$\boxed{\text{Diracology}}$

$$H\psi = \frac{i}{\hbar}\frac{\partial \psi}{\partial t}$$

The Dirac Equation: $-i\hbar c\, \vec{\alpha}\cdot\vec{\nabla}\psi + \beta mc^2\psi = \frac{i}{\hbar}\frac{\partial\psi}{\partial t}$

where $\beta = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ (ie 4 dimensional gives the minimal, unique solution)

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad 4\times 4$$

so ψ is a 4 component spinor.

Remember, ① $[H, H] = 0$

② $[H, p_i] = 0$

③ $i[H, \vec{p}\cdot\vec{S}] = 0$

can find solutions to D.E. which are simultaneously eigenfunctions of both H and p → free particle solutions.

where $\vec{S} \equiv \frac{1}{2}\vec{\Sigma}$

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

<u>Helicity</u> is the good quantum number

$[H, S_i] \neq 0$ not spin

$[H, L_i] \neq 0$ not angular momentum

The covariant form of D.E. is

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \qquad \{ (i\hbar \not{\partial} - mc)\psi = 0 \}$$

where $\gamma^0 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (4×4)

$$\vec{\gamma} = \beta\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

and

$$\gamma^\mu = [\gamma^0, \vec{\gamma}]$$

$$\left.\begin{array}{l}\{\gamma_0, \gamma_i\} = 0 \\ \{\gamma_i, \gamma_j\} = -2\delta_{ij}\end{array}\right\} \quad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1}$$

4x4 in Dirac space

and $\beta^+ = \beta \Rightarrow (\gamma^0)^+ = \gamma^0$ Hermitian.

$$\gamma_i^+ = (\beta\alpha_i)^+ = \alpha_i^+\beta^+$$
$$= \alpha_i^+\beta = \alpha_i\beta$$
$$\gamma_i^+ = -\beta\alpha_i = -\gamma_i \quad \text{anti Hermitian}$$

also, $\gamma^{\mu +} = \gamma^0\gamma^\mu\gamma^0$

for freeparticle solutions

In momentum space, $(\gamma^\mu p_\mu - m)\psi = 0$

and, defining conventionally, $\overline{\psi} = \psi^+\gamma^0$

$$\overline{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0 \quad \text{or} \quad \overline{\psi}(\gamma^\mu p_\mu + m) = 0$$

Conventional is "slash notation" $B_\mu\gamma^\mu \equiv \not{B}$

so,
$$(i\not{\partial} - m)\psi(x) = 0$$
$$\overline{\psi}(x)(i\overleftarrow{\not{\partial}} + m) = 0$$
$$(\not{p} - m)\psi(p) = 0$$
$$\overline{\psi}(p)(\overleftarrow{\not{p}} + m) = 0$$

The ψ's are 4 component spinors

$$\psi = \begin{pmatrix}\psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4\end{pmatrix}$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$i\gamma^\nu \partial_\nu \rightarrow$ $$(-\gamma^\nu\gamma^\mu \partial_\nu \partial_\mu - \underbrace{i\gamma^\nu \partial_\nu m)\psi}_{m^2\psi \quad \text{D.E.}} = 0$$

so $$(-\gamma^\nu\gamma^\mu \partial_\nu \partial_\mu - m^2)\psi = 0$$

use dummy indices and add another

$$(-\gamma^\nu\gamma^\mu \underset{\partial_\mu \partial_\nu}{\underset{\downarrow}{\partial_\nu \partial_\mu}} - m^2 - \gamma^\mu\gamma^\nu \partial_\mu \partial_\nu - m^2)\psi = 0$$

$$[-\underbrace{\underbrace{(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)}_{\{\gamma^\nu,\gamma^\mu\}}}_{2g^{\nu\mu}} \partial_\mu \partial_\nu - 2m^2]\psi = 0$$

$\times\ \mathbb{1}$ in spinor space.

$$(-2g^{\nu\mu}\partial_\mu\partial_\nu - 2m^2)\psi_a = 0$$

a particular component.

$$(\partial^\nu \partial_\nu + m^2)\psi_a = 0$$

$$(\Box + m^2)\psi_a = 0$$

each spinor component satisfies a K.G. equation -- a 4d wave equation.

So, we know we can write

$$\psi_i(x) = N u_i(p) e^{-ip\cdot x} \qquad i = 1,2,3,4$$
$$= N u_i(p) e^{-ip_0 x_0 + i\vec{p}\cdot\vec{x}}$$

In rest frame,

$$\psi_i^R(x) = N u_i(0) e^{-imx_0}$$

Use in Dirac Equation

$$i\frac{\partial \psi^R}{\partial t} = \underbrace{(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi^R} = \beta m\,\psi^R$$

$$i\frac{\partial}{\partial x^0} N u_i(0) e^{-imx_0} = i(-im)\underbrace{N u_i(0)\, e^{-imx_0}}_{\psi^R} = \beta m \psi^R$$

$$m\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}\psi^R = m\begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}\psi^R$$

$$\begin{pmatrix}\psi_1^R\\ \psi_2^R\\ \psi_3^R\\ \psi_4^R\end{pmatrix} = \begin{pmatrix}\psi_1^R\\ \psi_2^R\\ -\psi_3^R\\ -\psi_4^R\end{pmatrix} \Rightarrow \psi_3^R = \psi_4^R = 0$$

$\Rightarrow$ a natural segregation into the upper components and lower components.

$$u(p) = \begin{pmatrix}u_A\\ u_B\end{pmatrix} \equiv \begin{pmatrix}u_1\\ u_2\\ u_3\\ u_4\end{pmatrix}$$

So,

$$\psi(x) = N\begin{pmatrix}u_A(p)\\ u_B(p)\end{pmatrix} e^{-ip\cdot x}$$

Substitute into Dirac Equation

$$i\frac{\partial\psi}{\partial t} = (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi$$

$$i(-i)p^0 u(p) = -i\,\vec{\alpha}\cdot(i\vec{p})\,u(p) + \beta m\, u(p)$$

$$p^0 \mathbb{1}\, u(p) = \Big[\underbrace{\begin{pmatrix}0 & \vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & 0\end{pmatrix} + \begin{pmatrix}1&0\\0&-1\end{pmatrix}}\Big] u(p)$$

spinor space

like $Eu = Hu$ $\qquad \mathcal{H} = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -m\end{pmatrix}$ which is a complicated 4d object.

$$\mathcal{H} = \begin{pmatrix} m & 0 & p_3 & p_1 - ip_2\\ 0 & m & p_1+ip_2 & -p_3\\ p_3 & p_1 - ip_2 & -m & 0\\ p_1+ip_2 & -p_3 & 0 & -m\end{pmatrix}$$

So, $$\begin{pmatrix} p_0 & 0 \\ 0 & p_0 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$ 13

$$\begin{pmatrix} (p_0-m) & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & (p_0+m) \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

det = 0 $\Rightarrow$ $$(p_0-m)(p_0+m) - (-\vec{\sigma}\cdot\vec{p})(-\vec{\sigma}\cdot\vec{p}) = 0$$
$$p_0^2 - m^2 - \vec{\sigma}\cdot\vec{p}\,\vec{\sigma}\cdot\vec{p} = 0$$

remember $$\vec{\sigma}\cdot\vec{A}\,\vec{\sigma}\cdot\vec{B} = \vec{A}\cdot\vec{B} - \underbrace{i\vec{A}\cdot\vec{\sigma}\times\vec{B}}_{\vec{p}\cdot\vec{\sigma}\times\vec{p} = 0}$$

so, $$p_0^2 - m^2 - \vec{p}^{\,2} = 0$$
$$p_0^2 = \vec{p}^{\,2} + m^2$$

For a given momentum, energy eigenvalues will be
$$p_0 = \pm\sqrt{\vec{p}^{\,2}+m^2} = \pm E \quad \text{where } E>0$$

Choose $p_0 = +E$

$$\begin{pmatrix} (E-m) & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & (E+m) \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$ coupled equations

$$(E-m)u_A - \vec{\sigma}\cdot\vec{p}\, u_B = 0$$
$$-\vec{\sigma}\cdot\vec{p}\, u_A + (E+m)u_B = 0$$

$\Rightarrow$ $$u_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m} u_B \qquad u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m} u_A$$

$$(E^2-m^2)u_A = \vec{p}^{\,2} u_A \qquad \text{A}$$

Choose $p_0 = -E$ ... same steps

$$(E^2 - m^2)u_B = \vec{p}^{\,2} u_B \qquad (B)$$

(A) and (B) $\Rightarrow$ that within u_A and u_B, u_1, u_2 and u_3, u_4 are arbitrary.

from (A): fix u_A, determine u_B

(B): fix u_B, determine u_A

Choose (A) $u_1 = \phi \quad u_2 = 0$; $u_1 = 0 \quad u_2 = 1$ — up to an overall constant.. any other combination is linear combination of these

(B) ditto

$\boxed{p_0 = +E}$ $\qquad u_B = \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} u_A$

so ~~...~~ $u(p)_+ = N u(p) = N \begin{pmatrix} u_A \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} u_B \end{pmatrix}$

call 2 solutions $\quad u_A$ (solution 1) $\equiv \chi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\quad 2 \equiv \chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

so,

$$u(p)_+^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_1 + ip_2}{E+m} \end{pmatrix}$$

$$u(p)_+^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_1 - ip_2}{E+m} \\ \frac{-p_3}{E+m} \end{pmatrix}$$

$$\left.\right\} \quad u(p)_+^{(1,2)} = N \begin{pmatrix} \chi^{1,2} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \chi^{1,2} \end{pmatrix}$$

$\boxed{p_0 = -E}$

$$u_A = -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\, u_B$$

$$u(p)_- = N \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} u_B \\ u_B \end{pmatrix}$$

1 : $\quad \chi^3 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2 : $\quad \chi^4 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$u(p)_-^{(3,4)} = N \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \chi^{3,4} \\ \chi^{3,4} \end{pmatrix}$$

Normalization:

If ~~the~~ NRQM $\qquad \int \psi^\dagger \psi\, d^3x = 1$

$$\underbrace{d^3x}_{\frac{1}{\gamma} d^3x} \Rightarrow \psi^\dagger\psi \to \gamma$$

Rather, take

$\bar{u}(p)\gamma^\mu \to (\not{p}-m)u(p) = 0 \qquad \bar{u}(p)(\not{p}-m) = 0 \leftarrow \gamma^\mu u(p)$

add

$$2p^\mu \bar{u}(p)u(p) = 2m\bar{u}\gamma^\mu u \qquad \text{c general statement}$$

look at $\mu = 0$ $\qquad 2p^0 \bar{u}(p)u(p) = 2m u^\dagger \gamma^0 \gamma^0 u$
$\qquad = 2m u^\dagger u$

so $\qquad u^\dagger u = \frac{p^0}{m} \bar{u}(p)u(p)$

two conventions

① $\quad u^\dagger u = \gamma = \frac{p_0}{m} \Rightarrow \bar{u}(p)u(p) = 1$

② $\quad \bar{u}u = 2m\,, \quad u^\dagger u = \frac{p_0}{m} 2m = 2p^0$

$\qquad u^\dagger u = 2m\gamma$

cumbersome for massless particles

use this

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so, $\quad p^0 = +E \qquad \bar{u}_i(p)\, u_j(p) = 2m\, \delta_{ij} \qquad i,j = 1,2$

$\qquad\;\; = -E \qquad \bar{u}_i(p)\, u_j(p) = -2m\, \delta_{ij} \qquad = 3,4$

N? just from $\quad \bar{u}(p)_+^{(1,2)}\, u(p)_+^{(1,2)} = 2m$

$\vdots$

find $\quad N = \sqrt{E+m}$

$$\psi_1(x) = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_+}{E+m} \end{pmatrix} e^{-iEt + i\vec{p}\cdot\vec{x}}$$

$$\psi_2(x) = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ \frac{-p_3}{E+m} \end{pmatrix} e^{-iEt + i\vec{p}\cdot\vec{x}}$$

$$\psi_3(x) = \sqrt{E+m} \begin{pmatrix} \frac{-p_3}{E+m} \\ \frac{-p_+}{E+m} \\ 1 \\ 0 \end{pmatrix} e^{iEt + i\vec{p}\cdot\vec{x}}$$

$$\psi_4(x) = \sqrt{E+m} \begin{pmatrix} \frac{-p_-}{E+m} \\ \frac{p_3}{E+m} \\ 0 \\ 1 \end{pmatrix} e^{iEt + i\vec{p}\cdot\vec{x}}$$

where $\quad p_\pm \equiv p_1 \pm i p_2$

From Feynman's ideas, take ~~probability~~ current density .

$$j^\mu = [\rho, \vec{j}] \quad \text{where} \quad \rho = 2p^0 \text{ from } \psi^\dagger\psi$$

$$\vec{j} = 2\vec{p} \text{ from } \psi^\dagger \vec{\alpha} \psi$$

$$j^\mu = 2[p^0, \vec{p}]$$
$$j^\mu = 2p^\mu$$

probability 4-current.

Charge with ~~electromagnetic current~~ electric charge 4-current

for $E > 0$ $\quad j^\mu_{EM} = Q_{electron}\, j^\mu$

$$j^\mu_{EM}(e^-) = -2e\,p^\mu \qquad p^0 = \sqrt{\vec{p}^2 + m^2} = +E$$
$$= -2e[E, \vec{p}]$$

for antielectron state

$$j^\mu_{EM}(e^+) = Q_{positron}\, j^\mu$$
$$= +2e[E, \vec{p}] \qquad p^0 = E > 0$$

however, note,

$$j^\mu_{EM,e^+}(p^\mu) = -2e[-p^0, -\vec{p}]$$
$$= j^\mu_{EM\,e^-}(-p^\mu)$$

Free particle solutions appear like $e^{-ip^0 t}$

$$\underbrace{e^{-ip_0 t}}_{\substack{\text{electron}\\ \text{wavefunction}\\ +E}} = e^{-i(-p_0)(-t)}$$

$-E$ wavefunction & backwards in time

Consider Dirac Equation for electron interacting with electromagnetic field

$$(i\not\partial - e\not A - m)\psi(x) = 0$$

want to turn this into equation for charge conjugate counter part ... need to change $-e\not A \to +e\not A$ through some operation

suggests that ~~charge~~ complex conjugation will be involved.

can get $$(i\not\partial_\mu + e\not A - m]\psi_c(x) = 0$$

iff $$C'\psi^* = \psi_c$$

where after some work one finds $C = C'\gamma^0$

$$C \equiv -i\gamma^2\gamma^0$$

so, $$\psi_c = -i\gamma^2\psi^*$$

Take them one by one --

$$\psi_{c_1} = -i\gamma^2\psi_1^* = \sqrt{E+m}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{P_3}{E+m} \\ \frac{P_+^*}{E+m} \end{pmatrix} e^{iEt - i\vec{p}\cdot\vec{x}}$$

$P_+^* = P_-$

$$= -\sqrt{E+m}\begin{pmatrix} \frac{P_-}{E+m} \\ \frac{-P_3}{E+m} \\ 0 \\ 1 \end{pmatrix} e^{ip\cdot x}$$

remember $$\psi_4 = \sqrt{E+m}\begin{pmatrix} \frac{-P_-}{E+m} \\ \frac{P_3}{E+m} \\ 0 \\ 1 \end{pmatrix} e^{+iEt + i\vec{p}\cdot\vec{x}}$$

Reverse momentum direction and see that

$$\psi_{c_1}(\vec{p}) = -\psi_4(-\vec{p}) \quad \text{and} \quad E>0 \quad A^\mu \leftrightarrow +e$$

$$\psi_{c_2}(\vec{p}) = \psi_3(-\vec{p}) \quad E>0 \quad A^\mu \leftrightarrow +e$$

$$\psi_{c_3}(\vec{p}) = \psi_2(-\vec{p}) \quad E<0 \quad "$$

$$\psi_{c_4}(\vec{p}) = -\psi_1(-\vec{p}) \quad E<0 \quad "$$

Get rid of negative energy states altogether, use $\psi_1\,\psi_2\,\psi_{c_1}\,\psi_{c_2}$ as the physical states

$u_1(p)$	electron	$E>0$	"spin"	$\frac{1}{2}$
$u_2(p)$	"	"	"	$-\frac{1}{2}$
$v_1(p) \equiv u_{c_1}(\vec{p})$	positron	$E>0$	"	$\frac{1}{2}$
$v_2(p) \equiv u_{c_2}(\vec{p})$	"	"	"	$-\frac{1}{2}$

v's satisfy a D.E. but opposite p^μ

$$(\not{p}-m)\,u_i(p) = 0$$

$$(\not{p}+m)\,v_i(p) = 0$$

$$\bar{u}(p)(\not{p}-m) = 0 \qquad u_s(p) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\chi^s \\ \chi^s \end{pmatrix}$$

$$\bar{v}(p)(\not{p}+m) = 0$$

$$\bar{u}_i(p)\,u_j(p) = 2m\,\delta_{ij}$$

$$u_i^\dagger u_j = 2E\,\delta_{ij} \qquad v_s(p) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\chi^{-s} \\ \chi^{-s} \end{pmatrix}$$

$$\bar{v}_i(p)\,v_j(p) = -2m\,\delta_{ij}$$

$$v_i^\dagger v_j = 2E\,\delta_{ij} \qquad s = 1,2$$

$$\bar{v}u = \bar{u}v = 0 \qquad \chi^1 = \begin{pmatrix}1\\0\end{pmatrix} \to \tfrac{1}{2} \quad \chi^2 = \begin{pmatrix}0\\1\end{pmatrix} \to -\tfrac{1}{2}$$

rest frame

$$\chi^{-1} = \begin{pmatrix}1\\0\end{pmatrix} \to -\tfrac{1}{2} \quad \chi^{-2} = \begin{pmatrix}0\\1\end{pmatrix} \to +\tfrac{1}{2}$$

we chose $\chi = \binom{1}{0}$ etc originally → eigenstate of energy
not necessarily of helicity.

→ most general spinors are more complicated --
weyl problem

Completeness -

$$\sum_{j=1}^{4} u^{(j)}(\vec{p})_i \, u^{(j)}(p)^{\dagger}_m = 2p^0 \, \delta_{im} \mathbb{1} \qquad \text{for } +E \ \& \ -E$$

now, with positron states,

$$\sum_{j=1}^{2} u^{(j)}(p)_i \, \bar{u}^{(j)}(p)_m - \sum_{j=1}^{2} v^{(j)}(p)_i \, \bar{v}^{(j)}(p)_m = \delta_{im} \, 2m .$$

Remember $\qquad \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} &\gamma^{5\dagger} = \gamma^5 \\ &\gamma^5\gamma^5 = \mathbb{1} \\ &\{\gamma^5, \gamma^\mu\} = 0 \end{aligned}\right\} \text{useful later.}$$