Coordinate transformation is often ~~the~~ necessary and also, constraints figure in. For N particles with k constraints, the system can be described by $3N$ Cartesian coordinates $(i = x, y, z\ ;\ \rho = 1 \to N)$ -- then the system can be described by $3N-k$ generalized coordinates -- maybe nothing to do with dimensions of length.

$$\vec{x} = \vec{x}(q_1 q_2 q_3\, t)$$

For each $\qquad x_i = x_i(q_1 q_2 q_3\, t)$, generally --

or $\qquad x_{\rho,i} = x_{\rho i}(q_1 q_2 \ldots q_{3N}\, t)$

$$q_m = q_m(x_1 x_2 x_3\, t)$$

$$q_m = q_m(x_{1,1}, x_{1,2} \ldots x_{N,3}, t)$$

a small displacement, with t held fixed. $\quad = q_m(\vec{x}_1, \vec{x}_2 \ldots \vec{x}_N, t)$

$$\delta x_i = \frac{\partial x_i}{\partial q_m}\, \delta q_m$$

If the Jacobian determinant is $\neq 0$, $\left|\frac{\partial x_i}{\partial q_m}\right| \neq 0$,

then go back,

$$\delta q_m = \frac{\partial q_m}{\partial x_i}\, \delta x_i$$

and one could integrate to get q as function of x.

For coordinates varying in time,

$$\dot{x}_i = \frac{dx_i}{dt} \equiv v_i$$

or $$\dot{x}_{pi} = \frac{\partial x_{pi}}{\partial q_m}\dot{q}_m + \underbrace{\frac{\partial x_{pi}}{\partial t}}_{=0}$$ if no explicit t dep.

and
$$\ddot{x}_{pi} = \frac{\partial x_{pi}}{\partial q_m}\ddot{q}_m + \frac{\partial^2 x_{pi}}{\partial q_m \partial q_n}\dot{q}_m\dot{q}_n + \underbrace{2\frac{\partial^2 x_{pi}}{\partial q_m \partial t}\dot{q}_m + \frac{\partial^2 x_{pi}}{\partial t^2}}_{=0}$$

if no explicit t dependence.

If q_m and $\dot{q}_m$ are independent (okay, since dq_m in $\dot{q}_m$ don't depend on q_m)

$$\frac{\partial \dot{x}_{pi}}{\partial \dot{q}_m} = \frac{\partial x_{pi}}{\partial q_m}$$

The equations of motion, are

$$F_{pi} = m_p \ddot{x}_{pi}$$

Calculate work done by F_{pi} over displacement dx_{pi}

$$W = \sum_{pj} F_{pj}\, dx_{pj}$$

$$= \sum_{pi} m_p \ddot{x}_{pi}\, dx_{pi}$$

$$= \sum_{pi} m_p \ddot{x}_{pi} \left(\frac{\partial x_{pi}}{\partial q_m} dq_m + \frac{\partial x_{pi}}{\partial t} dt \right) \quad \text{A}$$

$$= \sum_{\rho i} m_\rho \left[\frac{d}{dt}\left(\dot{x}_{\rho i} \frac{\partial \dot{x}_{\rho i}}{\partial \dot{q}_m} \right) - \dot{x}_{\rho i} \frac{d}{dt} \frac{\partial x_{\rho i}}{\partial q_m} \right] dq_m$$

$$+ \sum_{\rho i} m_\rho \ddot{x}_{\rho i} \frac{\partial x_{\rho i}}{\partial t} dt$$

$$W = \sum_{\rho i} \left[\frac{d}{dt} \underline{\frac{\partial}{\partial \dot{q}_m}} \underbrace{\left(\tfrac{1}{2} m_\rho \dot{x}_{\rho i} \dot{x}_{\rho i} \right)}_{T} - \underline{\frac{\partial}{\partial q_m}} \underbrace{\left(\tfrac{1}{2} m_\rho \dot{x}_{\rho i} \dot{x}_{\rho i} \right)}_{T} \right] dq_m$$

$$+ \sum_{\rho i} m_\rho \ddot{x}_{\rho i} \frac{\partial x_{\rho i}}{\partial t} dt$$

From WETh -- the work done is related to ΔT

so,

$$W = \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_m} - \frac{\partial T}{\partial q_m} \right) dq_m + \sum_{\rho i} m_\rho \ddot{x}_{\rho i} \frac{\partial x_{\rho i}}{\partial t} dt$$

The original expression for W can be written,

$$W = \sum_{\rho i} F_{\rho i} \left(\frac{\partial x_{\rho i}}{\partial q_m} dq_m + \frac{\partial x_{\rho i}}{\partial t} dt \right) \qquad ☆$$

$$= Q_m dq_m + Q_t dt$$

$$Q_m \equiv \sum_{\rho i} F_{\rho i} \frac{\partial x_{\rho i}}{\partial q_m} \qquad Q_t = \sum F_{\rho i} \frac{\partial x_{\rho i}}{\partial t}$$

<u>generalized</u> <u>force</u> <u>components</u>.

Two expressions for work are equal--

so,

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_m} - \frac{\partial T}{\partial q_m} = Q_m \qquad \text{Euler Lagrange equations}$$

only applied forces, not constraint forces.

If: – no constraint involves time
– no moving coordinate system.

$$\Rightarrow \quad T = T(\dot{q}_m)$$

so,
$$\dot{x}_{pi} = \frac{\partial x_{pi}}{\partial q_m}\dot{q}_m$$

$$T = \frac{1}{2}\sum_{pi} m_p (\dot{x}_{pi})^2$$

$$= \frac{1}{2} a_{mn}\dot{q}_m\dot{q}_n$$

where
$$a_{nm} = \sum m_p \frac{\partial x_{pi}}{\partial q_m}\frac{\partial x_{pi}}{\partial q_n}$$

If conservative forces in F_{pi}, or Q_m, then

$$F_{pj} = -\frac{\partial U}{\partial x_{pj}}$$

so

$$Q_m = \sum F_{pj}\frac{\partial x_{pj}}{\partial q_m}$$

$$= -\sum_{pj}\frac{\partial U}{\partial x_{pj}}\frac{\partial x_{pj}}{\partial q_m}$$

$$= -\frac{\partial U}{\partial q_m}$$

and EL eq.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_m} - \frac{\partial T}{\partial q_m} + \frac{\partial U}{\partial q_m} = 0$$

$$= \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_m} - \frac{\partial}{\partial q_m}(T-U) = 0$$

and we're back to defining $L \equiv T-U$.
and writing,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m} - \frac{\partial L}{\partial q_m} = 0$$

So, $L(x) \longrightarrow L(q,\dot{q}) = T(q,\dot{q}) - U(q)$

just as for Cartesian coordinates,

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p, \text{ the momentum.}$$

for generalized coordinates, define:

generalized momentum $p_m \equiv \frac{\partial L}{\partial \dot{q}_m}$. p not regular momentum

This comes into play in Hamiltonian representation –

$$\cancel{H = \sum_{m=1}^{N} p_m \dot{q}_m - L(q,p)}$$

We've established the Euler-Lagrange Equations for generalized coordinates.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m} - \frac{\partial L}{\partial q_m} = 0 \qquad \text{one equation for each dof}$$

Further, we've established that we get the Euler Lagrange equations through Hamilton's Principle, originally interpreted as:

The motion of a system between times t_1 and t_2 is such that the line integral

$$S = \int_{t_1}^{t_2} (T - U)\,dt = \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt$$

is an extremum for the actual path of motion.

In general, the generalized momentum which is canonically conjugate to q_m is

$$p_m \equiv \frac{\partial L}{\partial \dot{q}_m}$$

for a particular particle of conservative forces. — 32

Consider

$$\frac{dL}{dt} = \sum_m \underbrace{\frac{\partial L}{\partial q_m}}_{\frac{d}{dt}\frac{\partial L}{\partial \dot q_m}} \frac{dq_m}{dt} + \frac{\partial L}{\partial \dot q_m}\frac{d\dot q_m}{dt} \quad \underset{\text{not explicit function of time.}}{= 0}$$

EL eq.

$$\frac{dL}{dt} = \sum_m \frac{d}{dt}\frac{\partial L}{\partial \dot q_m}\dot q_m + \frac{\partial L}{\partial \dot q_m}\frac{d\dot q_m}{dt}$$

$$= \sum_m \frac{d}{dt}\left(\dot q_m \frac{\partial L}{\partial \dot q_m}\right)$$

so

$$\frac{d}{dt}\underbrace{\left(L - \sum_m \dot q_m \frac{\partial L}{\partial \dot q_m}\right)}_{=} = 0$$

so

constant $= -H$

which can be turned around $\quad H = \sum_{m=1}^{N} p_m \dot q_m - L(q,p)$

— backwards: $\quad dH = \dot q_m dp_m + p_m d\dot q_m - \underbrace{\frac{\partial L}{\partial \dot q_m}}_{p_m} d\dot q_m - \frac{\partial L}{\partial q_m} dq_m$

$$= \dot q_m dp_m - \frac{\partial L}{\partial q_m} dq_m$$

- EL eq.

$$dH = \dot q_m dp_m - \dot p_m dq_m$$

~~looks like Heisenberg Equations of motion~~

so, H depends on p and q only. Generally,

$$dH = \frac{\partial H}{\partial p_m} dp_m + \frac{\partial H}{\partial q_m} dq_m$$

so we get by identification alone,

$$\left.\begin{aligned} \frac{\partial H}{\partial p_m} &= \dot{q}_m = \frac{dq_m}{dt} \\ \frac{\partial H}{\partial q_m} &= -\dot{p}_m = -\frac{dp_m}{dt} \end{aligned}\right\} \text{Hamilton's Equations.}$$

A hint of symmetry comes from noting—

$$\frac{dH}{dt} = \frac{\partial H}{\partial p_m}\frac{dp_m}{dt} + \frac{\partial H}{\partial q_m}\frac{dq_m}{dt}$$

$$= \dot{q}_m \dot{p}_m - \dot{p}_m \dot{q}_m = 0$$ which looks a lot like the Heisenberg equations of motion

As a trivial, but ubiquitous example: The 1d Harmonic Oscillator—

$$T = \tfrac{1}{2} m \dot{q}^2 \qquad U = \tfrac{1}{2} k q^2$$

$$L = T - U = \frac{m\dot{q}^2}{2} - \frac{kq^2}{2}$$

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \qquad \frac{\partial L}{\partial q} = -kq$$

so, $$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0$$ gives

$$-kq - \frac{d}{dt} m\dot q = 0$$

$$m\ddot q + kq = 0$$

For $\omega^2 = k/m$ $\quad \ddot q + \omega^2 q = 0 \quad \Rightarrow$ get back the Newtonian result.

For each dof, you get this result

The canonical momentum is $\quad p = \frac{\partial L}{\partial \dot q} = m\dot q$

and we get

$$H = p\dot q - L$$
$$= m\dot q^2 - \frac{m\dot q^2}{2} + \frac{kq^2}{2}$$

$$H = \tfrac{1}{2} m\dot q^2 + \frac{kq^2}{2}$$

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{kq^2}{2}$$ .. the total energy.

Hamilton's equations follow –

$$\frac{\partial H}{\partial p} = \frac{\partial H}{\partial (m\dot q)} = \underbrace{\frac{p}{m} = \frac{dq}{dt}}_{} \longrightarrow p = m\dot q$$

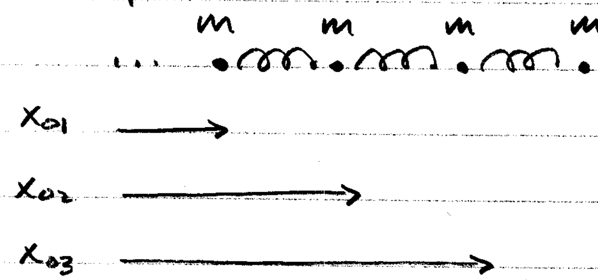
$$\frac{\partial H}{\partial q} = kq = -\dot p = -m\ddot q \longrightarrow m\ddot q = -kq$$

Newtonian result

The passage to continuous systems is relatively straightforward by identifying T & U in the limit..

Suppose we have a rod under compressional (or stretching) forces --

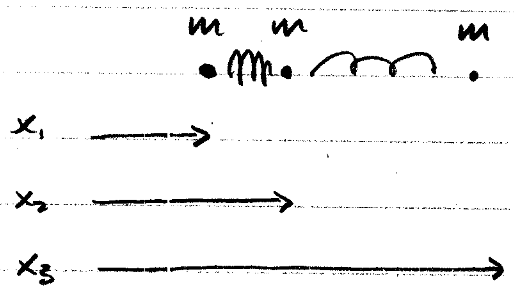
model as N point masses.--



relaxed.--

$$\left.\begin{array}{l} x_{02} - x_{01} = a \\ x_{03} - x_{02} = a \\ \text{etc} \end{array}\right.$$

an arbitrary unrelaxed situation might be,



Overall length: $(N-1)a$

mass : Nm

density: $\dfrac{Nm}{(N-1)a} = \dfrac{1}{(1-1/N)} \dfrac{m}{a}$

as $N \to$ large $\to m/a$

potential energy is

$$U = \tfrac{1}{2}h\,(x_2 - x_1 - a)^2 + \tfrac{1}{2}h\,(x_3 - x_2 - a)^2 + \cdots \tfrac{1}{2}h\,(x_N - x_{N-1} - a)^2$$

$$= \tfrac{1}{2}h \sum_{i=1}^{N-1} (x_{i+1} - x_i - a)^2$$

Use relative coordinates...

$$\eta_1 \equiv x_1 - x_{01} \quad \Rightarrow \quad x_1 = x_{01} + \eta_1$$
$$\eta_2 \equiv x_2 - x_{02} \quad \Rightarrow \quad x_2 = x_{02} + \eta_2$$
$$\vdots$$
$$\eta_i \equiv x_i - x_{0i} \quad \Rightarrow \quad x_i = x_{0i} + \eta_i$$

so

$$x_2 - x_1 - a = \underbrace{x_{02} + \eta_2 - x_{01}}_{\equiv a} - \eta_1 - a$$
$$= \eta_2 - \eta_1$$

etc

and $$U = \tfrac{1}{2}h \sum_{i=1}^{N-1} (\eta_{i+1} - \eta_i)^2$$

our generalized, or canonical coordinates, will be the $(\eta - a)$ displacement, or <u>disturbance</u>.

this idea of disturbance or excitation will reoccur lots in q. field theory

Kinetic energy - $$T = \tfrac{1}{2} \sum_{i=1}^{N} m \dot{x}_i^2$$

$$= \tfrac{1}{2} m \sum_{i=}^{N} \dot{\eta}_i^2$$

and

$$L = T - U$$
$$= \tfrac{1}{2} m \sum_{i=1}^{N} \dot{\eta}_i^2 - \tfrac{1}{2} h \sum_{i=1}^{N-1} (\eta_{i+1} - \eta_i)^2$$

note: $\frac{\partial L}{\partial \dot\eta_1} = m\dot\eta_1 \qquad \frac{\partial L}{\partial \eta_1} = k(\eta_2-\eta_1)$

and we could write, for the #1 link, EL eq:

$$\frac{\partial L}{\partial \eta_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot\eta_1} = 0$$

$$k(\eta_2-\eta_1) - m\ddot\eta_1 = 0 \qquad \text{coupled to } \frac{2}{3}$$

similarly: $\frac{\partial L}{\partial \dot\eta_2} = m\dot\eta_2 \qquad \frac{\partial L}{\partial \eta_2} = k(\eta_3-\eta_2) - k(\eta_2-\eta_1)$

$$\vdots \qquad\qquad \vdots$$

$$\frac{\partial L}{\partial \dot\eta_N} = m\dot\eta_N \qquad \frac{\partial L}{\partial \eta_N} = -k(\eta_N-\eta_{N-1})$$

multiply L by $1 = a/a$

$$L = \tfrac{1}{2} a \sum \frac{m}{a}\dot\eta_i^2 - \tfrac{1}{2} a \sum ka \frac{(\eta_{i+1}-\eta_i)^2}{a^2}$$

$$\frac{\partial L}{\partial \dot\eta_j} = a\left(\frac{m}{a}\right)\dot\eta_j$$

$$\frac{\partial L}{\partial \eta_j} = -\frac{a}{2}\sum_i ka \frac{\partial}{\partial \eta_j}\left(\frac{\eta_{i+1}-\eta_i}{a}\right)^2$$

$$= -a\sum_i ka\left(\frac{\eta_{i+1}-\eta_i}{a^2}\right)\left(\frac{\partial \eta_{i+1}}{\partial \eta_j} - \frac{\partial \eta_i}{\partial \eta_j}\right)$$

$$= -a\sum_i ka\left(\frac{\eta_{i+1}-\eta_i}{a^2}\right)\left(\delta_{j,i+1} - \delta_{ji}\right)$$

$$\frac{\partial L}{\partial \eta_j} = -ha^2\left[\left(\frac{\eta_j-\eta_{j-1}}{a^2}\right)-\left(\frac{\eta_{j+1}-\eta_j}{a^2}\right)\right]$$ not at ends, $j=1,N$

So,
EL eq:

$$a\left\{\frac{m}{a}\ddot{\eta}_j + ha\left[\left(\frac{\eta_j-\eta_{j-1}}{a^2}\right)-\left(\frac{\eta_{j+1}-\eta_j}{a^2}\right)\right]\right\}=0 \quad \forall_{j,N}$$

a set of coupled equations.

~~The continuum limit is~~

$a \to dx$, $m \to d\mu$ $\Big\}$ $\frac{m}{a} \to \rho$ (same as $N \to \infty$)

Suppose we try to extend one link..

$m_j \quad m_{j+1}$

generalized
↑ force required: $$F_j = -\frac{\partial V}{\partial \eta_j} = h(\eta_{j+1}-\eta_j)$$

$$= ha\frac{(\eta_{j+1}-\eta_j)}{a}$$

$$F_j = Y\xi_j$$

Young's Modulus $Y = ha$

ξ_j = elongation / unit length $\equiv \dfrac{\eta_{j+1}-\eta_j}{a}$

In continuum limit, discrete indexing goes away → a distribution in x,

$$a \to dx \quad \& \quad m \to dm \Rightarrow \frac{m}{a} \to \rho \text{, constant}$$

$$\eta_j \to \eta(x) \qquad [\text{in fact, } \eta(x,t)]$$

$$\xi_j \to \frac{\eta(x+a) - \eta(x)}{a} \to \frac{\partial \eta(x)}{\partial x}$$

$$\sum_i \to \int dx$$

nb: x (and t) is (are) parameters, not degrees of freedom

so,

$$L = \frac{1}{2} \int_{\text{system}} dx \left[\rho\, \dot{\eta}^2(x,t) - Y \left[\frac{\partial \eta(x,t)}{\partial x}\right]^2 \right]$$

From Hamilton's Principle,

$$S = \int_{t'}^{t''} dt\, L$$

$$\delta S = 0 = \delta \int_{t'}^{t''} dt\, L$$

$$= \delta \int_{t'}^{t''} dt \int_{x'}^{x''} dx\, \frac{1}{2} \left[\rho\, \dot{\eta}^2(x,t) - Y \left(\frac{\partial \eta}{\partial x}\right)^2 \right]$$

Define Lagrange Density, $\mathcal{L}$, such that

$$L = \int dx\, \mathcal{L} \Rightarrow \mathcal{L} = \frac{1}{2} \rho\, \dot{\eta}^2(x,t) - \frac{1}{2} Y \left(\frac{\partial \eta}{\partial x}\right)^2$$

so

$$\delta S = 0 = \delta \int_{t'}^{t''} dt \int_{x'}^{x''} dx\, \mathcal{L}\left(\eta, \dot{\eta}, \frac{\partial \eta}{\partial x}\right)$$

functional, no x or t dep.

Fix the end points

$$\delta S = \int dt \int dx \left\{ \underline{\frac{\partial \mathcal{L}}{\partial \eta}\delta\eta} + \overbrace{\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)}}^{B} \delta\left(\frac{\partial \eta}{\partial x}\right) + \overbrace{\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)}}^{A} \delta(\partial\eta/\partial t) \right\}$$

parts integration - twice eliminating the endpoint *

$$= \int dt \int dx \left\{ \underline{\frac{\partial \mathcal{L}}{\partial \eta}\delta\eta} - \frac{\partial}{\partial x}\left[\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)}\right]\delta\eta \right.$$

$$\left. - \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)}\right]\delta\eta \right\}$$

$$= \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial \eta} - \frac{\partial}{\partial x}\left[\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)}\right] - \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)}\right] \right\}\delta\eta$$

$$= 0$$

for arbitrary $\delta\eta$

$$\frac{\partial \mathcal{L}}{\partial \eta} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)}\right) = 0$$

Euler Lagrange Equation for continuous system.

For our system.

$$\frac{\partial \mathcal{L}}{\partial \eta} = 0 \; ; \quad \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)} = -Y\left(\frac{\partial \eta}{\partial x}\right) ; \quad \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)} = \rho \frac{\partial \eta}{\partial t}$$

A $\displaystyle\int \frac{\partial \mathcal{L}}{\partial \dot{\eta}} \delta \dot{\eta}\, dt = - \int \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) \delta \eta\, dt$ parts.

$\frac{1}{2}$ B $\displaystyle\int \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)} \delta\left(\frac{\partial \eta}{\partial x}\right) dx = \int \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)} \frac{\partial}{\partial x} \delta\eta\, dx$ parts.

$$= \frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)} \delta\eta \Big|_{\text{ends.}} - \int \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)}\right) \delta\eta\, dx$$

so, EL eq give,

$$Y \frac{\partial^2 \eta}{\partial x^2} - \rho \frac{\partial^2 \eta}{\partial t^2} = 0$$

a wave equation for disturbances which propagate with velocity $v = \sqrt{Y/\rho}$.

x is not a generalized coordinate ... the dynamics is about η.

We can also write a Hamiltonian formulation — by defining the Hamiltonian Density by

$$H = \int dx\, \mathcal{H}\left(\eta, \dot\eta, \frac{\partial \eta}{\partial x}\right)$$

where $$\mathcal{H} = \dot\eta \frac{\partial \mathcal{L}}{\partial \dot\eta} - \mathcal{L}$$

again, we have a canonical momentum conjugate to η

$$\pi(x,t) \equiv \frac{\partial \mathcal{L}}{\partial \dot\eta} = \rho \dot\eta$$ not a mechanical momentum

so

$$\mathcal{H} = \dot\eta \pi - \mathcal{L}$$

$$= \rho \dot\eta^2 - \frac{1}{2} \rho \dot\eta^2 + \frac{1}{2} Y \left(\frac{\partial \eta}{\partial x}\right)^2$$

$$\mathcal{H} = \frac{1}{2}\rho\dot{\eta}^2 + \frac{1}{2}Y\left(\frac{\partial\eta}{\partial x}\right)^2$$

$$= \frac{1}{2}\frac{\pi^2}{\rho} - \frac{1}{2}Y\left(\frac{\partial\eta}{\partial x}\right)^2$$

Ham Eq: ① $\frac{\partial\mathcal{H}}{\partial\pi} = \dot{\eta}$

$\qquad\qquad \parallel$

explicitly: $\frac{\pi}{\rho} \Rightarrow \pi = \rho\dot{\eta}$

② $\frac{\partial\mathcal{H}}{\partial\eta} = -\dot{\pi} = 0$

It's instructive to sketch a scalar field which is truly mechanical — <u>sound</u>.

longitudinal vibrations in a gas: (Goldstein)

generalized coordinates: $\vec{\eta}$, displacement

equilibrium pressure & density: P_0, ρ_0

pressure and density fluctuations are small.

$\mathcal{L} = \mathcal{T} - \mathcal{U}$ densities.

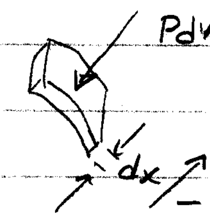
$$\mathcal{T} = \frac{\rho_0}{2}\dot{\vec{\eta}}^2 = \frac{\rho_0}{2}\left(\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2\right)$$

Potential energy is a measure of work that the gas can do in expanding against its pressure

$V_0 = \frac{M}{\rho_0} \Rightarrow \mathcal{U} = V_0 \mathcal{U}$ (potential energy) (V_0 small, so $\mathcal{U}$ constant in that volume)

the pressure disturbance causes $V_0 \to V_0 + \Delta V$

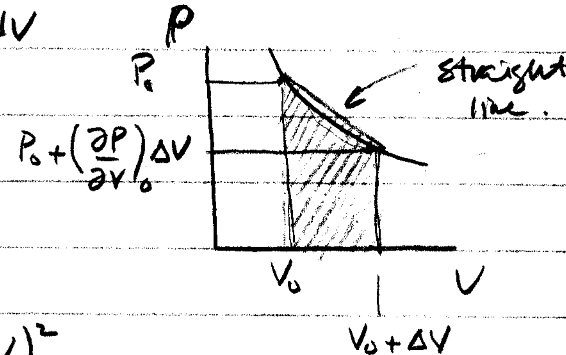
and since work is done on system, the potential energy increases $= -PdV$



$-PdAdx = dW$
$= -PdV$

so, PE

$$\mathcal{U} = \mathcal{U} V_0 = -\int_{V_0}^{V_0+\Delta V} PdV$$



$$\mathcal{U} = P_0 \Delta V + \frac{1}{2}\left(\frac{\partial P}{\partial V}\right)_0 (\Delta V)^2$$

need derivative

since contractions and expansions take place adiabatically, (not isothermally, so not Boyle's Law $PV = C$)

$$PV^\gamma = C \qquad \gamma = \frac{C_P}{C_V}$$

so $$\left(\frac{\partial P}{\partial V}\right)_0 = -\frac{\gamma P_0}{V_0}$$

also, $$\Delta V = -\frac{M}{\rho_0^2}\Delta\rho$$

where $$\rho = \rho_0(1+\sigma)$$ $\sigma = \frac{\rho - \rho_0}{\rho_0}$ fractional density change

$$\Delta V = -V_0 \sigma$$

so $$\mathcal{U} = \frac{P_0 \Delta V}{V_0} + \frac{1}{2}\left(-\frac{\gamma P_0}{V_0}\right)\frac{(\Delta V)^2}{V_0}$$

$$\mathcal{U} = -P_0\sigma + \frac{1}{2}\gamma P_0 \sigma^2$$

need σ in terms of η.

mass flowing out of small volume $= \rho_0 \int_V \vec{\eta}\cdot d\vec{A}$

which must be equal to mass transport as change in density

$$= -\rho_0 \int \sigma \, dV$$

Divergence theorem: $\quad -\int \sigma \, dV = \int \vec{\nabla}\cdot\vec{\eta}\, dV$

so $$\sigma = -\vec{\nabla}\cdot\vec{\eta}$$

and $$\mathcal{U} = -P_0 \vec{\nabla}\cdot\vec{\eta} + \frac{\gamma P_0}{2}(\vec{\nabla}\cdot\vec{\eta})^2$$

$$\mathcal{L} = \frac{1}{2}\left(\rho_0 \dot{\vec{\eta}}^2 + 2P_0 \vec{\nabla}\cdot\vec{\eta} - \gamma P_0 (\vec{\nabla}\cdot\vec{\eta})^2\right) = KE - PE$$

EL eq $-$ $$\frac{\partial \mathcal{L}}{\partial \dot{\eta}_j} = \rho_0 \dot{\eta}_j$$

$$\frac{\partial(\vec{\nabla}\cdot\vec{\eta})}{\partial\left(\frac{\partial \eta_j}{\partial x_k}\right)} = \delta_{jk}$$

$$\frac{\partial(\vec{\nabla}\cdot\vec{\eta})^2}{\partial\left(\frac{\partial \eta_j}{\partial x_k}\right)} = 2(\vec{\nabla}\cdot\vec{\eta})\,\delta_{jk}$$

so $$\rho_0 \frac{\partial^2 \eta_j}{\partial t^2} - \gamma P_0 \frac{\partial(\vec{\nabla}\cdot\vec{\eta})}{\partial x_j} = 0$$

as vector equation

$$\rho_0 \frac{\partial^2 \vec{\eta}}{\partial t^2} - \gamma P_0 \vec{\nabla}\, \vec{\nabla}\cdot\vec{\eta} = 0$$

take divergence,

$$\nabla^2 \sigma - \frac{\rho_0}{\gamma P_0} \frac{\partial^2 \sigma}{\partial t^2} = 0$$

a 3d wave equation in the fractional density w/

$$v = \sqrt{\frac{\gamma P_0}{\rho_0}} = v_s$$

NOTE: This started from KE and PE for a material substance — the gas (a rod, before).

* Doesn't need to be this way!

Suppose we get rid of the gas and just left behind the scalar quantity which is the value of σ at every x, y, z point —

→ scalar field, no mechanical system

go back and <u>find a $\mathcal{L}$</u> which gives you the right equation of motion. — THAT'S ALL YOU NEED TO DO.

Now $\underline{\underline{\sigma}}$ is the natural generalized coordinate for the field. This one works!

$$\mathcal{L} = \frac{1}{2}\left(\frac{\rho_0}{\gamma P_0} \dot{\sigma}^2 - (\vec{\nabla}\sigma)^2 \right)$$

$$= \frac{\rho_0}{2\gamma P_0} \dot{\sigma}^2 - \frac{1}{2} \sum_j \left(\frac{\partial \sigma}{\partial x_j} \right)^2 \longrightarrow \text{same equations of motion.}$$

not same $\mathcal{L}$ $\longrightarrow$ no mechanical kinetic or potential energies

this is where we start!