With that introduction we can similarly begin to create the Lagrangian formalism for relativistic, quantum fields.

We've encountered a scalar field, and we'll start there..

$\varphi(\vec{x}, t)$ .. which could have more than one component (distinct from space time components) but still transform as a Lorentz scalar.

$$\varphi_i(\vec{x}, t) = \varphi_i(x^\mu) = \varphi_i(x)$$

context $\Rightarrow$ spacetime

The action will be

$$S = \int_{t_1}^{t_2} dt \int_{\text{Volume}} d^3\vec{x}\ \mathcal{L}(\varphi_i, \partial_\mu \varphi_i)$$

The variation appropriate for Hamilton's Principle is

$$\delta\varphi_i = \varphi_i(x) - \varphi_i'(x) \qquad \text{vanishing at spacetime boundaries.}$$

$$\delta S = \int dt \int d^3\vec{x}\ \delta\mathcal{L} = 0 \qquad \text{in covariant form}$$

$$\delta\mathcal{L} = \sum_i \left\{ \frac{\partial \mathcal{L}}{\partial \varphi_i} \delta\varphi_i + \sum_\mu \frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^\mu)} \delta\left(\frac{\partial\varphi_i}{\partial x^\mu}\right) \right\}$$

interchange $\frac{\partial}{\partial x^\mu}$ and δ

$$\frac{\partial}{\partial x^\mu}(\delta\varphi_i) = \frac{\partial}{\partial x^\mu}(\varphi_i(x) - \varphi_i'(x)) = \delta\left(\frac{\partial\varphi_i}{\partial x^\mu}\right)$$

integrate the last term by parts,

$$\int dt \int d^3\vec{x}\ \frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^\mu)} \frac{\partial}{\partial x^\mu}(\delta\varphi_i) = \int dt \int d^3\vec{x}\ \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^\mu)}\delta\varphi_i\right]$$

$$- \int dt \int d^3\vec{x}\ \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^\mu)}\right]\delta\varphi_i$$

The first term: $= \int dt \int d^3\vec{x}\ \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial t)}\delta\varphi_i\right]$

$$- \int dt \int d^3\vec{x}\ \frac{\partial}{\partial x^j}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^j)}\delta\varphi_i\right]$$

like $\int d^3\vec{x}\ \vec{\nabla}\cdot\vec{F} = \int \vec{F}\cdot d\vec{A} = 0$

since $\delta\varphi_i = 0$ at surface.

integrate by dt : $\rightarrow \int d^3\vec{x} \left[\frac{\partial \mathcal{L}}{\partial(\frac{\partial\varphi_i}{\partial t})}\delta\varphi_i\right]_{t_1}^{t_2} = 0$

so,

$$\delta S = \int d^3\vec{x}\, dt \sum_i \left\{\frac{\partial \mathcal{L}}{\partial\varphi_i} - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^\mu)}\right]\right\}\delta\varphi_i = 0$$

and we get

$$\frac{\partial \mathcal{L}}{\partial\varphi_i} - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^\mu)}\right] = 0 \qquad \forall_i$$

the covariant form of E-L equation for scalar field.

Conjugate momentum field $\quad \Pi_i(x) \equiv \frac{\partial \mathcal{L}}{\partial\dot{\varphi}_i}$

.. nothing to do with momentum ...

The Hamiltonian density -- can be constructed as before.

We have an example of a simple, one-dimensional relativistic field — the Klein Gordon field, φ. We found the appropriate equation of motion to be

$$\left(\vec{\nabla}^2 - \frac{\partial^2}{\partial t^2} - m^2\right)\varphi(\vec{x},t) = 0$$

$$\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} - m^2\right)\phi(x) = 0$$

$$(\partial_\mu \partial^\mu - m^2)\,\phi(x) = 0$$

Guided by our previous examples, we might guess

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\,\partial^\mu\varphi - m^2\varphi^2\right)$$

$$= \frac{1}{2}\left(g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi - m^2\varphi^2\right)$$

$$= \frac{1}{2}\left(\dot{\varphi}^2 - \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi - m^2\varphi^2\right)$$

check: $$\frac{\partial \mathcal{L}}{\partial\varphi} = -m^2\varphi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\eta)} = \frac{1}{2}\left\{ g^{\mu\eta}\frac{\partial\varphi}{\partial x^\mu} + g^{\eta\nu}\frac{\partial\varphi}{\partial x^\nu}\right\} \qquad \text{general}$$

$$= g^{\eta\nu}\frac{\partial\varphi}{\partial x^\nu}$$

so, $$\frac{\partial}{\partial x^\eta}\left(\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\eta)}\right) = g^{\eta\nu}\frac{\partial^2\varphi}{\partial x^\eta \partial x^\nu}$$

$$= g^{00}\frac{\partial^2\varphi}{\partial x^0\partial x^0} + g^{11}\frac{\partial^2\varphi}{\partial x^1\partial x^1} + \cdots$$

$$= \frac{\partial^2\varphi}{\partial t^2} - \vec{\nabla}^2\varphi$$

and from E.L. eq.

$$\frac{\partial \mathcal{L}}{\partial\varphi} - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)}\right] = 0$$

$$-m^2\varphi - \frac{\partial^2\varphi}{\partial t^2} + \vec{\nabla}^2\varphi = 0 \qquad \checkmark$$

The Hamiltonian density:

$$\mathcal{H} = \pi\dot{\varphi} - \mathcal{L} \qquad\qquad \pi = \dot{\varphi}$$

$$= \dot{\varphi}^2 - \mathcal{L}$$

$$= \dot{\varphi}^2 - \frac{1}{2}\left[\dot{\varphi}^2 - \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi - m^2\varphi^2\right]$$

$$\mathcal{H} = \frac{1}{2}\left[\dot{\varphi}^2 + \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + m^2\varphi^2\right]$$

which is the energy density.

This is a positive quantity..! The Hamiltonian is

$$H = \int d^3x\,\mathcal{H}$$

Investigation of symmetries is important... and particularly easy in the Lagrangian framework

Suppose we consider a variation in φ induced by an arbitrary transformation

some function of the fields

$$\varphi \to \varphi' = \varphi + \epsilon(x) f(\varphi)$$

infinitesimal

$$\delta\varphi = \varphi' - \varphi = \epsilon(x) f(\varphi)$$

Our action is $\qquad S = \int dt \int d\vec{x}\, \mathcal{L}(\varphi, \partial_\mu \varphi)$

$$= \int d^4x\, \mathcal{L}(\varphi, \partial_\mu\varphi)$$

Using Hamilton's Principle, $\to$ not allowing $x^{\mu'} \to x^\mu + \delta x^\mu$ then $d^4x \to d^4x'$ etc. General discussion would do this.

$$\delta S = 0 = \int d^4x\, \delta\mathcal{L}$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)} \delta(\partial\varphi/\partial x^\mu) \right]$$

as before $\qquad \delta(\partial\varphi/\partial x^\mu) = \frac{\partial \varphi'}{\partial x^\mu} - \frac{\partial\varphi}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}(\varphi' - \varphi)$

$$= \frac{\partial(\delta\varphi)}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}(\epsilon f)$$

$$\delta S = \int d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)} \frac{\partial}{\partial x^\mu}(\delta\varphi) \right\}$$

$$= \int d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial}{\partial x^\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)}\delta\varphi\right] - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)}\right]\delta\varphi \right\}$$

$$= \int d^4x \left\{ \frac{\partial}{\partial x^\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)} \epsilon(x) f(\varphi)\right] \right.$$

ELU if φ satisfies an equation of motion

<u>Define</u> $$J^\mu(x) \equiv i \frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)} f(\varphi)$$

so $$\delta S = \int d^4x \frac{\partial}{\partial x^\mu}\left[J^\mu(x)\,\epsilon(x)\right] = 0$$

If ϵ is not a function of x^μ, then

$$\delta S = \int d^4x \frac{\partial}{\partial x^\mu}\left[J^\mu(x)\right]\epsilon = 0$$

so for arbitrary constant ϵ, a variation on φ leads to no change in dynamics (ie no path change $\Rightarrow \delta S = 0$) if

$$\frac{\partial J^\mu(x)}{\partial x^\mu} = 0$$

We call $J^\mu(x)$ a <u>current</u> and this statement says that it is a conserved current... defined in terms of a particular variation on the fields.

Go back,

$$\delta S = \int d^4x \left\{ \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial t)}\epsilon(x) f(\varphi)\right] + \frac{\partial}{\partial x^k}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^k)}\epsilon(x) f(\varphi)\right]\right\}$$

$$\text{second term} = \underbrace{\int dt \int d^3\vec{x}\, \frac{\partial}{\partial x^k}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^k)}\epsilon(x) f(\varphi)\right]}$$

from divergence theorem this = surface integral at boundaries $= 0$

$$\delta S = \int d^4x \left\{ \frac{\partial}{\partial t} \left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial t)} \epsilon(x) f(\varphi) \right] \right\}$$

again, let ϵ be constant

$$= \epsilon \int dt \int d^3\vec{x} \, \frac{\partial}{\partial t} \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial t)} f(\varphi)}_{\text{call this} \equiv \rho(x)} \Big]$$

$$= \epsilon \int dt \, \frac{\partial}{\partial t} \underbrace{\int d^3x \, \rho(x)}_{Q}$$

$$= \epsilon \int dt \left(\frac{\partial}{\partial t} Q \right)$$

where $\quad Q \equiv \int d^3x \, \rho(x) = \int d^3x \, J^0(x)$

then. $\quad \delta A = \epsilon \int_{t_1}^{t_2} dt \, \frac{\partial Q}{\partial t} = \epsilon \left[Q(t_2) - Q(t_1) \right] = 0$

ie Q doesn't change in time for arbitrary ϵ.

At this stage Q is just a function – later, an operator.

Noether's Theorem: Every continuous transformation leaving the Lagrangian invariant has, associated with it, a conserved current and a conservation law.

Two examples:

Simplest is a phase rotation. Notice, for our simple 1d Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi \, \partial^\mu \varphi - m^2 \varphi^2 \right)$$

one can see immediately that $\delta \mathcal{L} \neq 0$ for $\varphi \to e^{i\alpha}\varphi$ — so no conservation law and no conserved current. So, this is not a particularly useful theory.

The next simplest scalar theory is a 2d component complex field, degenerate in mass.

The Lagrangian is,

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{2} \left\{ \frac{\partial \varphi_i}{\partial x^\mu} \frac{\partial \varphi_i}{\partial x_\mu} - m^2 \varphi_i^2 \right\}$$

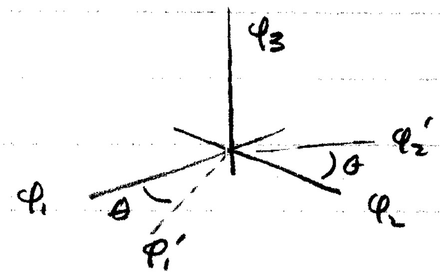
we could represent these as a vector, $\vec{\varphi} = \{\varphi_1, \varphi_2\}$

then,

$$\mathcal{L} = \frac{1}{2} \frac{\partial \vec{\varphi}}{\partial x^\mu} \cdot \frac{\partial \vec{\varphi}}{\partial x_\mu} - \frac{1}{2} m^2 \vec{\varphi} \cdot \vec{\varphi}$$

$\Rightarrow \quad \Box \varphi_i + m^2 \varphi_i = 0$ for each component is the equation of motion.

An obvious symmetry of this system is rotation in φ-space



$$\left.\begin{aligned}\varphi_1 \to \varphi_1' &= \varphi_1 \cos\theta - \varphi_2 \sin\theta \\ \varphi_2 \to \varphi_2' &= \varphi_1 \sin\theta + \varphi_2 \cos\theta\end{aligned}\right\} \text{mixing.}$$

This can't easily be put into the form we've been discussing.
of $\epsilon f(\phi) = \delta\varphi$

Another way to express this theory,

$$\left.\begin{aligned}\varphi &= \sqrt{1/2}\,(\varphi_1 + i\varphi_2) \\ \varphi^* &= \sqrt{1/2}\,(\varphi_1 - i\varphi_2)\end{aligned}\right\} \begin{aligned}&\varphi, \varphi^* \text{ — same K.G. eq.} \\ &\text{as } \varphi_1 \text{ and } \varphi_2.\end{aligned}$$

The Lagrangian:

$$\mathcal{L} = \left(\frac{\partial\varphi}{\partial x^\mu}\right)^* \left(\frac{\partial\varphi}{\partial x_\mu}\right) - m^2\varphi^*\varphi$$

The continuous transformation of interest is,

$$\varphi \to \varphi' = e^{-i\alpha}\varphi$$
$$\varphi^* \to \varphi^{*\prime} = e^{i\alpha}\varphi^*$$

so, we can write the transformation as an infinitesimal

$$\varphi \to \varphi' = (1 - i\alpha)\varphi \;\Rightarrow\; \epsilon = -i\alpha \quad f(\varphi) = \varphi$$
$$\varphi^* \to \varphi'^* = (1 + i\alpha)\varphi^* \;\Rightarrow\; \epsilon^* = i\alpha \quad f(\varphi) = \varphi^*$$

The Hamiltonian density is gotten from

$$\mathcal{L} = \frac{\partial \varphi^*}{\partial t}\frac{\partial \varphi}{\partial t} - \vec{\nabla}\varphi^* \cdot \vec{\nabla}\varphi - m^2 \varphi^* \varphi$$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial t)} = \frac{\partial \varphi^*}{\partial t} = \dot{\varphi}^* = \sqrt{1/2}\,(\dot{\varphi}_1 - i\dot{\varphi}_2)$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial\varphi^*/\partial t)} = \frac{\partial \varphi}{\partial t} = \dot{\varphi} = \sqrt{1/2}\,(\dot{\varphi}_1 + i\dot{\varphi}_2)$$

since, obviously,

$$\pi_{1,2} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_{1,2}} \Rightarrow \begin{array}{l} \pi = \sqrt{1/2}\,(\pi_1 - i\pi_2) \\ \pi^* = \sqrt{1/2}\,(\pi_1 + i\pi_2) \end{array}$$

so,

$$\mathcal{H} = \sum_{i=1}^{2} \pi_i \dot{\varphi}_i - \mathcal{L}$$

$$= \pi_1\dot{\varphi}_1 + \pi_2\dot{\varphi}_2 - \mathcal{L}$$

$$= \sqrt{1/2}\,(\pi+\pi^*)\sqrt{1/2}\,(\dot{\varphi}+\dot{\varphi}^*) + \sqrt{1/2}\,(\pi-\pi^*)\sqrt{1/2}\,(\dot{\varphi}-\dot{\varphi}^*) - \mathcal{L}$$

$$= (1/2)\,(\pi\dot{\varphi} + \pi\dot{\varphi}^* + \pi^*\dot{\varphi}^* + \pi^*\dot{\varphi}^*)$$

$$+ 1/2\,(\pi\dot{\varphi} - \pi\dot{\varphi}^* - \pi^*\dot{\varphi} + \pi^*\dot{\varphi}^*) - \mathcal{L}$$

$$= \pi\dot{\varphi} + \pi^*\dot{\varphi}^* - \mathcal{L}$$

$$= \pi\pi^* + \pi^*\pi^* - \dot{\varphi}^*\dot{\varphi} + \vec{\nabla}\varphi^* \cdot \vec{\nabla}\varphi + m^2\varphi^*\varphi$$

$$= \pi\pi^* + \pi^*\pi - \pi\pi^* + \vec{\nabla}\varphi^* \cdot \vec{\nabla}\varphi + m^2\varphi^*\varphi.$$

$$\mathcal{H} = \pi^*\pi + \vec{\nabla}\varphi^* \cdot \vec{\nabla}\varphi + m^2\varphi^*\varphi$$

— we'll need this later —

everything we did w/ single component field holds for 2 component field — just a Σ.

So, Hamilton's Principle becomes,

$$\delta S = \int d^4x \left\{ \frac{\partial}{\partial x^\mu} \left[\sum_i \frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^\mu)} \delta\varphi_i \right] \right\} = 0$$

$$= \int d^4x \, \frac{\partial}{\partial x^\mu} \left\{ \frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial\varphi^*/\partial x^\mu)} \delta\varphi^* \right\} = 0$$

$$= i\alpha \int d^4x \, \frac{\partial}{\partial x^\mu} \left\{ \frac{-\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)} \varphi + \frac{\partial \mathcal{L}}{\partial(\partial\varphi^*/\partial x^\mu)} \varphi^* \right\} = 0$$

and we identify,

$$J^\mu(x) = i \left\{ \frac{\partial \mathcal{L}}{\partial(\partial\varphi^*/\partial x^\mu)} \varphi^* - \frac{\partial \mathcal{L}}{\partial(\partial\varphi^*/\partial x^\mu)} \varphi \right\}$$

so that $\quad \partial_\mu J^\mu(x) = 0.$

and specifically,

$$J^\mu(x) = i \left\{ \frac{\partial\varphi}{\partial x^\mu} \varphi^* - \frac{\partial\varphi^*}{\partial x^\mu} \varphi \right\} = [J_0, \vec{J}]$$

so $\quad J^0 = i \left\{ \frac{\partial\varphi}{\partial t} \varphi^* - \frac{\partial\varphi^*}{\partial t} \varphi \right\} = -i \left[\pi^* \varphi^* - \pi \varphi \right]$

$$J^j = i \left\{ -\frac{\partial\varphi}{\partial x^j} \varphi^* + \frac{\partial\varphi^*}{\partial x^j} \varphi \right\}$$

$$\vec{J} = i \left[-\varphi^* \vec{\nabla}\varphi + \varphi \vec{\nabla}\varphi^* \right]$$

Again, the explicit vanishing of the space piece at the surface of δS leaves,

$$\delta S = i\alpha \int d^4x \, \frac{\partial}{\partial t} \left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi^*/\partial t)} \varphi^* - \frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial t)} \varphi \right]$$

$$= i\alpha \int d^4x \, \frac{\partial}{\partial t} \left[\pi^* \varphi^* - \pi \varphi \right] = 0$$

so we again identify,

$$i \left[\pi^* \varphi^* - \pi \varphi \right] = \rho(x) = J^0$$

and find
$$\delta S = \alpha \int dt \, \frac{\partial}{\partial t} \int d^3\vec{x} \, \rho(x)$$

$$= \alpha \int dt \, \frac{\partial Q}{\partial t}$$

This leads to a continuity equation, if all conserved.

So,

$$\vec{\nabla} \cdot \vec{J} = i \left\{ -\vec{\nabla}\varphi^* \cdot \vec{\nabla}\varphi + \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi^* + \varphi \nabla^2 \varphi^* - \varphi^* \nabla^2 \varphi \right\}$$
$$= i \left[\varphi \nabla^2 \varphi^* - \varphi^* \nabla^2 \varphi \right]$$

and
$$\frac{\partial J^0}{\partial t} = i \left[\dot{\pi}^* \varphi^* - \dot{\pi} \varphi + \pi^* \dot{\varphi}^* - \pi \dot{\varphi} \right]$$

$$= i \left[\dot{\pi}^* \varphi^* - \dot{\pi} \varphi + \cancel{\pi^* \pi} - \pi \pi^* \right.$$

$$= i \left(\dot{\pi}^* \varphi^* - \dot{\pi} \varphi \right)$$

So,

$$\frac{\partial J^0}{\partial t} + \vec{\nabla}\cdot\vec{J} = i\left[\dot{\pi}^*\varphi^* - \dot{\pi}\varphi + \varphi\nabla^2\varphi^* - \varphi^*\nabla^2\varphi\right]$$

$$= i\left[\ddot{\varphi}\varphi^* - \ddot{\varphi}^*\varphi + \varphi\nabla^2\varphi^* - \varphi^*\nabla^2\varphi\right]$$

$$= i\left[\varphi\left(\nabla^2\varphi^* - \frac{\partial^2}{\partial t^2}\varphi^*\right) - \varphi^*\left(\nabla^2\varphi - \frac{\partial}{\partial t^2}\varphi\right)\right]$$

$$= i\left[\varphi m^2\varphi^* - \varphi^* m^2\varphi\right] = 0$$

so $$\frac{\partial J^0}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

The lesson from Emmy Noether was that the conserved current → a conservation law. Whenever one has a complex theory like this and a photon, the current that is conserved is the electromagnetic current and the charge is electromagnetic charge.

Consider a different kind of transformation.

spacetime : $x^\mu \to x^{\mu'} = x^\mu + \xi^\mu$

infinitesimal.

This can induce a change in the field

$$\delta\varphi \to \varphi(x^\mu + \xi^\mu) - \varphi(x^\mu)$$

Taylor-expand $\delta\varphi = \varphi(x) + \xi^\mu \frac{\partial\varphi(x)}{\partial x^\mu} + \cdots - \varphi(x)$

$$\delta\varphi = \xi^\mu \partial_\mu \varphi(x)$$

This in turn induces a change in the Lagrange Density,

$$\delta\mathcal{L} = \mathcal{L}[\varphi(x'), \partial_\mu\varphi(x')] - \mathcal{L}[\varphi(x), \partial_\mu\varphi(x)]$$

$$= \mathcal{L}' - \mathcal{L} \qquad \text{but}$$

$$= \frac{\partial\mathcal{L}}{\partial x^\mu}\,\delta x^\mu \qquad \text{so}$$

$$\delta\mathcal{L} = \xi^\mu \frac{\partial\mathcal{L}}{\partial x^\mu}$$

also

$$= \frac{\partial\mathcal{L}}{\partial\varphi}\,\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)}\,\delta(\partial\varphi/\partial x^\mu)$$

as usual, $\delta(\partial_\mu\varphi) = \partial_\mu\,\delta\varphi$

$$= \frac{\partial}{\partial x^\mu}\left(\xi^\nu \frac{\partial\varphi}{\partial x^\nu}\right)$$

$$= \xi^\nu\,\partial_\mu\partial_\nu\varphi$$

so...

$$\delta\mathcal{L} = \xi^\mu\frac{\partial\mathcal{L}}{\partial x^\mu} = \frac{\partial\mathcal{L}}{\partial\varphi}\,\xi^\mu\frac{\partial\varphi}{\partial x^\mu} + \frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)}\,\xi^\nu\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\varphi$$

$$= \frac{\partial}{\partial x^\nu}\left[\frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial x^\nu)}\right]\xi^\mu\frac{\partial\varphi}{\partial x^\mu} + \frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)}\,\xi^\nu\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu}\varphi$$

since ξ^ν is a constant spacetime shift,

$$\delta\mathcal{L} = \xi^\mu\frac{\partial\mathcal{L}}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)}\,\xi^\nu\frac{\partial\varphi}{\partial x^\nu}\right]$$

$$0 = \xi^\nu\frac{\partial}{\partial x^\mu}\left[\quad\right] - \xi^\mu\frac{\partial\mathcal{L}}{\partial x^\mu}$$

$$= \xi^\nu\frac{\partial}{\partial x^\mu}\left[\quad\right] - \xi_\nu g^{\mu\nu}\frac{\partial\mathcal{L}}{\partial x^\mu} =$$

$$0 = \xi_\nu\frac{\partial}{\partial x^\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)}\frac{\partial\varphi}{\partial x_\nu} - g^{\mu\nu}\mathcal{L}\right]$$

$$0 = \xi_\nu \frac{\partial}{\partial x^\mu} \theta^{\mu\nu}$$

for arbitrary ξ, $\quad \dfrac{\partial \theta^{\mu\nu}}{\partial x^\mu} = 0$

where the conserved Noether current is

$$\theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)} \frac{\partial \varphi}{\partial x_\nu} - g^{\mu\nu}\mathcal{L}$$

Look at $\int \theta^{\mu\nu} d^3x$, where $\int \dfrac{\partial \theta^{\mu\nu}}{\partial x^\mu} d^3x = 0$

$$\int \frac{\partial \theta^{0\nu}}{\partial t} d^3x - \int \underbrace{\underbrace{\frac{\partial \theta^{i\nu}}{\partial x^i} d^3x}_{\vec{\nabla}\cdot\vec{\theta}\, d^3x}}_{\theta^{i\nu} dA_i \;=\; 0 \text{ @ surface}} = 0$$

so,

$$\frac{\partial}{\partial t} \int \theta^{0\nu} d^3x = 0 \quad \Rightarrow \quad \int \theta^{0\nu} d^3x \text{ is a constant}$$

of the motion — the conserved "charge" of Noether's theory.

n.b. $\theta^{0\nu}$ does not transform like a 4-vector.
$\int \theta^{0\nu} d^3x$ does

what about $$\theta^{00} = \frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)} \frac{\partial \varphi}{\partial x_\nu} - g^{\mu\nu}\mathcal{L} \Big|_{\mu=\nu=0}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)} = g^{\mu\eta}\frac{\partial\varphi}{\partial x^\eta} \quad \text{for our theory}$$

$$= g^{\mu\eta}\frac{\partial\varphi}{\partial x^\eta}\frac{\partial\varphi}{\partial x_\nu} - g^{\mu\nu}\mathcal{L} \Big|_{\mu=\nu=0}$$

$$= g^{0\eta}\frac{\partial\varphi}{\partial x^\eta}\frac{\partial\varphi}{\partial x_0} - \mathcal{L}$$

$$= \frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial t} - \mathcal{L} = \pi\dot{\varphi} - \mathcal{L}$$

which we recognize as the $\mathcal{H}$ for this theory.

$$\theta^{00} = \mathcal{H}$$

The continuous system leads to a conserved current, $\theta^{\mu\nu}$ and a conserved charge $\theta^{0\nu}$. The $\nu=0$ component of this charge is the Hamiltonian density $\Rightarrow$ the conservation law is energy.

What about θ^{0i} __

$$\theta^{0i} = \frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^0)}\frac{\partial\varphi}{\partial x_i} - 0$$

$$= \frac{\partial \mathcal{L}}{\partial(\dot{\varphi})}\frac{\partial\varphi}{\partial x_i}$$

$$= \pi\frac{\partial\varphi}{\partial x_i}$$

If $\int d^3x\, \theta^{0\nu}$ is a 4 vector...

$$\int d^3 \theta^{00} = \int d^3 \mathcal{H} = H \quad \text{is energy} ..$$

$\int d^3x\, \theta^{0i}$ must be momentum.

The whole 4 vector is

$$P^\nu = \int d^3x\, \theta^{0\nu}$$

and the whole $\theta^{\mu\nu}$ is the Stress Energy Tensor.

BUT remember from the Poincare Group : P^ν is generator.

the "Noether" charge is both → generator of the Lie group
→ constant of the motion

Noether demonstrated this

Lagrangians for spin ½ and spin 1 fields can be easily obtained. – Again, it's the equation of motion that counts.

Fermions –

The Dirac Equation leads to the following

$$\mathcal{L}(x) = \bar{\Psi}(x) \left(i \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \Psi(x) = \bar{\Psi}(x) [i \not{\partial} - m] \Psi(x)$$

keeping track of the <u>matrix</u> <u>indices</u> –

$$\mathcal{L}(x) = \bar{\Psi}_j(x) \left[i \gamma^\mu_{jk} \frac{\partial}{\partial x^\mu} - \delta_{jk} m \right] \Psi_k(x)$$

$\mu : 0-3$

$j, k = 1-4$

The Euler Lagrange prescription gives,

$$\bar{\Psi}(x) \left[i \frac{\overleftarrow{\partial}}{\partial x^\mu} \gamma^\mu + m \right] = 0$$

$$\left[i \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right] \Psi(x) = 0$$

The momentum conjugates are,

$$\pi_j(x) = i \Psi^+_j(x) \qquad \bar{\pi}_j(x) = 0$$

so that

$$\mathcal{H} = \Psi^+(x) (\vec{\alpha} \cdot \vec{p} + \beta m) \Psi(x)$$