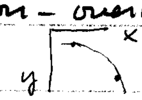
Lec3 Lagrangian, Classical Field Theory.

Have you ever thought about the Principle of Least Action? It's a strange notion in physics — really only shed of its weirdness by quantum mechanics and Feynman. It has a long history, with roots that go back to ancient times:

- Heron – 4^{th} century BC – proposed that light traveled in the path which was shortest. → to explain reflection.
- Fermat – 1657 – modified this to "shortest time" → to explain passage of light through media of varying indices
- Newton – 1696 – Johann Bernoulli challenged → brachistochrone (solved by Newton – overnight at Univ – Leibnitz, L'Hospital, Jakob B).

brach-ke-sto-chrone



curve joining 2 points – least time under gravity.

→ Jakob tried to embarrass Johann a number of times with similar geodesic problems. –

all involved trying to minimize a certain integral – a path integral,

$$I = \int_a^b F[y(x), y'(x), x]\, dx$$

by finding a function of x, $y(x)$, which will make I a minimum (or maximum).

- Euler – 1728 – worked in on a variety of problems.. brachistochrone w/ friction... found general equation.

based on differences..

$$\lim_{\Delta x \to 0} \left[\frac{\partial F}{\partial y} - \frac{\Delta}{\Delta x}\left(\frac{\partial F}{\partial y'}\right)\right] = 0$$

.. not satisfactory ⇒ lots of geometrical proof, finite sums – cumbersome

meanwhile

- Pierre-Louis Moreau de Maupertuis – 1744

  light – mechanics analogies. – straight lines refraction difficult –

  Nature – always acts in most simple ways. (quotes Fermat)

  → extrapolated recklessly F.P. to mechanics and preserved Descartes' conservation of momentum and Leibnitz's conservation of vis viva (kinetic energy)

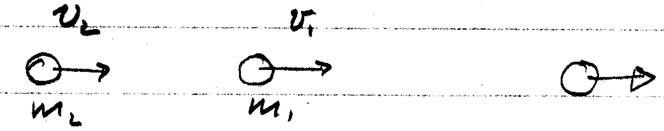
  → minimum principle → Unifying Principle

mathematician astronomer geographer biologist President of Berlin Academy of – foe of Voltaire

he defined Action = velocity × distance and declared that Nature worked to minimize Action.. "Nature's true storehouse"

(Euler showed central forces worked)

From momentum conservation (inelastic):



$$m_1 v_1 + m_2 v_2 = (m_1+m_2) v$$

$$v = \frac{m_1 v_1 + m_2 v_2}{m_1+m_2}$$

action: Maupertuis. variously –

$$S = m_1 (v_1 - v)^2 + m_2 (v - v_2)^2$$

$$m_1 v_1^2 + m_1 v^2 - 2m_1 v_1 v + m_2 v^2 + m_2 v_2^2 - 2m v v_2 = S$$

min S $\Rightarrow$ $\quad 2m_1 v + 2m_1 v_1 + 2m_2 v - 2m_2 v_2 = 0$

$$v(2m_1 + 2m_2) = 2m_2 v_2 + 2m_1 v_1$$

$$v = \frac{m_1 v_1 + m_2 v_2}{m_1+m_2}$$

$\Rightarrow$ Least Action: $\quad S = \int 2T\,dt$

$$= \int 2\left(\tfrac{1}{2} m v^2\right) dt$$

$$= \int m v^2\,dt$$

$$= \int m v\,ds$$

(for Maupertuis $\quad S = v \cdot s. \quad [M][L/T][L] = [\text{angular momentum}]$

(he confused velocity with distance and said

$$m\underbrace{(v_1 - v)}_{v}\underbrace{(v_1 - v)}_{\int v\,dt \;\text{distance}}$$

)

~~Descartes: "Action" $S = \int 2T\,dt$ where $T = 2mv^2$ or $mv^2 \rightarrow S = \int 2mv\,ds$~~

Maupertius chose $S = \int v\,ds$ and delighted in the fact that useful things emerged when demanding min(S) — collisions, lever, etc. — not Fermat's Principle, but Maupertius thought BIG.. declared the "Principle of Least Action" as universal, encompassing mechanics, optics (F.P), biology — indeed a proof for the existence of God.

→ the teleology of PLA is confusing: how does light, or do mechanical things, "know" to take the path of least action? This was troubling into the 20th century.

→ we owe Maupertius
  a) nothing for real work accomplished
  b) for influencing Euler
  c) patronizing Lagrange
  d) forming the first really Universal Principle.

admire
we can ~~admire~~ the application of it to all phenomena, in the movement of animals, in the vegetation of plants, in the revolution of the heavenly bodies — so much
worthier
~~vestiges~~ of its Author. These Laws, so beautiful and so simple, are perhaps the only ones which the Creator and Organizer of things has established in matter in order to effect ~~me~~ all the phenomena of the visible world"

in a letter (in Latin) to Euler as 23 year old,
he worked out the solution to minimization
(while a teenager) → completely new approach.

- Poisson (Useless principle) →
- Thomson and Tait ("strongly impressed") →
- Helmholtz (first systematic treatment) →
- Larmor and Schwarzschild (1900–1903 – E&M ... no mechanical medium) → Hilbert (general relativity)

- → Feynman

How does light "know" to take shortest time?

different phase
→ shortest
another different phase

total amplitude is sum of all -- phases cancel.

in QM -- each path is accompanied by a phase

$e^{iS/\hbar}$

(wrong path will have large S -- big phase difference.

Action has units of Planck's constant -- ever wonder why?

as $\hbar \to 0$ -- forget all amplitudes not in the correct path.

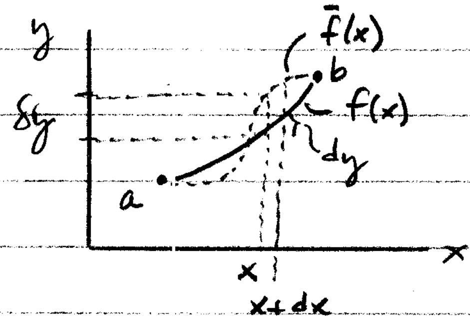
→ Maupertius was right ... it is a unifying principle.

Lagrange's idea was a new kind of calculus – not of variables, but of functions.

Take the integral,

$$I = \int_a^b F[\,y(x), y'(x), x\,]\, dx$$

and evaluate it "at" different functional forms (not points) for $y = f(x)$. These functional forms may be infinitesimally different.



new function

$$\bar{f}(x) = f(x) + \epsilon\,\eta(x)$$

infinitesimal parameter.

He called the difference $\bar{f}(x) - f(x) = \epsilon\,\eta(x) \equiv \delta y$ the <u>variation</u>. an arbitrary, infinitesimal change in f <u>at point x</u>

note: δy & dy are both infinitesimal changes in the function y

dy caused by $dx \rightarrow$ same function

δy new function

the variation δx here serves no purpose, so $\delta x = 0$

Two derivatives to distinguish:

$$\frac{d}{dx}\delta y = \frac{d}{dx}\left[\bar{f}(x) - f(x)\right] = \frac{d}{dx}\left[\epsilon\eta(x)\right] = \epsilon\eta'(x)$$

and

$$\delta \frac{d}{dx} f(x) = \bar{f}'(x) - f'(x) = y' + \epsilon\eta' - y' = \epsilon\eta'(x)$$

so,
$$\frac{d}{dx}\delta y = \delta \frac{dy}{dx}$$

Variation and integration are commutative,

int for mod. - int for original ≡ variation

$$\delta \int_a^b F[y, y', x]\,dx = \int_a^b \bar{F}\,dx - \int_a^b F\,dx$$

$$= \int_c^b (\bar{F} - F)\,dx$$

$$= \int_c^b \delta F\,dx$$

For I to be stationary,

$$\delta I = 0 = \delta \int_a^b F\,dx = \int_a^b \delta F\,dx$$

$$\delta F = F[y + \epsilon\eta, y' + \epsilon\eta', x] - F[y, y', x]$$

$$= \epsilon\left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right)$$ by Taylor's Expansion

so, $$\delta I = \epsilon \int_a^b \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \right) dx$$

For an extremum $\quad \dfrac{\delta I}{\epsilon} = 0$

η and η' are not independent — Integrate 2nd term by parts,

$$\int_a^b \frac{\partial F}{\partial y'}\eta'\,dx = \frac{\partial F}{\partial y'}\eta\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta\,dx$$

The usual condition, is that the F's are ~~go~~ equivalent at the endpoints, $\Rightarrow \eta = 0$ there

so,

$$\frac{\delta I}{\epsilon} = 0 = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta\,dx$$

Since $\eta(x)$ is arbitrary, this condition is equivalent to

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

and now called the Euler-Lagrange Equations

Hamilton - 1834 "Theory of Systems of Rays"

Optics ... devised:

1) surfaces of constant action at all times $\perp$ light rays

2) the "characteristic function"

$$C \equiv \int \underbrace{\eta}_{\text{index of ref.}} dx = \int \frac{c}{ds/dt}\,ds = c\int dt$$

so $\quad \delta C = c\,\delta \int dt \quad \Rightarrow$ Fermat's Princ. least time

3) Further generalized a bundle of light rays in analogy with paths of mechanical particles and defined the "Principle Function"

$$S = C - Ht$$ (solutions to Ham.-Jacobi Eq)

where H is the constant, total energy

To Hamilton, η (optics) ~ v (mechanics), so for constant mass, he surmised

$$C = m\int v\,ds = m\int \frac{ds}{dt}\frac{ds}{dt}\,dt = m\int v^2 dt$$

$$= \int 2T\,dt$$

So,

$$S = \int 2T\,dt - \int H\,dt$$
$$= \int 2T - (T+u)\,dt$$
$$= \int \underset{\sim}{(T-u)}\,dt$$

now called the Lagrangian.
Helmholtz: "Free Energy"

H now "Hamiltonian"

The demand $\delta S = 0$ results in $\int L\,dt = 0$ is Hamilton's Principle.

All of mechanics, not so much optics any more, follows.

So, from the general into,

$$F \rightarrow L$$
$$x \rightarrow t$$
$$y \rightarrow x \qquad x = x(t)$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$ one particle dynamics

ie For conservative systems, $V = V(x)$ only

$T = T(\dot{x})$ only

$$\frac{\partial L}{\partial x} = \frac{\partial V}{\partial x}$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} \qquad \sim \qquad \text{E-L:} \quad \frac{\partial V}{\partial x} - \frac{d}{dt}\frac{\partial T}{\partial \dot{x}} = 0$$

$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$

since $T = \frac{1}{2}m\dot{x}^2$ $\quad \frac{\partial T}{\partial \dot{x}} = m\dot{x}$; $\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p$

$$\frac{\partial V}{\partial x} - m\ddot{x} = 0 \Rightarrow m\ddot{x} = -\frac{\partial V}{\partial x} = f$$ N 2nd Law.

Two generalizations

1) non-conservative forces. — don't need.

2) "generalized coordinates" idea of Lagrange's. need