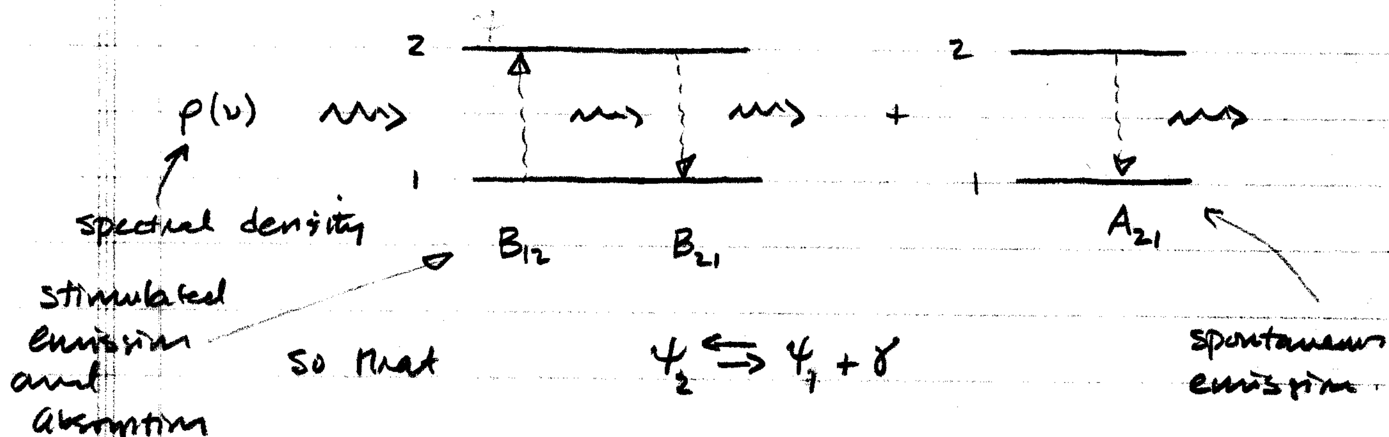
Lecture 7 Motivations for Field Theory 83

Before 1916 there were 2 uses of the quantum idea : 1) Planck's description, expanded upon by Einstein of quantized radiators in the blackbody situation (1905)

2) Bohr's quantized orbits seeming to solve the spectrum problem (1913).

In 1916 and 1917, Einstein brought them together by considering a "gas" of "molecules" bathed in radiation. Presuming equilibrium, it's the blackbody problem



so that $\psi_2 \rightleftarrows \psi_1 + \gamma$

rate $R_{12} = B_{12}\, \rho(\nu)$

$$R_{21} = B_{21}\, \rho(\nu) + A_{21}$$

presume N_1 atoms in state 1

N_2 " 2

Thermal equilibrium $\Rightarrow$ $$\frac{N_1}{N_2} = e^{(E_2 - E_1)/kT} = e^{\Delta E/kT}$$

and since this also implies,

$$N_1 R_{12} = N_2 R_{21}$$
$$N_1 B_{12} \rho(\nu) = N_2 \left[B_{21} \rho(\nu) + A_{21} \right]$$

solving for ρ

$$\rho(\nu) = \left(\frac{A_{21}}{B_{21}}\right) \left\{ \frac{1}{\left(\frac{B_{12}}{B_{21}}\right) e^{\Delta E/kT} - 1} \right\}$$

by appealing to Wein's displacement law, in the limit $\rho(\nu) \propto \nu^3 f(\nu/T)$ and plausibly arguing $B_{12} = B_{21}$ and $\Delta E = \nu h$

where "h is a constant"

he derived Planck's Law

$$\rho(\nu) = \frac{8\pi h \nu^3}{c^3} \left(\frac{1}{e^{h\nu/kT} - 1} \right)$$

and noted that the similarity of this suggests that some future theory would be necessary to explain it.

Of course, Dirac's the guy with the tools and the intuition to accomplish this, which he did in 1927.

What Dirac did was become amazed at the apparent fact that absorption of a photon caused that photon to

disappear "jumps into a zero state"

and when it is emitted, it

" it can be considered to jump from the zero state to one in which it is physically in evidence, so that it appears to have been created."

he went on.- " Since there is no limit to the number of light-quanta that may be created in this way, we must suppose that there are an infinite number of light-quanta in the zero state."

We'll do a modern treatment of a set of conclusions from Dirac's original approach.

The wave equation for $\vec{A}(\vec{x},t)$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A}(\vec{x},t) = 0$$

admits plane wave solutions

$$\vec{A}(\vec{x},t) = \sum_{\substack{\vec{k} \\ k>0}} N_{\vec{k}}\, \vec{C}_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}}$$

normalization.

The coulomb gauge is most appropriate here (used often when there are no sources)

$$\vec{\nabla}\cdot\vec{A} = 0 = \sum i\vec{k}\cdot\vec{C}_{\vec{k}}(t)\, N_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}}$$

independent

$$\Rightarrow \quad \vec{k}\cdot\vec{C}_{\vec{k}} = 0$$

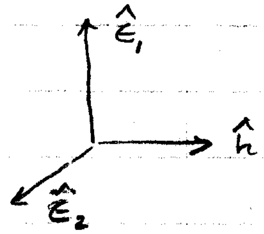
so the Fourier coefficients are $\perp$ to $\vec{k}$ which is why the coulomb gauge is also called the "transversality condition".

Since $\vec{E} = -\frac{\partial\vec{A}}{\partial t}$ the $\vec{C}_{\vec{k}}(t)$ directions are along $\vec{E}$, or the polarization direction.

(see note on p89 about gauging away ϕ)

87

For each momentum vector, $\vec{k}$, we can write a set of possible Cartesian polarization vectors:



$$\left.\begin{aligned} \hat{\epsilon}_{(1)} &= (1,0,0) \\ \hat{\epsilon}_{(2)} &= (0,1,0) \\ \hat{\epsilon}_{(3)} &= (0,0,1) \end{aligned}\right\} \text{plane polarized}$$

or, more conventionally, a spherical set

$$\left.\begin{aligned} \underset{(R)}{\hat{\epsilon}_+} &\equiv -\sqrt{1/2}\,(\hat{\epsilon}_1 + i\,\hat{\epsilon}_2) \\ \underset{(L)}{\hat{\epsilon}_-} &\equiv \sqrt{1/2}\,(\hat{\epsilon}_1 - i\,\hat{\epsilon}_2) \end{aligned}\right\} \quad \hat{\epsilon}^*_\lambda \cdot \hat{\epsilon}_{\lambda'} = \delta_{\lambda\lambda'} \quad \text{(circularly polarized)}$$

$$\hat{\epsilon}_+ = -\sqrt{1/2}\,(1, i, 0)$$
$$\hat{\epsilon}_- = \sqrt{1/2}\,(1, -i, 0)$$

$$\hat{\epsilon}_0 \equiv \hat{k} \qquad \times$$

(called "spherical" since they transform like Y_{1m} $m = 1, 0, -1$)

Remember, by convention, a particular gauge has been chosen which makes $\vec{G} \perp \vec{k}$, so in this case there are only 2 components of polarization relevant for a real photon.

So, the Fourier expansion must generally sum over $\sum_{\vec{k}}$ and $\sum_{\lambda}$

↑ sum is 2 whether plane or circular polarization

So, defining $\qquad \vec{C}_{\vec{k}}(t) = \hat{\epsilon}_{\vec{k}\lambda}\, C_{\vec{k}\lambda}(t)$

$$\vec{A}(\vec{x},t) = \sum_{\substack{\vec{k} \\ k>0}} \sum_{\lambda} N_{\vec{k}} \left[\hat{\epsilon}_{\vec{k}\lambda}\, C_{\vec{k}\lambda}(t)\, e^{i\vec{k}\cdot\vec{x}} + \hat{\epsilon}^*_{\vec{k}\lambda}\, C^*_{\vec{k}\lambda}(t)\, e^{-i\vec{k}\cdot\vec{x}} \right]$$

substituting this into the wave equation,

$$\frac{d^2}{dt^2} C_{k\lambda}(t) + \omega_k^2\, C_{k\lambda}(t) = 0 \qquad \text{where} \quad \omega_k = |\vec{k}|c$$

harmonic oscillator.

which has the general solution

$$C_{k\lambda}(t) = c_{k\lambda}(0)\, e^{-i\omega_k t} + c'_{k\lambda}(0)\, e^{i\omega_k t}$$

So, substituting,

$$\vec{A}(\vec{x}\, t) = \sum_{\substack{\vec{k} \\ k>0}} \sum_{\lambda} N_{\vec{k}} \Big[\hat{\epsilon}_{k\lambda}\, c_{k\lambda}(0)\, e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + \hat{\epsilon}_{k\lambda}\, c'_{k\lambda}(0)\, e^{i\vec{k}\cdot\vec{x} + i\omega_k t}$$

$$+ \hat{\epsilon}^*_{k\lambda}\, c^*_{k\lambda}(0)\, e^{-i\vec{k}\cdot\vec{x} + i\omega_k t} + \hat{\epsilon}^*_{k\lambda}\, c'^*_{k\lambda}(0)\, e^{-i\vec{k}\cdot\vec{x} - i\omega_k t} \Big]$$

We can get rid of the $k>0$ stipulation by defining

$$a_{k\lambda}(0) \equiv c_{k\lambda}(0) \qquad k>0$$

$$\equiv c'_{k\lambda}(0) \qquad k<0$$

then,

89

$$\vec{A}(\vec{x},t) = \sum_k \sum_\lambda N_k \left[\hat{\epsilon}_{k\lambda}\, a_{k\lambda}(0)\, e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + \hat{\epsilon}^*_{k\lambda}\, a^*_{k\lambda}(0)\, e^{-i\vec{k}\cdot\vec{x} + i\omega_k t} \right]$$

further define

$$a_{k\lambda}(t) \equiv a_{k\lambda}(0)\, e^{-i\omega_k t} \equiv a_{k\lambda}$$

$$\vec{A}(\vec{x},t) = \sum_k \sum_\lambda N_k \left[\hat{\epsilon}_{k\lambda}\, a_{k\lambda}\, e^{i\vec{k}\cdot\vec{x}} + \hat{\epsilon}^*_{k\lambda}\, a^*_{k\lambda}\, e^{-i\vec{k}\cdot\vec{x}} \right]$$

... an ∞ number of degrees of freedom $\rightarrow$ new quantum mechanical tools required to handle this.

The gauge transformation

$$A^{\mu\prime} = A^\mu + \partial^\mu \theta \longrightarrow \vec{A}' = \vec{A}' + \vec{\nabla}\theta \quad \&$$
$$\Phi' = \phi + \frac{\partial\theta}{\partial t}$$

can always have a solution for the gauge function, θ, such that $\phi' = 0$ and $\vec{\nabla}\cdot\vec{A}' = 0$ when there is no charge, ie $\vec{\nabla}\cdot\vec{E} = 0$.

So, we'll presume that, so that,

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} \quad \& \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

which allows us to calculate the Hamiltonian.

$$H = \int d^3x\, \tfrac{1}{2}\left(E^2 + B^2 \right)$$

$$= \int d^3x\, \tfrac{1}{2}\left[\left|\frac{\partial\vec{A}}{\partial t}\right|^2 + \left|\vec{\nabla}\times\vec{A}\right|^2 \right]$$

in turn.–

$$\frac{\partial \vec{A}}{\partial t} = \sum_k \sum_\lambda N_k \left[\hat{\epsilon}_{k\lambda} (-i\omega_k) a_{k\lambda} e^{i\vec{k}\cdot\vec{x}} + \hat{\epsilon}^*_{k\lambda} (i\omega_k) a^*_{k\lambda} e^{-i\vec{k}\cdot\vec{x}} \right]$$

$$= \sum\sum N_k (-i\omega_k) \left[\hat{\epsilon}_{k\lambda} a_{k\lambda} e^{i\vec{k}\cdot\vec{x}} - cc \right]$$

and

$$\frac{\partial \vec{A}}{\partial t} = \sum_{k'} \sum_{\lambda'} N_{k'} (-i\omega_{k'}) \left[\hat{\epsilon}_{k'\lambda'} a_{k'\lambda'} e^{i\vec{k}'\cdot\vec{x}} - cc \right]$$

so

$$\left|\frac{\partial \vec{A}}{\partial t}\right|^2 = -\sum_k \sum_{k'} \sum_\lambda \sum_{\lambda'} N_k N_{k'} \omega_k \omega_{k'} \left[\hat{\epsilon}_{k\lambda} a_{k\lambda} e^{i\vec{k}\cdot\vec{x}} - cc \right] \cdot \left[\hat{\epsilon}_{k'\lambda'} a_{k'\lambda'} e^{i\vec{k}'\cdot\vec{x}} - cc \right]$$

the polarizations must be the same $\hat{\epsilon}_{k\lambda} \cdot \hat{\epsilon}_{k'\lambda'} = \delta_{\lambda\lambda'}$ and normalization conditions for Fourier coefficients require

$$\int d^3x \, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = (\text{volume})\, \delta_{kk'}$$

so

$$\frac{1}{2} \int d^3x \left|\frac{\partial \vec{A}}{\partial t}\right|^2 = -\frac{1}{2} \sum\sum\sum\sum N_k N_{k'} \omega_k \omega_{k'}$$

$$\times \Big[\int d^3x \, \hat{\epsilon}_{k\lambda} \cdot \hat{\epsilon}_{\lambda' k'} a_{k\lambda} a_{k'\lambda'} e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}$$

$$- \int d^3x \, \hat{\epsilon}_{k\lambda} \cdot \hat{\epsilon}^*_{k'\lambda'} a_{k\lambda} a^*_{k'\lambda'} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} - \int d^3x \, \hat{\epsilon}^*_{k\lambda} \cdot \hat{\epsilon}_{k'\lambda'} a^*_{k\lambda} a_{k'\lambda'} e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$+ \int d^3x \, \hat{\epsilon}^*_{k\lambda} \cdot \hat{\epsilon}^*_{k'\lambda'} a_{k\lambda} a^*_{k'\lambda'} e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}} \Big]$$

$= $ lots of δ's $\Rightarrow$

$$= \frac{1}{2} \sum_{\vec{k}} \sum_{\lambda} N_k^2 \omega_k^2 \left[-a_k a_{-k} + a_k a_k^* + a_k^* a_k - a_k^* a_{-k}^* \right] (V)$$

$$= \frac{1}{2} \sum_{\vec{k}} \sum_{\lambda} N_k^2 \omega_k^2 \left[(a_{k\lambda} a_{k\lambda}^* + a_{k\lambda}^* a_{k\lambda}) - (a_{k\lambda} a_{-k\lambda} + a_{k\lambda}^* a_{-k\lambda}^*) \right] V$$

The other term is

$$\frac{1}{2} \int d^3x \, |\vec{\nabla} \times \vec{A}|^2$$

$c=1$

$$= \frac{1}{2} \sum_{k} \sum_{\lambda} N_k^2 \omega_k^2 \left[(a_{k\lambda} a_{k\lambda}^* + a_{k\lambda}^* a_{k\lambda}) + (a_{k\lambda} a_{-k\lambda} + a_{k\lambda}^* a_{-k\lambda}^* \right] V$$

and

$$H = \sum_{k} \sum_{\lambda} N_k^2 \omega_k^2 \, (a_{k\lambda} a_{k\lambda}^* + a_{k\lambda}^* a_{k\lambda}) V$$

familiar?

Remember the 1d quantum harmonic oscillator

$$H_{ho} |\psi\rangle = E |\psi\rangle \Rightarrow \quad H_{ho} = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} x^2$$

$$\text{w/} \quad [p_x, x] = -i\hbar$$

by defining $\quad \alpha \equiv \frac{\omega m}{\hbar} \quad \beta \equiv \frac{2mE}{\hbar^2} \quad \varepsilon \equiv \frac{\beta}{\alpha} = \frac{2E}{\omega\hbar}$

and generalized coordinates (dirac, again)

$$q \equiv \sqrt{\alpha}\, x$$

$$p \equiv \frac{1}{\hbar\sqrt{\alpha}} p_x = \frac{-i}{\sqrt{\alpha}} \frac{\partial}{\partial x} = \frac{-i}{\sqrt{\alpha}} \frac{\partial}{\partial q} \frac{\partial q}{\partial x} = -i \frac{\partial}{\partial q}$$

$$p^2 = -\frac{\partial^2}{\partial q^2}$$

then, $H'_{ho} = p^2 + q^2$ and Sch Eq $(p^2+q^2)\psi = \varepsilon\psi$

and $[p, q] = -i$

This will not simply factor, and the standard is to write

$$(q+ip)(q-ip) = q^2+p^2+1$$

$$(q-ip)(q+ip) = q^2+p^2-1$$

so $p^2+q^2 = \frac{1}{2}\left[(q+ip)(q-ip) + (q-ip)(q+ip) \right]$

Define

$$\left.\begin{aligned} a &\equiv \sqrt{1/2}\,(q+ip) \\ a^+ &\equiv \sqrt{1/2}\,(q-ip) \end{aligned}\right\} \quad [a, a^+] = 1$$

so $H'_{ho} = p^2+q^2 = \;\; (aa^+ + a^+a)$

where $H'_{ho}\psi = \varepsilon\psi$

suspicious... hmm.

In terms of the original variables. —

$$a = \sqrt{1/2}\left(\sqrt{\alpha}\, x + i \frac{1}{\hbar\sqrt{\alpha}} p\right)$$

$$a^{+} = \sqrt{1/2}\left(\sqrt{\alpha}\, x - i \frac{1}{\hbar\sqrt{\alpha}} p\right)$$

so,

$$a^{+}a = \frac{1}{2}(\quad + \quad)(\quad - \quad)$$

$$= \frac{1}{2}\Big[\alpha x^2 + \frac{p^2}{\hbar^2\alpha} \underbrace{- \frac{i}{\hbar} p x + \frac{i}{\hbar} x p}_{-\frac{i}{\hbar}\underbrace{[p,x]}_{-i\hbar}}\Big]$$

$$= \frac{1}{2}\left[\frac{\omega m}{\hbar} x^2 + \frac{p^2}{\hbar\omega m} - 1\right]$$

$$= \frac{1}{\hbar\omega}\left[\frac{m\omega^2}{2} x^2 + \frac{p^2}{2m} - \frac{\hbar\omega}{2}\right]$$

$$a^{+}a = \frac{1}{\hbar\omega} H_{ho} - \frac{1}{2}$$

or $$H_{ho} = \hbar\omega\left(a^{+}a + \frac{1}{2}\right)$$

From $H_{ho}''|\psi\rangle = \varepsilon|\psi\rangle$

$$(aa^{+} + a^{+}a)|\psi\rangle = \frac{2E}{\omega\hbar}|\psi\rangle$$

$$\frac{1}{2}\hbar\omega\,(aa^{+} + a^{+}a)|\psi\rangle = E|\psi\rangle$$

so $$H_{ho} = \tfrac{1}{2}\hbar\omega(aa^{+} + a^{+}a)$$

Remember these operators were called raising and lowering operators, or ladder operators and they had the property that

$$\tfrac{1}{2}\hbar\omega\,(aa^\dagger + a^\dagger a)\,|\psi\rangle = E|\psi\rangle$$

$$\tfrac{1}{2}\hbar\omega\,(aa^\dagger + aa^\dagger - 1)\,|\psi\rangle \Rightarrow \hbar\omega\, aa^\dagger|\psi\rangle = (E + \tfrac{1}{2}\hbar\omega)|\psi\rangle$$

$$\hbar\omega\, a^\dagger a|\psi\rangle = (E - \tfrac{1}{2}\hbar\omega)|\psi\rangle$$

since $\quad [a, a^\dagger] = 1 \qquad aa^\dagger - a^\dagger a = 1$.

plus, $\quad \hbar\omega\, a^\dagger(aa^\dagger|\psi\rangle) = (E + \tfrac{1}{2}\hbar\omega)\, a^\dagger|\psi\rangle$

$$\hbar\omega\,(aa^\dagger - 1)\,a^\dagger|\psi\rangle = (E + \tfrac{1}{2}\hbar\omega)\, a^\dagger|\psi\rangle$$

$$\hbar\omega\, aa^\dagger a^\dagger|\psi\rangle - \hbar\omega\, a^\dagger|\psi\rangle =$$

$$\hbar\omega\, aa^\dagger(a^\dagger|\psi\rangle) = (E + \tfrac{1}{2}\hbar\omega + \hbar\omega)\, a^\dagger|\psi\rangle$$

$$= (E + \tfrac{3}{2}\hbar\omega)\, a^\dagger|\psi\rangle$$

In the harmonic oscillator there is a ground state -- which is the state of lowest energy -- $|0\rangle$

This means that $\quad a|0\rangle = 0$

so $\quad \hbar\omega\, a^\dagger a|0\rangle = 0 = (E_0 - \tfrac{1}{2}\hbar\omega)|0\rangle$

so $\quad E_0 = \tfrac{1}{2}\hbar\omega$

The ground state energy for the quantum ho is at finite value,

The similarity of H_{ho} to H is suspicious and exploitable --
if

$$H_{ho} = H$$

$$\tfrac{1}{2}\hbar\omega(aa^{+} + a^{+}a) = N_k^2 \omega_k^2 V (aa^{*} + a^{*}a)$$

if $a^{*} \equiv a^{+}$, then, we can recover our normalization.

$$N_k^2 = \frac{\omega}{2\omega^2 V}$$

$$N_k = \frac{1}{\sqrt{2\omega V}}$$

and we can write the H as

$$H = \sum_k \sum_\lambda \frac{1}{2\cancel{\omega}\cancel{V}} \omega^{\cancel{2}} (a_{k\lambda} a^{+}_{k\lambda} + a^{+}_{k\lambda} a_{k\lambda}) \cancel{V}$$

$$H = \frac{1}{2} \sum_k \sum_\lambda \hbar\omega (a_{k\lambda} a^{+}_{k\lambda} + a^{+}_{k\lambda} a_{k\lambda})$$

where now we're obligated to treat the a's as operators with the same commutation relation as for the h.o.:

$$[\hat{a}_{k\lambda}, \hat{a}^{+}_{k'\lambda'}] = \delta_{kk'}\delta_{\lambda\lambda'} \qquad \text{so}$$

$$\hat{H} = \frac{1}{2} \sum_{k\lambda} \hbar\omega (a^{+}a + 1 + a^{+}a)$$

$$\hat{H} = \sum_k \sum_\lambda \hbar\omega (\hat{a}^{+}_{k\lambda} \hat{a}_{k\lambda} + \tfrac{1}{2})$$

This is a harmonic oscillator - with:

define $$X_{k\lambda} \equiv \frac{a_{k\lambda} + a_{k\lambda}^{+}}{\sqrt{2\omega/\hbar}}$$

$$\Pi_{k\lambda} \equiv \frac{i\,(a_{k\lambda} + a_{k\lambda}^{+})}{\sqrt{2\omega/\hbar}}$$

then, $$H = \sum_{k}\sum_{\lambda} \tfrac{1}{2}\left(\Pi_{k\lambda}^2 + \omega^2 X_{k\lambda}^2\right)$$

and Π and X are the canonical variables,

$$\frac{\partial H}{\partial X_{k\lambda}} = -\dot{\Pi}_{k\lambda} \qquad \frac{\partial H}{\partial \Pi_{k\lambda}} = \dot{X}_{k\lambda}$$

$\Rightarrow$ the radiation field can be thought of as a collection of independent oscillators $\rightarrow$ the dynamical variables are a linear set of the Fourier expansion coefficients.

What Dirac did was propose that:

just as in "regular QM" (only 2 y old!)

p & q $\rightarrow$ operators

the dynamical variables Π & X also become operators so, $$[\hat{X}_{k\lambda}, \hat{\Pi}_{k'\lambda'}] = i\hbar\, \delta_{kk'}\, \delta_{\lambda\lambda'}$$

was proposed along with

$$[\hat{\Pi}, \hat{\Pi}] = [\hat{X}, \hat{X}] = 0$$

<u>now</u> a and $a^+ \longrightarrow \hat{a}$ and $\hat{a}^+$

and, remembering, $a = a(t)$, we get

$$[a_{k\lambda}(t), a^+_{k'\lambda'}(t)] = \delta_{kk'}\, \delta_{\lambda\lambda'}$$ (equal times)

just like the regular h.o.

back to 96