Lecture 8 Groundstate of Field Theory 96

The absolute energy scale is arbitrary-- , so we can define the state of lowest energy to have value zero

$$\text{so} \qquad H|0\rangle = 0$$

This is the same as subtracting $\frac{1}{2}\hbar\omega$ from the radiation Hamiltonian, so:

$$\hat{H} = \sum_k \sum_\lambda \hbar\omega\, \hat{a}^\dagger_{k\lambda}\, \hat{a}_{k\lambda} \, . \longrightarrow 438.1 \ \& \ 438.2$$

~~96a~~ 96b

Also, $\vec{A}$ is changed.. now

$$\hat{\vec{A}}(\vec{x},t) = \sum_k \sum_\lambda \sqrt{\frac{\hbar}{2\omega V}}\, c\left[\vec{\epsilon}_{k\lambda}\,\hat{a}_{k\lambda}\, e^{i\vec{k}\cdot\vec{x}} + \vec{\epsilon}^{\,*}_{k\lambda}\,\hat{a}^\dagger_{k\lambda}\, e^{-i\vec{k}\cdot\vec{x}}\right]$$

is the box-normalized photon field operator.

What do these operators operate on? A many-field space called Occupation Number Representation or Fock Space.... clearly motivated by the harmonic oscillator model of Dirac.

In Fock Space, the state vector

$$|\psi\rangle = |\, n(k_1), n(k_2), n(k_3) \cdots n(k_i) \cdots \rangle$$

each "slot" denotes the number of quanta which have a given momentum.

The state vector is an eigenstate of combinations of the a's —

$\hat{a}(k_i)$ will annihilate (absorb in Dirac's original view) a quantum of momentum k_i and not affect any other states

remember the problem was one of a radiation field losing and gaining individual photons + Einstein's A's & B's ...

These operators do that

so $$\hat{a}(k_i)|\psi\rangle = \hat{a}(k_i)|n(k_1), n(k_2), \ldots, n(k_i), \ldots\rangle$$
$$= \sqrt{n(k_i)}\,|n(k_1), n(k_2), \ldots, n(k_i)-1, \ldots\rangle$$

$\hat{a}^{+}(k_i)$ will create ("emit") a photon of momentum k_i

$$\hat{a}^{+}(k_i)|\psi\rangle = \hat{a}^{+}(k_i)|n(k_1), n(k_2), \ldots, n(k_i), \ldots\rangle$$
$$= \sqrt{n(k_i)+1}\,|n(k_1), n(k_2), \ldots, n(k_i)+1, \ldots\rangle$$

also: $$\langle n(k_i)|\hat{a}^{+}(k_i) = \sqrt{n(k_i)}\,\langle n(k_i)-1|$$
$$\langle n(k_i)|\hat{a}(k_i) = \sqrt{n(k_i)+1}\,\langle n(k_i)+1|$$

clearly $|\psi\rangle$ is not an eigenstate of either a or a^{+}. However guided by the h.o.

$$\hat{a}(k_i)\,\hat{a}^+(k_i)\,|n(k_i)\rangle = \sqrt{n(k_i)+1}\;\hat{a}(k_i)\,|n(k_i)+1\rangle$$
$$= \sqrt{(n(k_i)+1)}\sqrt{n(k_i)+1}\;|n(k_i)\rangle$$

and $\hat{a}^+(k_i)\hat{a}(k_i)\,|n(k_i)\rangle = \sqrt{n(k_i)}\;\hat{a}^+(k_i)\,|n(k_i)-1\rangle$
$$= \sqrt{n(k_i)}\sqrt{n(k_i)-1+1}\;|n(k_i)\rangle$$
$$= n(k_i)\,|n(k_i)\rangle$$

so, $\hat{a}^+_\lambda(k_i)\,\hat{a}_\lambda(k_i) \equiv N_\lambda(k_i)$ .. the number operator, which counts the number of photons in momentum state k_i and with λ

note

so, $$\hat{H} = \sum_k \sum_\lambda \hbar\omega\, N_\lambda(k) \qquad \omega = |\vec{k}|c$$

and total energy is the sum of individual energies

note $\hat{H}\,|n_\lambda(k_i)\rangle = |n_\lambda(k_i)\rangle\,\hbar\omega_i \qquad \omega_i = |\vec{k}_i|c$.

Just as w/ ho, there is a ground state — the vacuum — in which there are no quanta.

$|\Psi_0\rangle = |0,0,\dots 0,\dots\rangle \equiv |0\rangle$ and unique normalization $\langle 0|0\rangle = 1$

so, $\hat{a}(k_i)\,|0\rangle = 0 \qquad N(k)\,|0\rangle = 0$

It is the state of lowest energy.

$$H|0\rangle = \sum\sum \hbar\omega\,\hat{a}^+\hat{a}\,|0\rangle = 0 \quad \text{or}$$

$$\langle 0|H|0\rangle \propto \langle 0|\hat{a}^+\hat{a}\,|0\rangle = 0$$

either way

Lecture 6 2/7/03

Suppose we know the energy of a state,

$$H|\psi\rangle = E|\psi\rangle$$

and make a new one $\hat{a}^{+}(k')|\ldots n(k')\ldots\rangle = \sqrt{n(k')+1}\,|\ldots, n(k')+1, \ldots\rangle$

what's the energy of it?

ie what's $H\,\hat{a}^{+}(k')|n\rangle \equiv H|n'\rangle$

Find.
$$[\hat{H}, \hat{a}^{+}_{\lambda'}(k')] = \sum_{k}\sum_{\lambda} \hbar\omega\,[\hat{N}_{\lambda}(k), \hat{a}^{+}_{\lambda'}(k')]$$

$$= \sum\sum \hbar\omega\,[\,a^{+}aa^{+\prime} - a^{+\prime}a^{+}a \;\; \underbrace{-\,a^{+}a^{+\prime}a + a^{+}a^{+\prime}a}_{0}\,]$$

$$= \sum\sum \hbar\omega\,\{\,\hat{a}^{+}_{\lambda}(k)\underbrace{[\hat{a}_{\lambda}(k), \hat{a}^{+}_{\lambda'}(k')]}_{=\,\delta_{kk'}\delta_{\lambda\lambda'}} + \underbrace{[\hat{a}^{+}_{\lambda}(k), \hat{a}^{+}_{\lambda'}(k')]}_{=\,0}\hat{a}_{\lambda}(k)\,\}$$

$$\hat{H}\hat{a}^{+}_{\lambda'}(k') - \hat{a}^{+}_{\lambda'}(k')\hat{H} = \hbar\omega'_{k'}\,\hat{a}^{+}_{\lambda'}(k')$$

so
$$\hat{H}\hat{a}^{+}_{\lambda'}(k') = \hat{a}^{+}_{\lambda'}(k')H + \hbar\omega'_{k'}\,\hat{a}^{+}_{\lambda'}(k')$$

and
$$\hat{H}\,\hat{a}^{+}_{\lambda'}(k')|\ldots n_{\lambda}(k')\ldots\rangle = \hat{a}^{+}_{\lambda'}(k')H|\ldots n_{\lambda}(k')\ldots\rangle + \hbar\omega'_{k'}\,\hat{a}^{+}_{\lambda'}(k')|\ldots n_{\lambda}(k')\ldots\rangle$$

$$= \hat{a}^{+}_{\lambda'}(k')|\ldots n_{\lambda}(k')\ldots\rangle E + \hat{a}^{+}_{\lambda'}(k')|\ldots n_{\lambda}(k')\ldots\rangle\hbar\omega'_{k'}$$

$$= (E + \hbar\omega'_{k'})|n'\rangle$$

— the energy of the new state increases by $\hbar\omega'$.

So, now to radiation. A standard treatment was to take the Hamiltonian (non-relativistic) and through the minimal subtraction method $\vec{p} \to \vec{p} - e\vec{A}/c$

now the ~~radiation~~ field operator

A many body Hamiltonian

$$H = \sum_{i=\text{electrons}} \frac{\vec{p}_i^{\,2}}{2m_i} + \frac{1}{2} \sum_{i \neq j} \frac{e_i e_j}{|\vec{x}_i - \vec{x}_j|}$$

V

add $\vec{A}$

$$= \sum_i \frac{(\vec{p}_i - \frac{e_i}{c}\vec{A})(\vec{p}_i - \frac{e_i}{c}\vec{A})}{2m_i} + V + \int \frac{\hat{E}\cdot\hat{E}^* + \hat{B}\cdot\hat{B}^*}{8\pi} d^3x$$

the interaction $\underline{H_{int}}$ inside

$$\hat{H}_{int} = \sum_i \left\{ \frac{-e_i}{2m_i c} \left[\hat{\vec{p}}_i \cdot \hat{\vec{A}}(\vec{x}_i, t) + \hat{\vec{A}}(\vec{x}_i, t) \cdot \hat{\vec{p}}_i \right] + \frac{e_i^2}{2m_i c^2} \hat{\vec{A}}(x_i t) \cdot \hat{\vec{A}}(\vec{x}_i t) \right\}$$

$\vec{A}$ operator which acts on many photon state at $\vec{x}_i$

$\vec{p}$ is a differential operator which operates on everything to the right.

so $\quad \vec{p}_i \cdot \vec{A} \,|\psi\rangle = (\vec{p}_i \cdot \vec{A})\,|\psi\rangle + \vec{A}\cdot\vec{p}_i\,|\psi\rangle$

in the coulomb gauge $\quad \vec{p}\cdot\vec{A} \sim \vec{\nabla}\cdot\vec{A} = 0$

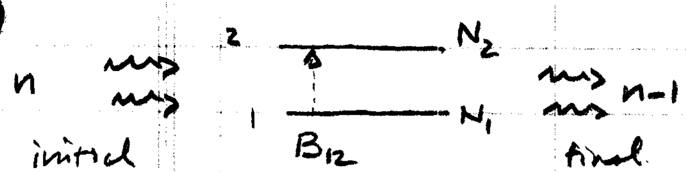
so the term [ ]

can be written $\quad [\ 2\vec{A}(\vec{x}_i,t)\cdot\vec{p}_i\ ]$

$$|\psi\rangle = |\text{many atomic electrons}\rangle \otimes |\text{many photons}\rangle$$

for one two level system exposed to a bath of radiation (leading to stimulated absorption or emission)

$$|\psi_{initial}\rangle = |A_{initial}\rangle \otimes |\ldots n_{k\lambda}\ldots\rangle$$



frequency distribution characterized by $\rho(\nu)$

The transition probability is obviously proportional to

$$|\langle \psi_{final} | H_{int} | \psi_{initial}\rangle|^2$$

treated as a 1st order perturbation.

suppose that only photons with $\omega_k = |\vec{k}|c$ & that only one atom makes the transition.
The process is

$$\langle \ldots, n_{k\lambda}-1, \ldots | \langle A_{final} | H_{int} | A_{initial}\rangle | \ldots n_{k\lambda} \ldots\rangle$$

so we need the parts of $\vec{A}\cdot\vec{p}$ and $\vec{A}\cdot\vec{A}$ which annihilate 1 photon

$$A \sim a + a^+$$

only need this piece why? $\Rightarrow$ $\quad a^+|\ldots n\rangle = |n+1\rangle$, $\quad \langle n-1|n+1\rangle = 0$

... AA will create and annihilate 0 or ±2 photons

$$\langle \ldots, n_{k\lambda}-1, \ldots ; A_{final} \,|\, \frac{-e}{2mc}\left(\sqrt{\frac{\hbar}{2\omega V}}\; \vec{\epsilon}_{k\lambda}\, a_{k\lambda}(0)\, e^{-i\omega_k t}\, e^{i\vec{k}\cdot\vec{x}} \cdot \vec{p}\right) \,|\, A_{initial}; \ldots n_{k\lambda} \ldots \rangle$$

so $\quad a_{k\lambda}(0)\,|\ldots n_{k\lambda} \ldots\rangle = \sqrt{n_{k\lambda}}\,|\ldots n_{k\lambda}-1 \ldots\rangle$

and there are no other fock space operators, so

$$\langle \ldots n-1 \ldots | \ldots n-1 \ldots \rangle = 1$$

$$\frac{-e}{2mc}\sqrt{\frac{\hbar\, n_{k\lambda}}{2\omega V}}\;\langle A_{final}\,|\, e^{i\vec{k}\cdot\vec{x}}\; \vec{\epsilon}_{k\lambda}\cdot\vec{p}\,|\,A_{initial}\rangle\, e^{-i\omega_k t}$$

Now, what would have done for a classical EM field?

you would have used an harmonic perturbation like

$$\vec{A}(\vec{x},t) = \int A(\omega)\,\vec{\epsilon}\, e^{-i\omega t + \vec{x}\cdot\vec{k}}\, d\omega$$

(Merzbacher)

why didn't you see this before? Because in the real field-theoretic approach $A(\omega)$ contains the $\sqrt{n_{k\lambda}}$ and you never see it – you get exactly the same answer – for arbitrarily small or large numbers of photons, either way.

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Emission is a whole other story

$$\langle \ldots, n_{k\lambda}+1, \ldots; A_{final} \,|\, H_{int} \,|\, A_{initial}; \ldots n_k \ldots \rangle$$

$$A \sim a + a^{+}$$

want this one

$$\frac{-e}{2mc}\sqrt{\frac{\hbar}{2\omega V}}\,\sqrt{n_{k\lambda}+1}\,\langle A_{final} \,|\, e^{-i\vec{k}\cdot\vec{x}}\,\vec{\epsilon}^{\,*}_{k\lambda}\cdot\vec{p} \,|\, A_{initial}\rangle e^{i\omega_k t}$$

ahah! see this?

for large numbers of photons, $\underline{\sqrt{n+1}} \sim \sqrt{n}$ and you have the regular, classical result. But, for very small numbers of photons, you get a decidedly non-classical result -- that's spontaneous emission. → also influence the flux.

In a field-theoretic, which means many body, formalism you get both stimulated and spontaneous emissions.

definition of a functional variation of $\delta F[f(x)]$

$$\delta F[f] = F[f+\delta f] - F[f]$$
$$= \int d^3x \; \frac{\delta F[f]}{\delta f(x)} \, \delta f(x)$$

↑ functional derivative of F wrt f @ x.

For field quantities ϕ and π, where generally

$$L(t) = L[\phi(x,t), \dot{\phi}(x,t)]$$

$$\delta L[\phi, \dot{\phi}] = \int d^3x \left(\frac{\delta L}{\delta \phi(x)} \delta \phi(x) + \frac{\delta L}{\delta \dot{\phi}(x)} \delta \dot{\phi}(x) \right)$$

We saw generally, for a field theory, density

$$L(t) = \int d^3x \, \mathcal{L}(\phi(x,t), \vec{\nabla}\phi(x,t), \dot{\phi}(x,t)) \qquad 1^{st} \text{ order deriv.}$$

$$\delta L(t) = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \phi(x,t)} \delta\phi(x,t) + \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\phi(x,t))} \delta \vec{\nabla}\phi(x,t) \right.$$
$$\left. + \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x,t)} \delta \dot{\phi}(x,t) \right)$$

since $\delta \vec{\nabla}\phi = \vec{\nabla}\delta\phi$ and integrating by parts

$$= \int d^3x \left(\left[\frac{\partial \mathcal{L}}{\partial \phi(x,t)} - \vec{\nabla} \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\phi(x,t))} \right] \delta\phi(x,t) \right.$$
$$\left. + \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x,t)} \delta\dot{\phi}(x,t) \right)$$

comparing

$$\frac{\delta L(t)}{\delta \phi(x,t)} = \frac{\partial \mathcal{L}}{\partial \phi(x,t)} - \vec{\nabla} \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\phi(x,t))}$$

$$\frac{\delta L(t)}{\delta \dot{\phi}(x,t)} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x,t)}$$ variation L = regular derivative $\mathcal{L}$

The canonical field is defined by

$$\pi(x,t) \equiv \frac{\delta L(t)}{\delta \dot{\phi}(x,t)} \longrightarrow \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x,t)}$$

From the E.L. equations for L, $\frac{\delta L}{\delta \phi} - \frac{\partial}{\partial t}\frac{\delta L}{\delta \dot{\phi}} = 0$

$$\dot{\pi}(x,t) = \frac{\delta L}{\delta \phi(x,t)}$$

The Hamiltonian:

$$H(t) = \int d^3x \; \pi(x,t)\,\dot{\phi}(x,t) - L(t)$$

or. $$\mathcal{H}(x) = \pi(x)\,\dot{\phi}(x) - \mathcal{L}(x)$$

and

$$\delta H = \int d^3x \left(\dot{\phi}\,\delta\pi + \pi\,\delta\dot{\phi} \right) - \delta L$$

$$\delta L = \int d^3x \left(\frac{\delta L}{\delta \phi}\delta\phi + \frac{\delta L}{\delta \dot{\phi}}\delta\dot{\phi} \right)$$

$$= \int d^3x \left(\dot{\pi}\,\delta\phi + \pi\,\delta\dot{\phi} \right)$$

so

$$\delta H = \int d^3x \left(\dot{\phi}\,\delta\pi + \underline{\pi\,\delta\dot{\phi}} - \dot{\pi}\,\delta\phi - \underline{\pi\,\delta\dot{\phi}} \right)$$

$$= \int d^3x \left(\dot{\phi}\,\delta\pi - \dot{\pi}\,\delta\phi \right)$$

and $\dot{\phi} = \frac{\delta H}{\delta \pi}$ and $\dot{\pi} = -\frac{\delta H}{\delta \phi}$

Remember for classical mechanics,

$$\dot{p} = -\frac{\partial H}{\partial q} \qquad \dot{q} = \frac{\partial H}{\partial p}$$ and Poisson Brackets:

$$\{A,B\}_{PB} \equiv \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$$

Using Ham. Eqs, n

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial p}\dot{p} + \frac{\partial A}{\partial q}\dot{q}$$

$$= \frac{\partial A}{\partial t} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial A}{\partial q}\frac{\partial H}{\partial p}$$

$$= \frac{\partial A}{\partial t} + \{A,H\}_{PB}$$

in particular $\{q,p\} = -\cancel{\frac{\partial p}{\partial q}\frac{\partial q}{\partial p}}^{\,0} + \frac{\partial p}{\partial p}\frac{\partial q}{\partial q}$

$$= 1$$

all referenced in canonical quantization to reg. QM.

Now, for Functionals, $A(\phi,\pi)$, $B(\phi,\pi)$

$$\{A,B\}_{PB} = \int d^3x \left(\frac{\delta A}{\delta\phi(x)} \frac{\delta B}{\delta\pi(x)} - \frac{\delta A}{\delta\pi(x)} \frac{\delta B}{\delta\phi(x)} \right)$$

How about

$$\dot{A}(t) = \int d^3x \left(\frac{\delta A(t)}{\delta\phi(x)} \underbrace{\dot{\phi}(x)}_{\frac{\delta H}{\delta\pi}} + \frac{\delta A(t)}{\delta\pi(x)} \underbrace{\dot{\pi}(x)}_{-\frac{\delta H}{\delta\phi}} \right)$$

$$= \int d^3x \left(\frac{\delta A(t)}{\delta\phi(x)} \frac{\delta H}{\delta\pi} - \frac{\delta A(t)}{\delta\pi(x)} \frac{\delta H}{\delta\phi} \right)$$

$$\dot{A}(t) = \{A,H\}_{PB}$$

An important special case:

$$\phi(\vec{x}\,t) = \int d^3y \, \phi(\vec{y}\,t)\, \delta^3(\vec{x}-\vec{y})$$

$$\frac{\delta\phi(x,t)}{\delta\phi(y,t)} = \delta^3(\vec{x}-\vec{y}) \quad \text{since} \quad \delta\phi(xt) \equiv \int d^3x \, \frac{\delta\phi(x,t)}{\delta\phi(y\,t)} \delta(y\,t)$$

Likewise $\quad \dfrac{\delta\pi(x,t)}{\delta\pi(y,t)} = \delta^3(x-y) \qquad \dfrac{\delta\pi(x\,t)}{\delta\phi(y\,t)} = \dfrac{\delta\phi}{\delta\pi} = 0$

so. $\dot{\phi}(x\,t) = \{\phi(x\,t), H(t)\}_{PB}$

$$= \int d^3y \; \frac{\delta\phi(x\,t)}{\delta\phi(y\,t)} \frac{\delta H}{\delta\pi(y,t)} - \cancel{\frac{\delta\phi(x\,t)}{\delta\pi(y\,t)}} \frac{\delta H}{\delta\phi(y\,t)}$$

0

$$= \int d^3y \; \frac{\delta\phi(x\,t)}{\delta\phi(y\,t)} \frac{\delta H(t)}{\delta\pi(y\,t)}$$

$\delta(x-y)$

$$\dot{\phi}(x,t) = \frac{\delta H(t)}{\delta\pi(x\,t)}$$

and

$$\dot{\pi}(x,t) = -\frac{\delta H(t)}{\delta\phi(x\,t)}$$

Also,

$\delta(x-z)$ $\delta(y-z)$

$$\{\phi(x\,t), \phi(y,t)\}_{PB} = \int d^3z \; \frac{\delta\phi(x\,t)}{\delta\phi(z\,t)} \frac{\delta\pi(y\,t)}{\delta\pi(z\,t)}$$

$= \delta(x-y)$ @ equal times.

The idea is now in parallel —

<u>Classical Mechanics</u> $\longrightarrow$ <u>quantum mechanics</u>

q & p $\rightarrow$ satisfy Ham Eq $\longrightarrow$ operators

$\{q,p\} = 1$ $\longrightarrow$ commutators $[p,q] = i$

<u>classical field theory</u> $\longrightarrow$ <u>quantum field theory</u>

ϕ and π $\rightarrow$ satisfy Ham Eq $\longrightarrow$ operators

$\{\phi, \pi\}_{PB} = \delta(\quad)$ $\longrightarrow$ commutators $[\phi, \pi] = i\delta(\quad)$

Motivation? The Harmonic Oscillator —

<u>Now</u>, it's a many-body theory with <u>states</u> corresponding to the normal modes of an harmonic distribution.

We see the operator-character of the ϕ by

$$a, a^\dagger \longrightarrow \phi = \underbrace{a + a^\dagger}_{\text{operators}} \longrightarrow$$

operators $\Rightarrow$ ϕ-operators