Lecture 6 Spin 1

For spin 1, .. go back.

The measurable quantities for E&M are $\vec{E}$ and $\vec{B}$

Maxwell's equations:

(A) $\vec{\nabla}\cdot\vec{E} = \rho$ $\qquad$ $\vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}$

(B) $\vec{\nabla}\cdot\vec{B} = 0$ $\qquad$ $\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

The potentials are obtained as derived quantities by

$$\left.\vec{B} = \vec{\nabla}\times\vec{A} \quad \& \quad \vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}\right\} \begin{array}{l}\text{automatically} \Rightarrow \\ \text{(B) M.E.'s}\end{array}$$

Although its not necessary, it's useful and standard to deal with the vector and scalar potentials in a field-theoretic framework, founded on a Lagrangian formalism.

Let's establish a covariant formalism for E&M.

presume $\qquad A^\mu = [\phi, \vec{A}] \quad \& \quad j^\mu = [\rho, \vec{J}]$

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}A^0 - \frac{\partial \vec{A}}{\partial x^0} \quad :$$

remember that $\quad \partial_\mu \equiv \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$

so

$$E^j = -g^{jk}\frac{\partial A^0}{\partial x^k} - \frac{\partial A^j}{\partial x^0} = +\frac{\partial A^0}{\partial x_j} - \frac{\partial A^j}{\partial x_0} \equiv F^{0j}$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad :$$

$$B^i = \varepsilon^{jki} \frac{\partial A_j}{\partial x^k} = \varepsilon^{jki} g_{jm} \frac{\partial A^m}{\partial x^k}$$

eg

$$B^1 = \varepsilon^{jk1} g_{jm} \frac{\partial A^m}{\partial x^k}$$

$$= \varepsilon^{231} g_{22} \frac{\partial A^2}{\partial x^3} + \varepsilon^{321} g_{33} \frac{\partial A^3}{\partial x^2}$$

$$= (1)(-1) \frac{\partial A^2}{\partial x^3} + (-1)(-1) \frac{\partial A^3}{\partial x^2}$$

$$B^1 = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}$$

or

$$B^i = \frac{\partial A^k}{\partial x^j} - \frac{\partial A^j}{\partial x^k} \qquad i, j, k \text{ cyclic}$$

$$B^i = -\frac{\partial A^k}{\partial x_j} + \frac{\partial A^j}{\partial x_k} = \frac{\partial A^j}{\partial x_k} - \frac{\partial A^k}{\partial x_j} \equiv F^{jk}$$

so, these can be put together to make the "Field Strength Tensor"

$$F^{\mu\nu} = \frac{\partial A^\mu}{\partial x_\nu} - \frac{\partial A^\nu}{\partial x_\mu} \qquad \text{antisymmetric tensor}$$

$$\begin{array}{c c} & 0 \longrightarrow 3 \\ \begin{array}{c} 0 \\ \downarrow \\ 3 \end{array} & \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix} \end{array}$$

or $$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$$

Since F is antisymmetric, form

$$T^{\sigma\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x_\sigma} + \frac{\partial F^{\sigma\mu}}{\partial x_\nu} + \frac{\partial F^{\nu\sigma}}{\partial x_\mu}$$

$$= \frac{\partial}{\partial x_\sigma}\left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) + \frac{\partial}{\partial x_\nu}\left(\partial^\mu A^\sigma - \partial^\sigma A^\mu\right) + \frac{\partial}{\partial x_\mu}\left(\partial^\sigma A^\nu - \partial^\nu A^\sigma\right)$$

cancel, etc.

So, $T^{\sigma\mu\nu} = 0$

look at some...

$$T^{123} = 0 = \frac{\partial F^{23}}{\partial x_1} + \frac{\partial F^{12}}{\partial x_3} + \frac{\partial F^{31}}{\partial x_2}$$

$$= \frac{\partial B^1}{\partial x_1} + \frac{\partial B^3}{\partial x_3} + \frac{\partial B^2}{\partial x_2}$$

$$= -\left(\frac{\partial B^1}{\partial x^1} + \frac{\partial B^2}{\partial x^2} + \frac{\partial B^3}{\partial x^3}\right) = -\vec{\nabla}\cdot\vec{B}$$

so $T^{123} \Rightarrow \vec{\nabla}\cdot\vec{B} = 0$ (B)

$$T^{021} = 0$$

$$= \frac{\partial F^{21}}{\partial x_0} + \frac{\partial F^{02}}{\partial x_1} + \frac{\partial F^{10}}{\partial x_2}$$

$$= \frac{\partial F^{21}}{\partial x^0} - \frac{\partial F^{02}}{\partial x^1} - \frac{\partial F^{10}}{\partial x^2}$$

$$= -\frac{\partial B^3}{\partial t} - \frac{\partial E^2}{\partial x^1} + \frac{\partial E^1}{\partial x^2} = -\frac{\partial B^3}{\partial t} - (\vec{\nabla}\times\vec{E})^3 \qquad \text{(B)}$$

So, the (B) M.E. are satisfied by the defining relation derivative of the

(sometimes one sees $\frac{\partial \mathcal{F}^{\mu\nu}}{\partial x^\mu} = 0$ where $\mathcal{F}^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$)

Consider, $\frac{\partial F^{\mu\nu}}{\partial x^\nu}$

$\mu = 0$
$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \cancel{\frac{\partial F^{00}}{\partial x^0}} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3}$$
$$= \vec{\nabla} \cdot \vec{E} = \rho \qquad \text{(A)}$$

$\mu = i$
$$\frac{\partial F^{i\nu}}{\partial x^\nu} = \frac{\partial F^{i0}}{\partial x^0} + \frac{\partial F^{i1}}{\partial x^1} + \frac{\partial F^{i2}}{\partial x^2} + \frac{\partial F^{i3}}{\partial x^3}$$

in order to be non-zero, $\frac{\partial F^{ij}}{\partial x^j} \Rightarrow i \neq j$

$i \neq 0 \Rightarrow 3$ terms.

so.
$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \cancel{\frac{\partial F^{11}}{\partial x^1}} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3}$$
$$= -\frac{\partial E^1}{\partial x^0} + 0 + \frac{\partial B^3}{\partial x^2} - \frac{\partial B^2}{\partial x^3}$$
$$= -\frac{\partial E^1}{\partial t} + (\vec{\nabla} \times \vec{B})^1$$

so,
$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \left(-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} \right)^1 = J^1 \qquad \text{(A)}$$

and the Maxwell equations in the presence of sources can both be imbedded in

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = j^\mu$$

and $T^{\sigma\mu\nu} = 0$

Remember that the A^μ are ambiguous. There exist gauge transformations that leave the measurables unchanged. There are the following gauges, each with its own use and subsequent formalism:

Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0$
Lorentz gauge: $\partial_\mu A^\mu = 0$
Temporal gauge: $A_0 = 0$

The Lorentz gauge is covariant and the ambiguity is expressed by

$$A^\mu \longrightarrow A^{\mu\prime} = A^\mu + \frac{\partial\theta(x)}{\partial x_\mu}$$

where $\theta(x)$ is some scalar function of x^μ.

note:
$$F'^{\mu\nu} = \frac{\partial A'^\mu}{\partial x_\nu} - \frac{\partial A'^\nu}{\partial x_\mu}$$
$$= \partial^\nu(A^\mu + \partial^\mu\theta) - \partial^\mu(A^\nu + \partial^\nu\theta)$$
$$= \partial^\nu A^\mu + \frac{\partial^2\theta}{\partial x_\nu \partial x_\mu} - \partial^\mu A^\nu - \frac{\partial\theta}{\partial x_\mu \partial x_\nu}$$
$$= \partial^\nu A^\mu - \partial^\mu A^\nu = F^{\mu\nu} \quad \rightarrow \quad E \,\&\, B \text{ don't change}$$
they're measurables.

So, if $\theta(x)$ is chosen such that,

$$\frac{\partial A^\mu}{\partial x^\mu} = -\frac{\partial^2\theta(x)}{\partial x^\mu \partial x_\mu} \qquad (\vec{\nabla}\cdot\vec{A} = \Box\theta)$$

then, $$\frac{\partial A'^{\mu}}{\partial x^{\mu}} = \frac{\partial}{\partial x^{\mu}}\left(A^{\mu} + \frac{\partial \theta}{\partial x_{\mu}}\right)$$

$$= \frac{\partial A^{\mu}}{\partial x^{\mu}} + \frac{\partial^2 \theta}{\partial x^{\mu} \partial x_{\mu}} = 0$$

so, $\dfrac{\partial A'^{\mu}}{\partial x^{\mu}} = 0$ is a choice, (L.G.)

Then, Maxwell's equations give, (no source)

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = 0$$

$$\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial A^{\mu}}{\partial x_{\nu}} - \frac{\partial A^{\nu}}{\partial x_{\mu}}\right) = \frac{\partial^2 A^{\mu}}{\partial x^{\nu} \partial x_{\nu}} - \frac{\partial A^{\nu}}{\partial x^{\nu} \partial x_{\mu}} = 0$$

$$= \frac{\partial^2 A^{\mu}}{\partial x^{\nu} \partial x_{\nu}} - \frac{\partial}{\partial x_{\mu}}\Bigg(\underbrace{\frac{\partial A^{\nu}}{\partial x^{\nu}}}_{L.G = 0}\Bigg) = 0$$

$$\frac{\partial^2 A^{\mu}}{\partial x^{\nu} \partial x_{\nu}} = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) A^{\mu} = 0$$

$$\Box A^{\mu} = 0$$

} a Klein Gordon wave equation for A^{μ}.

From the above, the addition of sources results in

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = j^{\mu} \quad \Rightarrow$$

$\Box A^{\mu} = j^{\mu}$ ... the source of A^{μ}

The correct Lagrangian density for the classical discussion is.

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

which can be seen by.—

$$F^{\mu\nu} F_{\mu\nu} = F^{\mu\nu} F^{\alpha\beta} g_{\alpha\mu} g_{\beta\nu}$$

$$= \underset{A}{F^{\mu 0} F^{\alpha 0} g_{\alpha\mu} g_{00}} + \underset{B}{F^{\mu 1} F^{\alpha 1} g_{\alpha\mu} g_{11}} + \underset{C}{F^{\mu 2} F^{\alpha 2} g_{\alpha\mu} g_{22}} + \underset{D}{F^{\mu 3} F^{\alpha 3} g_{\alpha\mu} g_{33}}$$

$$A_\mu = F^{10} F^{\alpha 0} g_{\alpha 1} g_{00} + F^{20} F^{\alpha 0} g_{\alpha 2} g_{00} + F^{30} F^{\alpha 0} g_{\alpha 3} g_{00} \qquad (F^{00} = 0)$$

$$= F^{10} F^{10} g_{11} g_{00} + F^{20} F^{20} g_{22} g_{00} + F^{30} F^{30} g_{33} g_{00}$$

$$= (F^{01})^2 (-1)(1) + (F^{20})^2 (-1)(1) + (F^{30})^2 (-1)(1)$$

$$= -E_1^2 - E_2^2 - E_3^2$$

$$B_\mu = F^{01} F^{\alpha 1} g_{\alpha 0} g_{11} + F^{21} F^{\alpha 1} g_{\alpha 2} g_{11} + F^{31} F^{\alpha 1} g_{\alpha 3} g_{11} \qquad (F^{11} = 0)$$

$$= F^{01} F^{01} g_{00} g_{11} + \cdots$$

$$= -(F^{01})^2 + (F^{21})^2 + (F^{31})^2 = -E_1^2 + B_3^2 + B_2^2$$

etc. C & D

$$\mathcal{L} = \frac{1}{2} \left(\vec{E}^2 - \vec{B}^2 \right) \quad \checkmark$$

The Euler Lagrange equations are complicated in covariant form:

for each β: $$\frac{\partial}{\partial x_\alpha} \left(\frac{\partial \mathcal{L}}{\partial (\partial A^\beta / \partial x_\alpha)} \right) - \frac{\partial \mathcal{L}}{\partial A^\beta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial A^\beta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial A^\beta/\partial x_\alpha)} = -\frac{1}{4}\left\{ \frac{\partial F^{\mu\nu}}{\partial\left(\frac{\partial A^\beta}{\partial x_\alpha}\right)} F_{\mu\nu} + F^{\mu\nu} \frac{\partial F_{\mu\nu}}{\partial\left(\frac{\partial A^\beta}{\partial x_\alpha}\right)} \right\}$$

from $F^{\mu\nu} = \frac{\partial A^\mu}{\partial x_\nu} - \frac{\partial A^\nu}{\partial x_\mu}$ or, $F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}$

$$\frac{\partial F^{\mu\nu}}{\partial\left(\frac{\partial A^\beta}{\partial x_\alpha}\right)} = \delta_\beta^{\ \mu}\delta_\alpha^{\ \nu} - \delta_\beta^{\ \nu}\delta_\alpha^{\ \mu} = \frac{\partial F^{\mu\nu}}{\partial(\partial^\alpha A^\beta)}$$

$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu \partial A^\nu$

and $$\frac{\partial F_{\mu\nu}}{\partial\left(\frac{\partial A^\beta}{\partial x_\alpha}\right)} = g_{\mu\eta}\, g_{\nu\rho} \frac{\partial F^{\eta\rho}}{\partial\left(\frac{\partial A^\beta}{\partial x_\alpha}\right)}$$

$$= g_{\mu\eta}\, g_{\nu\rho} \left(\delta^\eta_{\ \beta}\delta^\rho_{\ \alpha} - \delta^\rho_{\ \beta}\delta^\eta_{\ \alpha} \right)$$

$$= g_{\mu\beta}\, g_{\nu\alpha} - g_{\mu\alpha}\, g_{\nu\beta}$$

so,

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A^\beta}{\partial x_\alpha}\right)} = -\frac{1}{4}\left\{ \left(\delta^\mu_\beta \delta^\nu_\alpha - \delta^\nu_\beta \delta^\mu_\alpha\right) F_{\mu\nu} + \left(g_{\mu\beta}\, g_{\nu\alpha} - g_{\mu\alpha}\, g_{\nu\beta}\right) F^{\mu\nu} \right\}$$

$$= -\frac{1}{4}\left\{ F_{\beta\alpha} - F_{\alpha\beta} + F_{\beta\alpha} - F_{\alpha\beta} \right\}$$

$$= -\frac{1}{4}\left\{ 2F_{\beta\alpha} - 2F_{\alpha\beta} \right\} = -\frac{1}{2}\left\{ F_{\beta\alpha} - F_{\alpha\beta} \right\}$$

$$= -F_{\alpha\beta} = F_{\beta\alpha}$$ X

73

so, the equations of motion are. (for each A^β)

$$\frac{\partial F_{\beta\alpha}}{\partial x_\alpha} = 0 \qquad \text{which we know give 2 ME's}$$

To get the Hamiltonian,

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\dot{A}_\mu)} = \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial A_\mu}{\partial x^0}\right)}$$

from above * $\dfrac{\partial \mathcal{L}}{\partial(\partial A_\mu/\partial x^0)} = F^{\mu 0} \Rightarrow F^{i0}$

so $\qquad \pi^\mu(x) = \pi^i(x) = E^i$

Then,

$$\begin{aligned}\mathcal{H}(x) &= \pi^\mu \dot{A}_\mu - \mathcal{L} \\ &= \pi^i \dot{A}_i - \mathcal{L} \\ &= \pi^i \frac{\partial A_i}{\partial x^0} - \mathcal{L} \quad = -\pi^i \frac{\partial A^i}{\partial x^0} - \mathcal{L}.\end{aligned}$$

$$\frac{\partial A^i}{\partial x^0} = (-\vec{\nabla}\phi)^i - (\vec{E})^i$$

$$\mathcal{H}(x) = -(E^i)(-E^i) + (E^i)\left(\frac{\partial \phi}{\partial x_i}\right) - \tfrac{1}{2}E^2 + \tfrac{1}{2}B^2$$

$$= \tfrac{1}{2}E^2 + \tfrac{1}{2}B^2 + \vec{E}\cdot\vec{\nabla}\phi$$

Interaction with em field -- modern way.

Imagine a global $U(1)$ rotation on Hilbert Space vectors which stand for spin ½ particles. -

$$|\psi(x)\rangle \longrightarrow |\psi'(x)\rangle = R(\theta)\,|\psi(x)\rangle = e^{iq\theta}\,|\psi(x)\rangle$$

$$e^{iQ\theta}\,|\psi\rangle = Q|\psi\rangle + \cdots = |\psi\rangle q + \cdots = e^{-iq\theta}|\psi\rangle$$

θ is some constant ("global") phase. What happens?

D.E. $\quad (i\not\partial - m)\,\psi(x) = 0$

transform it.

$$(i\not\partial - m)\psi(x) \longrightarrow (i\not\partial - m)\psi'(x) = (i\not\partial - m)e^{iq\theta}\psi(x) = 0$$

$$= \underbrace{(i\not\partial - m)\psi(x)}_{\text{m, same}}$$

Dirac's theory is invariant wrt a global $U(1)$ transformation -- operator of conserved operation is Q and eigenvalues are conserved quantum numbers - electric charge

(or baryon charge, or lepton charge --)

Now consider a <u>local</u> $U(1)$ transformation

$$|\psi(x)\rangle \longrightarrow |\psi'(x)\rangle = R[\theta(x)]\,|\psi(x)\rangle = e^{iq\theta(x)}\,|\psi(x)\rangle$$

θ a function of spacetime.

The Dirac Equation:

$$(i\not\partial - m)\,\psi'(x) = 0$$
$$(i\not\partial - m)\,e^{iq\theta(x)}\,\psi(x) = 0$$
$$i\,\partial_\mu \gamma^\mu e^{iq\theta(x)}\psi(x) - m e^{iq\theta(x)}\,\psi(x) = 0$$
$$i\left\{ iq\,[\partial_\mu\theta]\,\gamma^\mu e^{iq\theta(x)}\psi(x) + [\partial_\mu\psi]\,e^{iq\theta(x)}\gamma^\mu\right\} - m e^{iq\theta(x)}\,\psi(x) = 0$$

$$\left\{-q\,[\partial_\mu\theta]\,\gamma^\mu + i\,\partial_\mu\gamma^\mu - m\right\}\psi(x) = 0$$

$$i(\not\partial - m)\,\psi(x) - \underbrace{q\,\not\partial\theta\,\psi}_{\text{spoils the invariance.}} = 0$$

Suppose you decide that an acceptable theory <u>must</u> be invariant wrt local $U(1)$ -- need to fix the gradient.

Define covariant derivative, $D^\mu \equiv \partial^\mu + X^\mu$

some 4 vector

determine X^μ so that

76

$$(i\not{\partial}-m)\psi(x) = 0$$

$$(i\not{D}-m)\psi(x) = 0$$

$$\left[i\gamma_\mu(\partial^\mu + X^\mu) - m\right]e^{iq\theta(x)}\psi(x) = 0$$

$$i\gamma_\mu\, iq\left[\partial^\mu\theta\right]\psi e^{iq\theta(x)} + i\gamma_\mu\left[\partial^\mu\psi\right]e^{iq\theta(x)} + i\gamma_\mu X^\mu\psi e^{iq\theta(x)} - m\psi e^{iq\theta(x)} = 0$$

$$-q\,\not{\partial}\theta(x)\,\psi + i\not{\partial}\psi + i\cancel{X}\psi - m\psi = 0$$

$$i\gamma_\mu\left[\partial^\mu + X^\mu + iq\partial^\mu\theta\right]\psi - m\psi = 0$$

still not quite right -- so need to be accompanied by an arbitrariness in X^μ:

$$X^\mu \longrightarrow X^{\mu\prime} = X^\mu - iq\partial^\mu\theta$$

Then,

$$i\gamma_\mu\left[\partial^\mu + X'^\mu + iq\partial^\mu\theta\right]\psi - m\psi = 0$$

$$i\gamma_\mu\left[\partial^\mu + X^\mu - iq\partial^\mu\theta + iq\partial^\mu\theta\right]\psi - m\psi = 0$$

$$i\gamma_\mu\left[\partial^\mu + X^\mu\right]\psi - m\psi = 0$$

$$(i\not{D}-m)\psi(x) = 0$$

Remember the time-honored way to add an electromagnetic field?

$$\vec{p} \longrightarrow \vec{p} - q\vec{A}$$

$$p^\mu \longrightarrow p^\mu - qA^\mu$$

~~where~~

when relativistic $A^\mu = [\phi, \vec{A}]$

quantize: $\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + iq\phi$

$E \xrightarrow{Q_1} i\frac{\partial}{\partial t}$ $-\vec{\nabla} \rightarrow -\vec{\nabla} + iq\vec{A}$

$\vec{p} \xrightarrow{Q_1} -i\vec{\nabla}$

since $\partial^\mu = [\frac{\partial}{\partial t}, -\vec{\nabla}]$

then, the minimum coupling rule says.

$$\partial^\mu = [\frac{\partial}{\partial t}, -\vec{\nabla}] \longrightarrow [\frac{\partial}{\partial t} + iq\phi, -\vec{\nabla} + iq\vec{A}]$$

$$\longrightarrow \partial^\mu + iqA^\mu \equiv D^\mu$$

and we identify $iqA^\mu \equiv X^\mu$. The subsidiary condition on X^μ chosen to make things invariant wrt $U(1)$, hence,

$$X^\mu \rightarrow X^\mu - iq\partial^\mu\theta$$
$$iqA^\mu \rightarrow iqA^\mu - iq\partial^\mu\theta$$
$$A^\mu \rightarrow A^\mu - \partial^\mu\theta$$

which is gauge invariance $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\theta$

$\phi \rightarrow \phi - \frac{\partial\theta}{\partial t}$

78

So, the U(1) Gauge Invariant Dirac equation is

$$(i\not{\partial} - m)\psi(x) = 0$$

$$i\gamma_\mu(\partial^\mu + iqA^\mu)\psi - m\psi = 0$$

$$(i\not{\partial} - q\not{A} - m)\psi(x) = 0$$

$$\underbrace{(i\not{\partial} - m)\psi(x)}_{\text{free equation.}} - q\underbrace{\not{A}(x)\psi(x)}_{\text{extra}} = 0$$

$$(i\not{\partial} - m)\psi(x) = \underbrace{q\not{A}\psi(x)}_{\text{source.}}$$

electric charge has snuck in via symmetry

~~Dynamics has~~ snuck in via symmetry. — the photon has snuck in via symmetry!

write it out —

$$(i\gamma_0 \tfrac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m)\psi - q\phi\gamma^0\psi + q\vec{A}\cdot\vec{\gamma}\psi = 0$$

$$(i\gamma_0 \tfrac{\partial}{\partial t} + i\gamma_0\vec{\alpha}\cdot\vec{\nabla} - m)\psi - q\phi\gamma^0\psi + \gamma^0 q\vec{A}\cdot\vec{\alpha}\psi = 0$$

$$(i\tfrac{\partial}{\partial t} + i\vec{\alpha}\cdot\vec{\nabla} - m\beta)\psi - q\phi\psi + q\vec{A}\cdot\vec{\alpha}\psi = 0$$

$$i\frac{\partial\psi}{\partial t} = \cdot\,\vec{\alpha}\cdot\vec{p} + m\beta\psi + q\phi\psi - q\vec{\alpha}\cdot\vec{A}\psi$$

$$i\frac{\partial\psi}{\partial t} = (\vec{\alpha}\cdot\vec{p} - e\,\vec{\alpha}\cdot\vec{A} + \beta m + e\phi)\psi(x)$$

BACK

and.
$$H = \int d^3x\, \mathcal{H}(x)$$

$$= \int d^3x\, \tfrac{1}{2}\left(E^2 + B^2\right) + \underbrace{\int d^3x\, \vec{E}\cdot\vec{\nabla}\phi}$$

integrate by parts
use $\vec{\nabla}\cdot\vec{E} = 0$ for no source
* next page

From our earlier discussion we learned that the correct way to account for the presence of the photon is to replace gradient operators for the "matter" with a covariant derivative,

$$\partial_\mu \rightarrow \partial_\mu + iqA_\mu \equiv D_\mu$$

This came from the demand of local gauge invariance

$$\psi \rightarrow \psi' = e^{-iq\theta(x)}\psi$$

which is satisfied if $\qquad A^\mu \rightarrow A^\mu - \frac{1}{q}\frac{\partial\theta(x)}{\partial x_\mu}$

which we now recognize as "regular" gauge invariance.

From the Lagrangian density for fermions, let's add the EM field this way,

$$\mathcal{L} = \bar{\psi}(x) \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi(x)$$

$$\partial_\mu \rightarrow D_\mu$$

$$\mathcal{L} = \bar{\psi}(x) \left[i\gamma^\mu \frac{\partial}{\partial x^\mu} + i\gamma^\mu (iqA_\mu) - m \right] \psi(x)$$

$$= \bar{\psi}(x) \left[i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right] \psi(x) - \bar{\psi}\gamma^\mu q A_\mu \psi$$

$$= \bar{\psi}(x) \left[i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right] \psi(x) - q\,\bar{\psi}(x)\gamma^\mu \psi(x) A_\mu$$

if we accept that this really is the addition of the photon, then we add the free EM piece to get the complete Lagrangian density for fermions, photons, and their interaction.

$$+ \quad -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

correction! should have defined Noether current,

$$J^\mu = -i \frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial x^\mu)} f(\phi)$$

convention

For a $U(1)$ global phase invariance: $\psi \rightarrow \psi' = e^{iq\theta}\psi$

then the Noether current is:

$$-i\left[\frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial x^\mu)}\right]\psi = (-i)\left[(i)\,\bar{\psi}(x)\gamma^\mu\right]\psi$$

$$= \bar{\psi}(x)\gamma^\mu\psi(x) = j^\mu$$

81

the charge $$Q = \int d^3x\, j^0$$
$$= \int d^3x\, \psi^\dagger \gamma^0 \gamma^0 \psi = \int d^3x\, \psi^\dagger \psi$$

we'll need quantum field theory to interpret.

nb. For a massive spin 1 field, the appropriate addition to the $\mathcal{L}$ density is

$$+ \frac{1}{2} M A^\mu A_\mu$$

However, This is <u>not</u> locally gauge invariant and this was a serious problem discovered in the late 1950's which blocked a workable theory of the weak interactions until 1967.

So, if we follow the EL eq prescription for our $\mathcal{L}$ (massless) we find

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi - q\gamma^\mu \psi A_\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial\bar{\psi}/\partial x)} = 0 \quad \Rightarrow \quad \left(i\gamma^\mu \frac{\partial}{\partial x_\mu} - m\right)\psi(x) = q\gamma^\mu A_\mu \psi(x)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -\bar{\psi} m - q\bar{\psi}\gamma^\mu A_\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial x)} = i\bar{\psi}\gamma^\mu \quad \Rightarrow \quad \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial x)}\right) = i\gamma^\mu \frac{\partial \bar{\psi}}{\partial x^\mu}$$

so

$$\left(i\gamma^\mu \partial_\mu + m\right)\bar{\psi}(x) = q\bar{\psi}\gamma^\mu A_\mu .$$

$$\left\{\begin{array}{l}
\dfrac{\partial \mathcal{L}}{\partial(\partial A_\nu/\partial x^\mu)} = F^{\nu\mu} \quad \Rightarrow \quad \partial_\mu\left(\dfrac{\partial \mathcal{L}}{\partial(\partial A_\nu/\partial x^\mu)}\right) = \dfrac{\partial F^{\nu\mu}}{\partial x_\mu} = \dfrac{\partial^2 A^\nu}{\partial x^\mu \partial x_\mu} \\[2ex]
\dfrac{\partial \mathcal{L}}{\partial A_\nu} = -q\bar{\psi}\gamma^\mu\psi \\[2ex]
\text{so} \qquad \dfrac{\partial^2 A^\nu}{\partial x^\mu \partial x_\mu} = q\bar{\psi}\gamma^\mu\psi = q j^\mu
\end{array}\right.$$

.. the electron current responsible for (the source of) the EM field.