Lecture 9 Quantize Spin 0

SPIN 0

$$\hat{\mathcal{L}}(x) = \frac{1}{2} \frac{\partial\hat{\phi}}{\partial x^\mu} \frac{\partial\hat{\phi}}{\partial x_\mu} - \frac{1}{2} m^2 \hat{\phi}^2$$

and we get the K.G. equation $\frac{\partial^2 \varphi}{\partial x_\mu \partial x^\mu} + m^2 \varphi = 0$

and Hamiltonian

$$\hat{\mathcal{H}} = \hat{\pi}\hat{\dot{\phi}} - \hat{\mathcal{L}}$$
$$= \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\vec{\nabla}\hat{\varphi}\cdot\vec{\nabla}\hat{\varphi} + \frac{1}{2} m\hat{\varphi}^2$$ $\rightarrow$ operators all give + definite eigenvalues $\Rightarrow$ no negative E

-- like relativistic Harmonic Oscillator $\Pi^2 + \omega^2 X^2$
w/ $\omega^2 = k^2 + m^2$

so, we quantize by demanding

$$\left.\begin{aligned} &[\pi_i(\vec{x},t), \varphi_j(\vec{x}',t)] = -i\,\delta_{ij}\,\delta(\vec{x}-\vec{x}') \\ &[\pi_i(\vec{x},t), \pi_j(\vec{x},t)] = [\varphi_i(\vec{x},t), \varphi_j(\vec{x}',t)] = 0 \end{aligned}\right\}\ \text{same time}$$

calculate $[\hat{H}, \hat{\varphi}(\vec{x}',t)]$

drop $\hat{}$ $\rightarrow$ operators, now.

$$= \int d^3x\, [\mathcal{H}, \phi(x',t)]$$
$$= \int d^3x \left[\left(\frac{1}{2}\pi(\vec{x},t)\pi(\vec{x},t) + \frac{1}{2}\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + \frac{1}{2}m^2\phi(\vec{x},t)\phi(\vec{x},t)\right), \phi(\vec{x}',t)\right]$$

suppress spacetime dependence: $\phi' \equiv \phi(\vec{x}',t)$ etc

$$= \frac{1}{2}\int d^3x \,\Big(\pi\pi\phi' + \underline{\vec\nabla\phi\cdot\vec\nabla\phi\,\phi'} + \underline{m^2\phi\phi\phi'}$$
$$- \phi'\pi\pi - \underline{\phi'\vec\nabla\phi\cdot\vec\nabla\phi} - \underline{m^2\phi'\phi\phi} \Big)$$

since $[\phi, \phi'] = 0$

$$= \frac{1}{2}\int d^3x \,\big(\pi\underline{\pi\phi'} - \underline{\phi'}\pi\pi \big)$$

$$[\pi, \phi'] = \pi\phi' - \phi'\pi = -i\,\delta^3(\vec x - \vec x')$$
$$\pi\phi' = \phi'\pi - i\delta^3(\vec x - \vec x')$$

$$H = \frac{1}{2}\int d^3x \,\big(\underline{\pi\phi'\pi} - i\pi\,\delta^3(\vec x - \vec x') - \underline{\pi\phi'\pi} - \pi\, i\delta^3(\vec x - \vec x') \big)$$

$$= \frac{1}{2}\int d^3x \;\; 2\pi(\vec x, t)\,\delta^3(\vec x - \vec x')$$

$$= -i\pi(\vec x', t) \;=\; -i\dot\phi(\vec x', t)$$

so $\quad [H, \phi(\vec x, t)] = -i\,\dfrac{\partial\phi}{\partial t}(\vec x', t)$

$\Rightarrow$ field operators satisfy Heisenberg's equations of motion — which characterized the time dependence.

Normalization: many conventions to deal with and the need to pass from a box normalization (Σ) to continuous normalization ($\int$).

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Remember that we naturally found a normalization for $\vec{A} \propto \frac{1}{\sqrt{2\omega V}} \left(\quad \right)$. This is part of the story, but

not enough. By allowing the "box" to become unbounded, in essence, our Fourier series representation with standing wave solutions $\longrightarrow$ Fourier integral representation for traveling wave solutions, so this would suggest

$$\phi(\vec{x}, t) \propto \int_{-\infty}^{\infty} \frac{d^3 k}{2\pi \cdot \text{stuff}} \left(a(k) e^{-ik\cdot x} + hc \right)$$

The most modern way to assign "stuff" is to use

$$\int \frac{d^3k}{(2\pi)^3 2\omega_k} \qquad \text{because} \qquad \int \frac{d^3k}{2\omega_k} \text{ is}$$

Lorentz invariant. which can be seen by noting

$$d^3k\, dk_0$$

$$\frac{d^3k}{2\omega_k} = d^4k\, \delta(k^2 - m^2)\, \theta(k_0) \qquad \text{positive energies only}$$

remember that, under an integral,

$$\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|df/dx|_{x=x_i}} \qquad \text{where } x_i \text{ are the simple zeros of } f(x_i) = 0$$

here, $f = k^2 - m^2 = f(k_0)$

$$= k_0^2 - \vec{k}^2 - m^2$$

$$k_0 = \pm\sqrt{\vec{k}^2 + m^2}$$

$$= \pm \omega_k \qquad \text{zero.}$$

so $\quad df/dk_0 = 2k_0$

$$\left| df/dk_0 \right|_{k_0 = \omega_k} = 2\omega_k$$

and $\quad \delta(k^2 - m^2) = \dfrac{\sum_{i=\pm} \delta(k_0 - \omega_{k_i})}{2\omega_k}$

$$= \frac{\delta(k_0 + \omega_k) + \delta(k_0 - \omega_k)}{2\omega_k}.$$

$$(2\pi)\frac{d^4k}{(2\pi)^4}\,\delta(k^2 - m^2)\,\theta(k_0) = \frac{d^4k}{2\omega_k(2\pi)^3}\left[\delta(k_0 + \omega_k) + \delta(k_0 - \omega_k)\right]\theta(k_0)$$

$$= \frac{d^3\vec{k}\,dk_0}{(2\pi)^3 2\omega_k}\,\delta(k_0 - \omega_k)$$

$$= \frac{d^3k}{(2\pi)^3 2\omega_k}\Bigg|_{k_0 = \underline{\omega_k} \text{ as a condition.}} \qquad = \sqrt{\vec{k}^2 + m^2}$$

So, we have

$$\phi(x) = \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2 - m^2)\,f_k(x)$$

where $\quad f_k(x) = A(k)\,e^{-ik\cdot x}$ $\quad$ a general solution to K.G. equation

$$\phi(x) = \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2 - m^2)\,A(k)\,e^{-ik\cdot x}$$

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from above

$$\phi(x) = \int \frac{d^3k\, dk_0}{(2\pi)^3 2\omega_k} A(k_0, \vec{k}) \left[\delta(k_0+\omega_k) + \delta(k_0-\omega_k) \right] e^{-ik\cdot x}$$

$$e^{-ik\cdot x} = e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[A(-\omega_k, \vec{k}) e^{i(\omega_k t + \vec{k}\cdot\vec{x})} + A(\omega_k, \vec{k}) e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} \right]$$

write

$$A(-\omega_k, \vec{k})\, e^{i\vec{k}\cdot\vec{x}} = A(-\omega_k, -\vec{k})\, e^{-i\vec{k}\cdot\vec{x}}$$

since $k_0^2 = \vec{k}^2 + m^2$ for either sign of k_0,

$$A(\omega_k, \vec{k}) = A(\vec{k}) \equiv a(\vec{k})$$
$$A(-\omega_k, -\vec{k}) = A(-\vec{k}) \equiv a(-\vec{k})$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(-\vec{k}) e^{+ik\cdot x} + a(\vec{k}) e^{-ik\cdot x} \right]$$

also $$\phi^+(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a^+(-\vec{k}) e^{-ik\cdot x} + a^+(\vec{k}) e^{ik\cdot x} \right]$$

for a <u>real</u> field, $\phi = \phi^+$, so $a^+(-\vec{k}) = a(\vec{k})$

$$a^+(\vec{k}) = a(-\vec{k})$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \Big[\underbrace{a(\vec{k}) e^{-ik\cdot x}}_{+\text{ energy}} + \underbrace{a^+(\vec{k}) e^{ik\cdot x}}_{-\text{ energy}} \Big]$$

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We can figure out the algebra of the a's. It's standard to define the "positive (and negative) frequency solutions" — to precisely define the a's:

$$f_k(x) \equiv \frac{1}{\left[(2\pi)^3 2\omega_k\right]^{1/2}} e^{-ik\cdot x}$$

+ frequency component

− freq comp.

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a(\vec{k}) f_k(x) + a^\dagger(\vec{k}) f_k^*(x) \right]$$

$$\equiv \phi^{(+)}(x) + \phi^{(-)}(x)$$

form, $\displaystyle \int f_{k'}^*(x)\, \phi(x)\, d^3x = \int d^3x \frac{1}{\sqrt{}} \int \frac{d^3k}{(2\pi)^3 2\omega_k}$

$$\left[a(\vec{k})\, \underbrace{e^{ik'\cdot x} e^{-ik\cdot x}}_{} + a^\dagger(\vec{k})\, e^{ik'\cdot x} e^{ik\cdot x} \right]$$

$$e^{ik'_0x^0 - i\vec{k}'\cdot\vec{x}}\ \underbrace{e^{-ik^0x^0 + \vec{k}\cdot\vec{x}}}_{\text{etc.}}$$

so $\left(\text{define } dK \equiv \dfrac{d^3k}{(2\pi)^3 2\omega_k}\right)$

$$= \int d^3x \frac{1}{\sqrt{}} \int dK \Big[a(\vec{k})\, e^{i(k^{0\prime} - k^0)x^0 - i(\vec{k}' - \vec{k})\cdot\vec{x}}$$

$$+ a^\dagger(\vec{k})\, e^{i(k^{0\prime} + k^0)x^0 - i(\vec{k}' + \vec{k})\cdot\vec{x}} \Big]$$

integrate $\int d^3\vec{x}$

$$= \frac{1}{\sqrt{}} \int dK \Big[a(\vec{k})\, e^{i(k_0'-k_0)t} (2\pi)^3 \delta(\vec{k}'-\vec{k})$$

$$+ a^\dagger(\vec{k})\, e^{i(k_0'+k_0)t} (2\pi)^3 \delta(\vec{k}'+\vec{k}) \Big]$$

$$= \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} \frac{1}{(2\pi)^3 2\omega_k} (2\pi)^3 \Big[a(\vec{k}')\, e^{i(k_0'-k_0)t} + a^\dagger(-\vec{k})\, e^{i(k_0'+k_0)t} \Big]$$

note -- δ integration forces $\vec{k} = \vec{k}'$

and $m^2 = \vec{k}'^2 + k^{0\prime 2} = \vec{k}^2 + k^{0\prime 2}$

$\Rightarrow k^{0\prime} = k^0 = \omega_k$.

$$= \frac{(2\pi)^3}{[\quad]^{3/2}} \Big[a(\vec{k}') + a^\dagger(-\vec{k})\, e^{2i\omega_k t} \Big]$$

$$= \frac{1}{\sqrt{}} \frac{1}{2\omega_k} \Big[a(\vec{k}') + a^\dagger(-\vec{k})\, e^{2i\omega_k t} \Big]$$

Similarly, calculate

$$\int f_{k'}^*(x)\, \dot{\phi}(x)\, d^3x = \frac{-i}{2} \frac{1}{\sqrt{}} \Big[a(\vec{k}') - a^\dagger(-\vec{k})\, e^{2i\omega t} \Big]$$

now, isolate the a,

$$\int f_{k'}^*(x) \big[\omega_k \phi(x) + i \dot{\phi}(x) \big] d^3x = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}}\, a(\vec{k}')$$

so,

$$a(\vec{k}') = \int f_{k'}^*(x)\left[\omega_{k'}\phi(x) + i\dot{\phi}(x)\right]\sqrt{(2\pi)^3 2\omega_k}\; d^3x$$

$$= i\int d^3x \sqrt{}\left[-\frac{\partial f_{k'}^*}{\partial t}\phi(x) + f_{k'}^*(x)\frac{\partial\phi(x)}{\partial t}\right]$$

$$\equiv i\int d^3x\sqrt{}\left(f_{k'}^*(x)\overleftrightarrow{\partial_0}\phi(x)\right)$$

where $\quad A\overleftrightarrow{\partial_0}B \equiv A\partial_0 B - (\partial_0 A)B$

Likewise, $\quad a^\dagger(\vec{k}) = -i\int d^3x\sqrt{}\; f_k(x)\overleftrightarrow{\partial_0}\phi(x)$

so, now work out the algebra of the a's,

$$[a(k), a^\dagger(k')] = +\int d^3x\int d^3x'\,(2\pi)^3\sqrt{2\omega_k 2\omega_{k'}}\left[f_k^*(x)\overleftrightarrow{\partial_0}\phi(x), f_{k'}(x')\overleftrightarrow{\partial_0}\phi(x')\right.$$

look at commutator,

$$[\ =,\ ] = \left[f_k^*(x)\frac{\partial\phi(x)}{\partial t} - \frac{\partial f_k^*(x)}{\partial t}\phi(x),\quad \text{etc.}\right]$$

use shorthand $\quad f_{k'}(x') \equiv f'$

$\qquad\qquad\qquad \phi(x') \equiv \phi'$

$$= [f^*\dot{\phi} - \dot{f}^*\phi,\ f'\dot{\phi}' - \dot{f}'\phi']$$

$$= f^*\dot{\phi}f'\dot{\phi}' - f^*\dot{\phi}\dot{f}'\phi' - \dot{f}^*\phi f'\dot{\phi}' + \dot{f}^*\phi\dot{f}'\phi'$$

$$- f'\dot{\phi}'f^*\dot{\phi} + \dot{f}'\phi'f^*\dot{\phi} + f'\dot{\phi}'\dot{f}^*\phi - \dot{f}'\phi'\dot{f}^*\phi$$

f's are functions. $\quad \dot{\phi} = \pi$

$$= f^* f' \pi\pi' - f^* \dot{f}' \pi \phi' - \dot{f}^* f' \phi \pi' + \dot{f}^* \dot{f}' \phi\phi'$$
$$- f' f^* \pi' \pi + \dot{f}' f^* \phi' \pi + f' \dot{f}^* \pi' \phi - \dot{f}' \dot{f}^* \phi' \phi$$
$$= -\dot{f}' f^* [\pi, \phi'] + f' \dot{f}^* [\pi', \phi] + 0$$

$$= i \dot{f}' f^* \delta(\vec{x} - \vec{x}') - i f' \dot{f}^* \delta(\vec{x} - \vec{x}')$$

so,

$$[a(\vec{k}), a^\dagger(\vec{k}')] = +i \int d^3x \int d^3x' \, (2\pi)^3 \sqrt{2\omega_k 2\omega_{k'}}$$

$$\left[\dot{f}' f^* \delta(\vec{x} - \vec{x}') - f' \dot{f}^* \delta(\vec{x} - \vec{x}') \right]$$

$$= i \int d^3x \, (2\pi)^3 \sqrt{2\omega_k 2\omega_{k'}} \left[\dot{f}' f^* - f' \dot{f}^* \right]$$

$$= i \int d^3x \, (2\pi)^3 \sqrt{2\omega_k 2\omega_{k'}} \times$$

$$\left[\left(\frac{-i\omega_{k'} e^{-ik' \cdot x}}{\sqrt{'}} \right) \left(\frac{e^{ik \cdot x}}{\sqrt{}} \right) - \left(\frac{e^{-ik \cdot x}}{\sqrt{'}} \right) \left(\frac{i\omega_k e^{ik \cdot x}}{\sqrt{}} \right) \right]$$

$$= i \left[-i\omega_{k'} e^{-i(k^{0\prime} - k^0)t} (2\pi)^3 \delta^3(\vec{k}' - \vec{k}) - i\omega_k e^{-i(k^{0\prime} - k^0)t} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \right]$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = 2\omega_k (2\pi)^3 \delta^3(\vec{k}' - \vec{k})$$

(by looking closed and realizing that $\delta(\vec{k}' - \vec{k})$ forces $\vec{k}' = \vec{k}' \Rightarrow k^0 = k^{0\prime}$)

Likewise $\quad [a(\vec{k}), a(\vec{k}')] = [a^\dagger(\vec{k}), a^\dagger(\vec{k})] = 0$

The Hamiltonian ...

$$\mathcal{H} = \tfrac{1}{2}\left[\pi^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2\right]$$

so $\quad H = \int d^3x\, \mathcal{H} = \frac{1}{2}\int d^3x \left[\qquad\qquad\right]$

$$\phi(x) = \int d\tilde{K}\left[a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right]$$

$$\pi(x) = \dot{\phi}(x) = \int d\tilde{K}(-i\omega_k)\left[a(k)e^{-ik\cdot x} - a^\dagger(k)e^{ik\cdot x}\right]$$
$$\vec{\nabla}\phi = \int d\tilde{K}\,(i\vec{k})\left[a(k)e^{-ik\cdot x} - a^\dagger(k)e^{ik\cdot x}\right]$$

so, $\quad H = \frac{1}{2}\int d^3x \int d\tilde{K}\int d\tilde{K}'$

$$\times\Big\{ (-i\omega_k)(-i\omega_{k'})\left[a e^{-ik\cdot x} - a^\dagger e^{ik\cdot x}\right]\left[a' e^{-ik'\cdot x} - a^{\dagger\prime} e^{ik'\cdot x}\right]$$
$$+ (i\vec{k})\cdot(i\vec{k}')\left[\quad\right]\left[\quad\right]$$
$$+ m^2\left[a e^{-ik\cdot x} + a^\dagger e^{ik\cdot x}\right]\left[a' e^{-ik'\cdot x} + a^{\dagger\prime} e^{ik\cdot x}\right]\Big\}$$

we know that H is a constant of the motion $\Rightarrow \dot{H} = 0$

so we can choose a convenient time to do this at --

$t = 0$

$$H = \frac{1}{2}\int d^3x \int dK \int dK'$$

$$\times \Big\{ (-i\omega_k)(-i\omega_{k'}) \left[a e^{i\vec{k}\cdot\vec{x}} - a^\dagger e^{-i\vec{k}\cdot\vec{x}} \right]\left[a' e^{i\vec{k}'\cdot\vec{x}} - a^{\dagger\prime} e^{-i\vec{k}'\cdot\vec{x}} \right]$$

$$+ \quad (i\vec{k})\cdot(i\vec{k}') \left[\quad \right]\left[\quad \right]$$

$$+ \quad m^2 \left[a e^{i\vec{k}\cdot\vec{x}} + a^\dagger e^{-i\vec{k}\cdot\vec{x}} \right]\left[a' e^{i\vec{k}'\cdot\vec{x}} + a^{\dagger\prime} e^{-i\vec{k}'\cdot\vec{x}} \right] \Big\}$$

$$[\ ][\ ] = a a' e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} - a a^{\dagger\prime} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$- a^\dagger a' e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} + a^\dagger a^{\dagger\prime} e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}$$

again, recognize δ's coming out

$$\int d^3x\, e^{\pm i(\vec{k}+\vec{k}')\cdot\vec{x}} = (2\pi)^3 \delta(\vec{k}+\vec{k}') \Rightarrow \vec{k} = -\vec{k}'$$

$$\int d^3x\, e^{\pm i(\vec{k}-\vec{k}')\cdot\vec{x}} = (2\pi)^3 \delta(\vec{k}-\vec{k}') \Rightarrow \vec{k} = \vec{k}'$$

so, integrating over the $\int d^3x$ gives, after some work...

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[\frac{1}{2}\omega_k \left(a^\dagger(\vec{k}) a(\vec{k}) + a(\vec{k}) a^\dagger(\vec{k}) \right) \right]$$

which looks like our EM oscillator Hamiltonian, so we're free to make the same interpretation of the operators $a(\vec{k})$ and $a^\dagger(\vec{k})$

... we create and annihilate massive spin ϕ quanta one at a time starting from the vacuum ~~having~~ ~~moment~~

We need to rearrange our energy scale again -- just a little. --

$$[a(k), a^{\dagger}(k)] = (2\pi)^3 2\omega_k \delta(0)$$

$$H = \int dK \left\{ \tfrac{1}{2}\omega_k \left[a^{\dagger}a + a^{\dagger}a + (2\pi)^3 2\omega_k \delta(0) \right] \right.$$

$$= \int dK \, \omega_k \, a^{\dagger}(\vec{k}) a(\vec{k}) + \underbrace{\int_{-\infty}^{\infty} d^3k \, \delta(0)}_{\text{an infinite offset } \equiv H_\infty}$$

subtract H_∞ -- from now on

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k \, a^{\dagger}(\vec{k}) a(\vec{k})$$

We can insure this adjustment of the energy scale through an overt ordering of the product of operators... to wit:

our new choice of energy scale ensures $\langle 0|H|0\rangle = 0$

since this is $\langle 0|a^{\dagger}a|0\rangle$

to insure this, we used the commutator, then shifted the energy scale by H_∞

the same thing could be insured by changing the order of all products of a and a^+ so that always:

a's are to the RIGHT of a^+'s

This is called Normal Ordering --

Remember $\phi = \phi^{(+)} + \phi^{(-)}$

$\phi^{(+)}$: + freq, a $\qquad$ $\phi^{(-)}$: − freq, a^+

so arranging for + to be RIGHT of −

eq. $$\phi(x)\phi(y) = \left(\phi^{(+)}(x) + \phi^{(-)}(x)\right)\left(\phi^{(+)}(y) + \phi^{(-)}(y)\right)$$

$$\equiv (x^+ + x^-)(y^+ + y^-)$$

$$= x^+y^+ + x^+y^- + x^-y^+ + x^-y^-$$

like $aa^+ \Rightarrow \langle 0|x^+y^-|0\rangle \neq 0$.

normal ordered products are designated

$$: \phi(x)\phi(y): \;=\; x^+y^+ + y^-x^+ + x^-y^+ + x^-y^-$$

$$= \phi^{(+)}(x)\,\phi^{(+)}(y) + \phi^{(-)}(y)\,\phi^{(+)}(x) + \phi^{(-)}(x)\,\phi^{(+)}(y) + \phi^{(-)}(x)\,\phi^{(-)}(y)$$

so by shifting the energy scale as we did we've done

$$H \rightarrow \;:H: \qquad \text{— this will happen everywhere}$$

One particle normalization:

let $\qquad |\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle$

then $\qquad \langle \vec{k}|\vec{k}'\rangle = \langle 0|a(\vec{k})a^\dagger(\vec{k}')|0\rangle$

$$= \langle 0|\,[a(\vec{k}), a^\dagger(\vec{k}')]\,|0\rangle + \underbrace{\langle 0|a^\dagger(\vec{k}')a(\vec{k})|0\rangle}_{=0}$$

$$= (2\pi)^3 2\omega_k\, \delta(\vec{k}-\vec{k}')\underbrace{\langle 0|0\rangle}_{=1}$$

$$\langle \vec{k}|\vec{k}'\rangle = (2\pi)^3 2\omega_k\, \delta(\vec{k}-\vec{k}')$$

called covariant normalization