E_4 \quad$ (from phase space)

$$2E = 2E' \Rightarrow E = E'$$

so

$$\{\} = 64\,E^4(1+\cos^2\theta)$$

We've done the phase space.

$$\frac{d\sigma}{d\Omega} = \frac{64E^4}{4s^2}\frac{e^4}{64s\pi^2}\left(\frac{p'}{p}\right)(1+\cos^2\theta)$$

Since $s = (p_1+p_2)^2$

$= 2p_1\cdot p_2 = 4E^2$

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}(1+\cos^2\theta)$$

and we can identify $\sigma_T = \frac{4\pi\alpha^2}{3s}$

Remembering the separate Mandelstam designation,

$$\frac{d\sigma}{d\Omega} = \frac{e^4 s}{128\pi^2}\left(\frac{t^2+u^2}{s^2}\right)$$

The calculation is the same, regardless of the fermion pair produced.

$$\sigma(f\bar{f}) = \alpha^2 \frac{4\pi}{3s}$$

so, for quarks with charges

$$q_\tau = e = Q_\tau e$$
$$q_{u,c,t} = 2/3 e \equiv Q_i e$$
$$q_{d,s,b} = -1/3 e \equiv Q_i e$$

we can conventionally relate

$$\sigma(e^+e^- \to q_i\bar{q}_i) = \frac{Q_i^2}{e^2}\,\sigma(e^+e^- \to \mu^+\mu^-)$$

why not e^+e^-?

The ratio $\dfrac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \equiv R = \sum_i Q_i^2$

a radiatively-corrected calculation, not a measurement

all $f\bar{f}$ which decay or fragment into hadrons.

.. however, phase space steps in as the masses of some fermions are not trivial, and so there is a threshold for production

$m_u \simeq m_d \simeq 1/3\, m_p \sim 300$ MeV

$m_s \simeq m_K \simeq 400$ MeV

$m_c \simeq 1/2\, m_\psi = 1.5$ GeV

$m_b \simeq 1/2\, m_\Upsilon = 4.7$ GeV

$m_t = 173$ GeV

$m_\tau = 1.78$ GeV.

So, the measurement of R, scanned as a function of s, has steps as different channels open up.

$$R = \sum_{i=1}^{n(s)} Q_i^2$$

where $n(s)$ is the number of species of $q\bar{q}$ channels which open as a function of s.

Below the $\psi(c\bar{c})$ of $\sqrt{s} \simeq 3$ GeV, only u, d, s are open.

$$R = (2/3)^2 + (-1/3)^2 + (-1/3)^2 = 4/9 + 1/9 + 1/9 = 6/9 = 2/3$$

However, R is larger due to the quantum number color, thought to be 3

so really $R = N_c \sum_{i=1}^{n(s)} Q_i^2$

$= 2$ below ψ.

Above $\psi(c\bar{c})$ threshold

$$R = 3\left[\,2/3 + (2/3)^2\right]$$
$$= 3\left[\,2/3 + 4/9\right]$$
$$= (2 + 4/3) \simeq 3.3$$

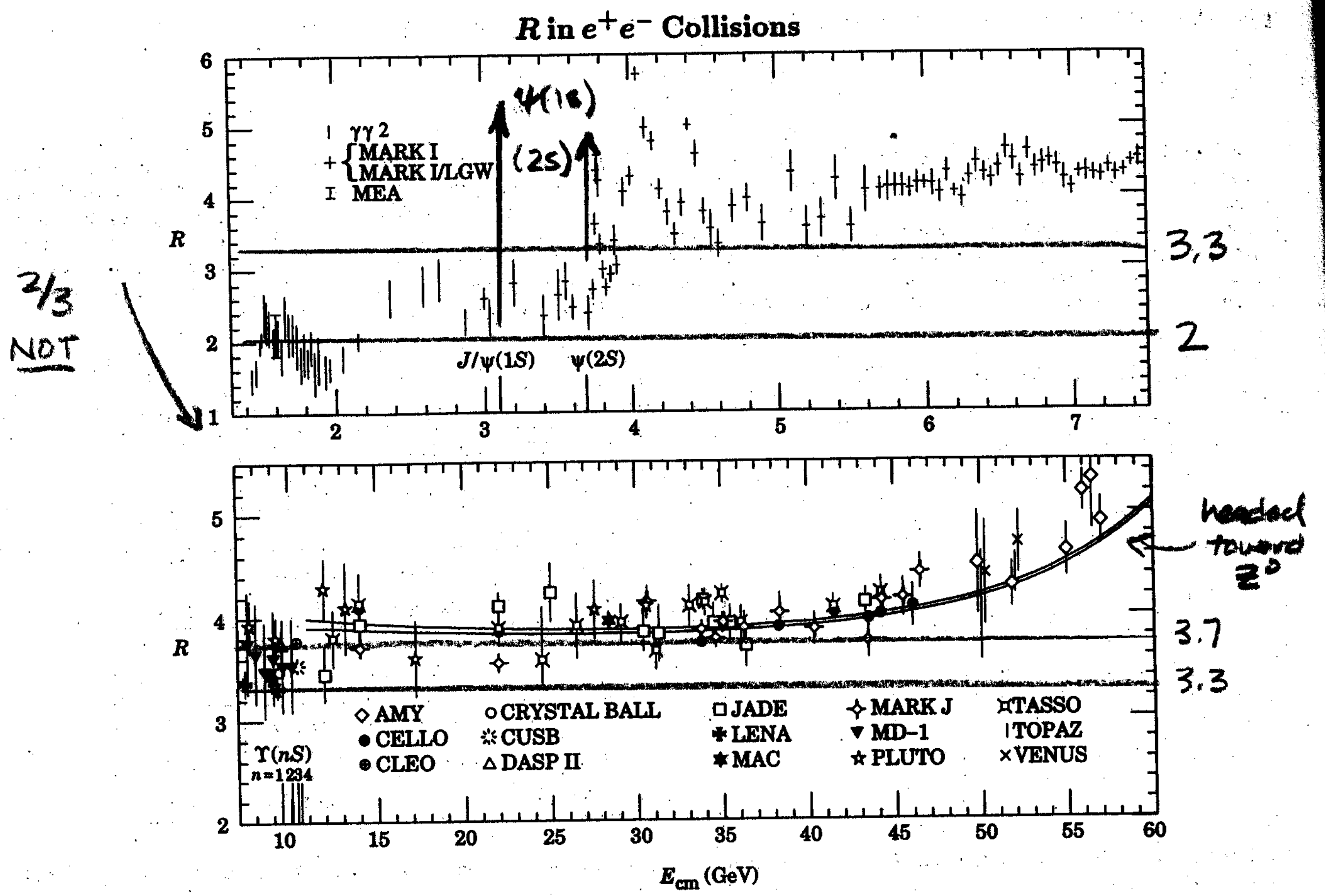


Figure 38.16: Selected measurements of $R \equiv \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$, where the annihilation in the numerator proceeds via one photon or via the Z. Measurements in the vicinity of the Z mass are shown in the following figure. The denominator is the calculated QED single-photon process; see the section on Cross-Section Formulae for Specific Processes. Radiative corrections and, where important, corrections for two-photon processes and τ production have been made. Note that the ADONE data ($\gamma\gamma$2 and MEA) is for ≥ 3 hadrons. The points in the $\psi(3770)$ region are from the MARK I—Lead Glass Wall experiment. To preserve clarity only a representative subset of the available measurements is shown—references to additional data are included below. Also for clarity, some points have been combined or shifted slightly ($< 4\%$) in E_{cm}, and some points with low statistical significance have been omitted. Systematic normalization errors are not included; they range from $\sim$5–20%, depending on experiment. We caution that especially the older experiments tend to have large normalization uncertainties. Note the suppressed zero. The horizontal extent of the plot symbols has no significance. The positions of the $J/\psi(1S)$, $\psi(2S)$, and the four lowest Υ vector-meson resonances are indicated. Two curves are overlaid for $E_{\text{cm}} > 11$ GeV, showing the theoretical prediction for R, including higher order QCD [M. Dine and J. Sapirstein, Phys. Rev. Lett. **43**, 668 (1979)] and electroweak corrections. The Λ values are for 5 flavors in the $\overline{\text{MS}}$ scheme and are $\Lambda^{(5)}_{\overline{\text{MS}}} = 60$ MeV (lower curve) and $\Lambda^{(5)}_{\overline{\text{MS}}} = 250$ MeV (upper curve). (Courtesy of F. Porter, 1992.) References (including several references to data not appearing in the figure and some references to preliminary data):

AMY: T. Mori *et al.*, Phys. Lett. **B218**, 499 (1989);
CELLO: H.-J. Behrend *et al.*, Phys. Lett. **144B**, 297 (1984); and H.-J. Behrend *et al.*, Phys. Lett. **183B**, 400 (1987);
CLEO: R. Giles *et al.*, Phys. Rev. **D29**, 1285 (1984); and D. Besson *et al.*, Phys. Rev. Lett. **54**, 381 (1985);
CUSB: E. Rice *et al.*, Phys. Rev. Lett. **48**, 906 (1982);
CRYSTAL BALL: A. Osterheld *et al.*, SLAC-PUB-4160; and Z. Jakubowski *et al.*, Z. Phys. **C40**, 49 (1988);
DASP: R. Brandelik *et al.*, Phys. Lett. **76B**, 361 (1978);
DASP II: Phys. Lett. **116B**, 383 (1982);
DCI: G. Cosme *et al.*, Nucl. Phys. **B152**, 215 (1979);
DHHM: P. Bock *et al.* (DESY-Hamburg-Heidelberg-MPI München Collab.), Z. Phys. **C6**, 125 (1980);
$\gamma\gamma$2: C. Bacci *et al.*, Phys. Lett. **86B**, 234 (1979);
HRS: D. Bender *et al.*, Phys. Rev. **D31**, 1 (1985);
JADE: W. Bartel *et al.*, Phys. Lett. **129B**, 145 (1983); and W. Bartel *et al.*, Phys. Lett. **160B**, 337 (1985);
LENA: B. Niczyporuk *et al.*, Z. Phys. **C15**, 299 (1982).
MAC: E. Fernandez *et al.*, Phys. Rev. **D31**, 1537 (1985);
MARK J: B. Adeva *et al.*, Phys. Rev. Lett. **50**, 799 (1983); and B. Adeva *et al.*, Phys. Rev. **D34**, 681 (1986);
MARK I: J.L. Siegrist *et al.*, Phys. Rev. **D26**, 969 (1982);
MARK I + Lead Glass Wall: P.A. Rapidis *et al.*, Phys. Rev. Lett. **39**, 526 (1977); and P.A. Rapidis, thesis, SLAC-Report-220 (1979);
MARK II: J. Patrick, Ph.D. thesis, LBL-14585 (1982);
MD-1: A.E. Blinov *et al.*, Z. Phys. **C70**, 31 (1996);
MEA: B. Esposito *et al.*, Lett. Nuovo Cimento **19**, 21 (1977);
PLUTO: A. Bäcker, thesis Gesamthochschule Siegen, DESY F33-77/03 (1977); C. Gerke, thesis, Hamburg Univ. (1979); Ch. Berger *et al.*, Phys. Lett. **81B**, 410 (1979); and W. Lackas, thesis, RWTH Aachen, DESY Pluto-81/11 (1981);
TASSO: R. Brandelik *et al.*, Phys. Lett. **113B**, 499 (1982); and M. Althoff *et al.*, Phys. Lett. **138B**, 441 (1984);
TOPAZ: I. Adachi *et al.*, Phys. Rev. Lett. **60**, 97 (1988); and
VENUS: H. Yoshida *et al.*, Phys. Lett. **198B**, 570 (1987).

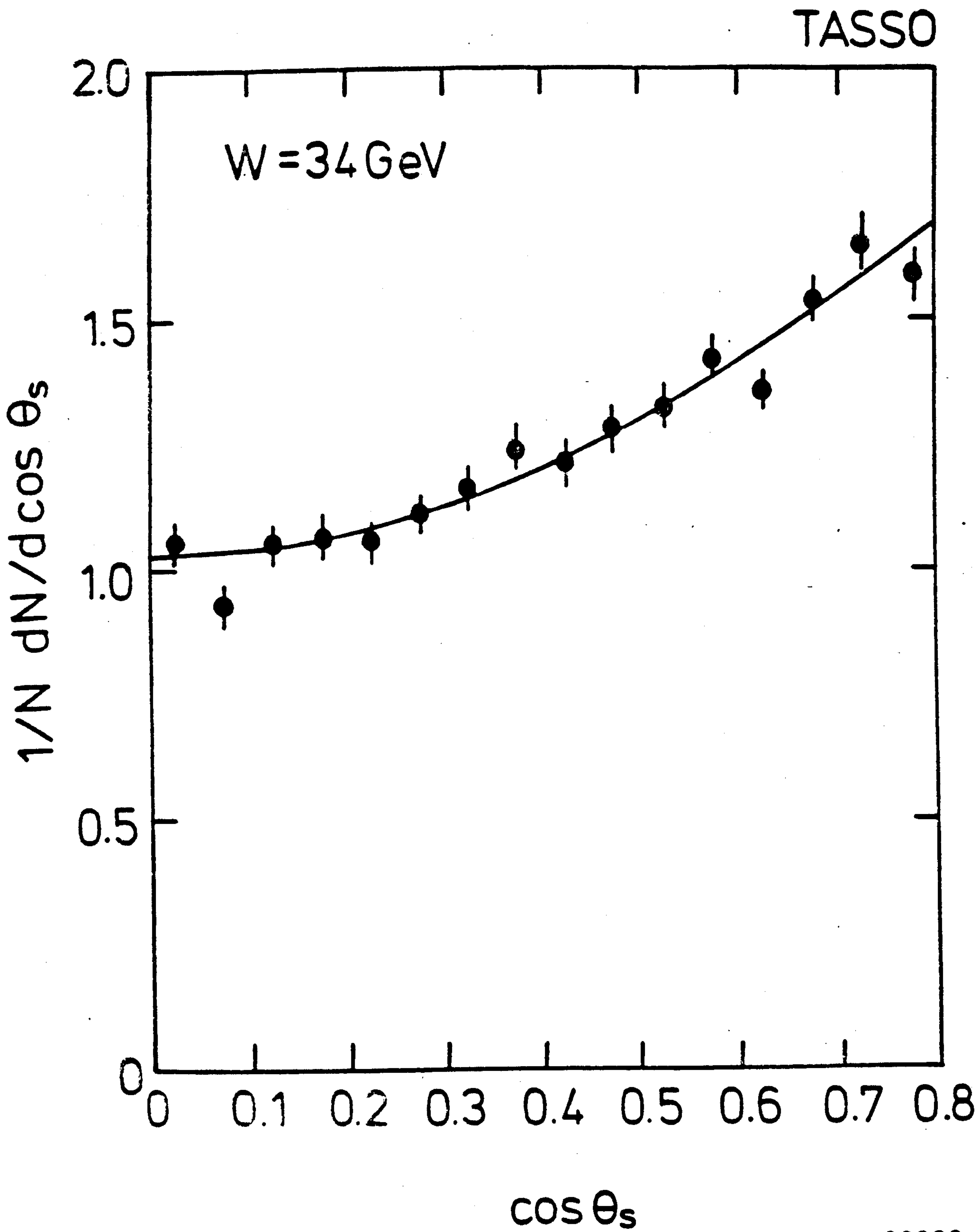


Fig. 2.21 36366

The angular distribution of the jet axis determined by sphericity. The curves are proportional to $1 + \cos^2\theta_s$.

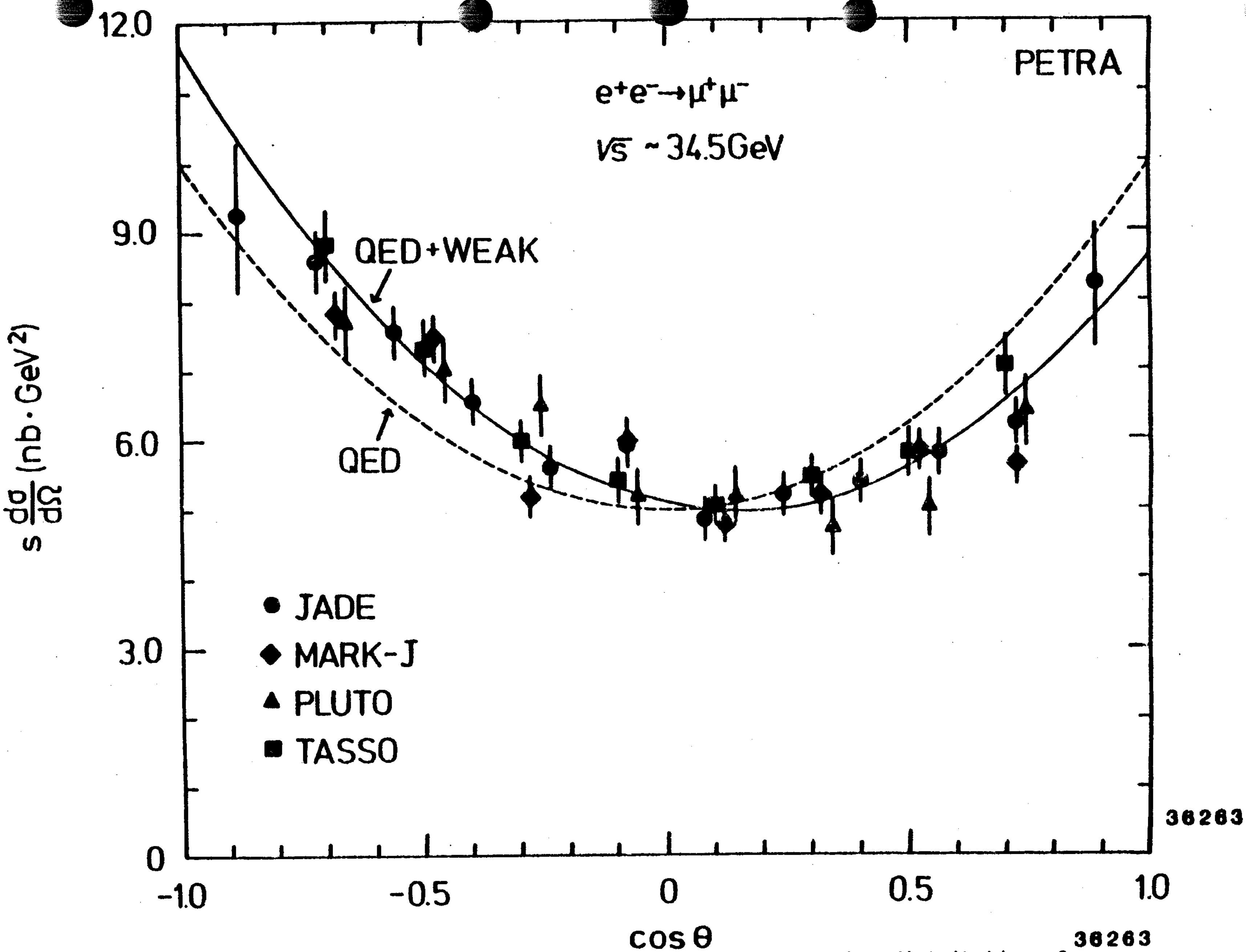


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Fig. 5.13 Compilation [5-36] of PETRA high-energy data for the angular distribution of $e^+e^- \rightarrow \mu^+\mu^-$ at $\sqrt{s}$-34.5 GeV. The data are corrected for effects α^3. The full curve shows a fit to the data allowing for an asymmetry; the dashed curve is the symmetric QED prediction.