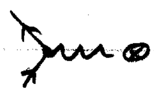
We'll work on the radiative corrections to Coulomb scattering — as simpler and historically the first calculation (by Schwinger).

Our diagram is  ← essentially a Feynman rule:

$\frac{4\pi Ze}{|q|^2}$ for the effective potential

$-ie\gamma^0$ for the vertex

This is of order e^2 (α) and contributes to $S^{(1)}$.

Quantum corrections will involve the addition of another γ-e set of vertices. Write,

$$\mathcal{H}_I = \mathcal{H}_I(x) + \mathcal{H}_I^{ext}(x)$$

usual $\bar\psi\gamma^\mu\psi A_\mu$; $\bar\psi\gamma^\mu\psi A_\mu^{ext}$ — classical

The next order correction comes from

$$S^{(2)} = \frac{(-i)^2}{2!}\int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2 \; P\left[\left(\mathcal{H}_I(x_1) + \mathcal{H}_I^{ext}(x_1)\right)\left(\mathcal{H}_I(x_2) + \mathcal{H}_I^{ext}(x_2)\right)\right]$$

$$= \frac{(-i)^2}{2}\int d^4x_1 \int d^4x_2 \; P\left[\mathcal{H}_I(x_1)\mathcal{H}_I(x_2) + \mathcal{H}_I(x_1)\mathcal{H}_I^{ext}(x_2) + \mathcal{H}_I^{ext}(x_1)\mathcal{H}_I(x_1) + \mathcal{H}_I^{ext}(x_1)\mathcal{H}_I^{ext}(x_1)\right]$$

We'll concentrate on the $\mathcal{H}\mathcal{H}^{ext}$ etc pieces. Wick's Theorem instructs us...

$$S^{(2)} = \frac{(-i)^2}{2}\, 2 \int d^4x_1 \int d^4x_2\, P\left[\mathcal{H}_I^{ext}(x_1)\mathcal{H}_I(x_2)\right]$$

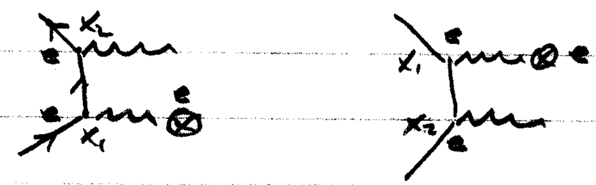
$$= (-ie)^2 \gamma^{\mu}_{ij}\gamma^{\nu}_{\ell m} \int d^4x_1 \int d^4x_2\, P\left[\bar\Psi_i(x_1)\Psi_j(x_1) A^{ext}_{\mu}(x_1) \bar\Psi_\ell(x_2)\Psi_m(x_2) A_\nu(x_2)\right]$$

$$= (-ie)^2 \gamma^{\mu}_{ij}\gamma^{\nu}_{\ell m} \int d^4x_1 \int d^4x_2$$

$$\times \Big\{ :\underbrace{\bar\Psi_i(x_1)\Psi_j(x_1)\bar\Psi_\ell(x_2)\Psi_m}(x_2):\,:A^{ext}_{\mu}(x_1)A_\nu(x_2):$$

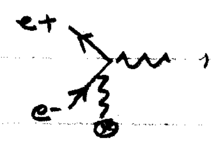
$$+\; :\bar\Psi_i(x_1)\underbrace{\Psi_j(x_1)\bar\Psi_\ell}(x_2)\Psi_m(x_2):\,:A^{ext}_{\mu}(x_1)A_\nu(x_2): \Big\}$$

are the only non-zero contributions. From this, we infer the diagrams:



$\sim e^3$

which is Bremsstrahlung (of course other $S^{(2)}$ reactions occur like



, but these aren't our specific process..)

The next order is

$$S^{(3)} = \frac{(-i)^3}{3!} \int d^4x_1 \int d^4x_2 \int d^4x_3\, P\Big\{ \mathcal{H}_I(x_1)\mathcal{H}_I(x_2)\mathcal{H}_I^{ext}(x_3) + \mathcal{H}_I^{ext}\mathcal{H}_I(x_2)\mathcal{H}_I(x_3) + \text{etc.}\Big\}$$

again, keeping only terms with one $\mathcal{H}_I^{ext}$,

$$S^{(3)} = \frac{(-i)^3}{2} \int d^4x_1 \int d^4x_2 \int d^4x_3\, P\Big\{ \underbrace{\mathcal{H}_I(x_1)\mathcal{H}_I(x_2)\mathcal{H}_I^{ext}(x_3)}_{\text{3 vertices}} \Big\}$$

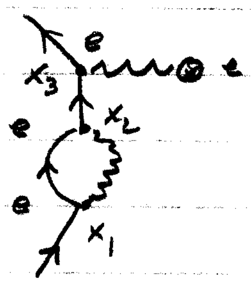
sketching the solutions...

$$\gamma^\mu\gamma^\nu\gamma^\lambda\, P\{ \underbrace{\bar\psi(x_1)\psi(x_1)A_\mu(x_1)\,\bar\psi(x_2)\psi(x_2)A_\nu(x_2)\,\bar\psi(x_3)\psi(x_3)A^{ext}_\lambda(x_3)}_{\text{doubly and triply contracted terms}} \}$$

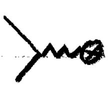
for example

$$\gamma^\mu_{ij}\gamma^\nu_{kl}\gamma^\lambda_{mn}\int d^4x_1\int d^4x_2\int d^4x_3 \Big[:\bar\psi_i(x_1)\psi_j(x_1)A_\mu(x_1)\,\bar\psi_k(x_2)\psi_l(x_2)A_\nu(x_2)\,\bar\psi_m(x_3)\psi_n(x_3)A^{ext}_\lambda(x_3):$$

- 2 electron propagators $x_1 - x_2$ & $x_2 - x_3$
- 1 photon propagator: $x_1 - x_2$
- incoming e annihilated at x_1
- outgoing e created at x_3
- external field connected at x_3

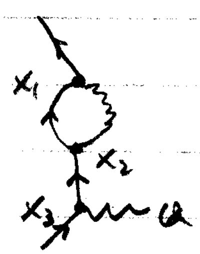


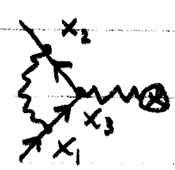
$\Theta(e^4) \sim \alpha^2$

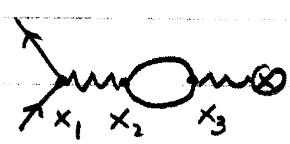
smaller than  by α^2 in σ

smaller than  by α in σ

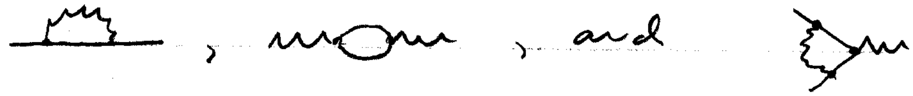
can also get



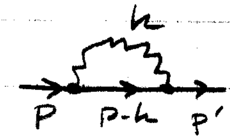




These diagrams all have within them a set of subdiagrams which can be treated as pieces of larger diagrams. They're called "primative divergences"



Look at the first one in momentum space.



$$T = \int \frac{d^4k}{(2\pi)^4} \; \bar{u}(p')(-ie\gamma_\mu) \frac{i}{\not{p} - \not{k} - m} (-ie\gamma_\nu)\left(-\frac{ig^{\mu\nu}}{k^2}\right) u(p)$$

$$T = \bar{u}(p') \int \frac{d^4k}{(2\pi)^4} \left[-i(-ie)^2 \gamma_\mu \frac{1}{\not{p}-\not{k}-m} \gamma_\nu \frac{g^{\mu\nu}}{k^2} \right] u(p)$$

This has a name -- the "self energy" of the electron,

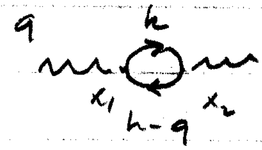
$$-i\Sigma(p) \equiv (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu \frac{1}{\not{p}-\not{k}-m} \gamma_\nu \frac{g^{\mu\nu}}{k^2}$$

so $\quad T = \bar{u}(p')\left[-i\Sigma(p)\right] u(p)$

Collect powers of k. -- $\quad \int dk \frac{k^3}{k^3} \quad$ or $\quad \int dk \frac{k^3}{k^2}$

$$\int dk$$

$\Rightarrow$ at least linearly divergent @ high k.

look at  ... "Vacuum polarization"

$$T \sim \left(\frac{-ie}{2}\right)^2 \int d^4x_1 \int d^4x_2 \; : \bar{\psi}_i(x_1)\, \psi_j(x_1) A_\mu(x_1)\, \bar{\psi}_k(x_2)\, \psi_l(x_2) A_\nu(x_2) : \gamma^\mu_{ij}\, \gamma^\nu_{kl}$$

$$= -\left(\frac{-ie}{2}\right)^2 \int d^4x_1 \int d^4x_2 \quad \psi_l(x_2)\, \bar{\psi}_i(x_1)\, \gamma^\mu_{ij}\, \psi_j(x_1)\, \bar{\psi}_k(x_2)\, \gamma^\nu_{kl} \; : A_\mu A_\nu :$$

trace

remember

$$\psi_l(x_2)\, \bar{\psi}_i(x_1) = i S_F(x_2 - x_1)_{li}$$

$$\psi_j(x_1)\, \bar{\psi}_k(x_2) = i S_F(x_1 - x_2)_{jk}$$

$$= -\frac{1}{2} \int d^4x_1 \int d^4x_2 \; \mathrm{Tr}\left[i S_F(x_2 - x_1)(-ie\gamma^\mu)\, i S_F(x_1 - x_2)(-ie\gamma^\nu) \right]$$

$$\times : A_\mu(x_1)\, A_\nu(x_2) :$$

so in momentum space,

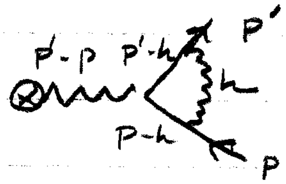
$$\propto \quad \varepsilon^\alpha(q) \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\left[(-ie\gamma_\alpha) \frac{i}{\not{k} - m} (-ie\gamma_\beta) \frac{i}{\not{k} - \not{q} - m} \right] \varepsilon^\beta(q')$$

define $$ie^2\, \Pi(q^2)_{\alpha\beta} = \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\left[(-ie\gamma_\alpha) \frac{i}{\not{k} - m} (-ie\gamma_\beta) \frac{i}{\not{k} - \not{q} - m} \right]$$

collecting powers of k..

$$\int dk \frac{k^3}{k^2} \sim \int dk\, k$$ at least quadratic divergence

Finally,



"vertex correction"

$$\int \frac{d^4k}{(2\pi)^4}\, \bar u(p')(-ie\gamma_\alpha)\frac{i}{\not p' - \not k - m}(-ie\gamma_\mu)\frac{i}{\not p - \not k - m}(-ie\gamma_\beta)\frac{(-ig^{\alpha\beta})}{k^2}\, u(p) A_\mu(p-p')$$

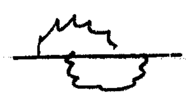
define $$-ie\Lambda_\mu(p,p') \equiv \int \frac{d^4k}{(2\pi)^4}(-ie\gamma_\alpha)\frac{i}{\not p' - \not k - m}(-ie\gamma_\mu)\frac{i}{\not p - \not k - m}(-ie\gamma_\beta)\left(\frac{-ig^{\alpha\beta}}{k^2}\right)$$

$$\int dk\, \frac{k^3}{k^4} \sim \int \frac{dk}{k}$$ ~~at least~~ logarithmically divergent

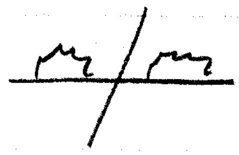
Dealing with these divergent objects in a consistent and physically meaningful way constitutes the program of Renormalization. We can get an early idea of the strategy by considering the electron self energy graph.

Terminology:

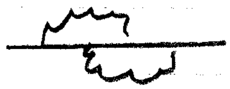
Take a single fermion line and attach self-energy diagrams in a variety of ways.

 ;  ;  ; etc.

One particle irreducible graphs (1PI) cannot be separated into primatively divergent graphs with a single cut. Reducible graphs can be.

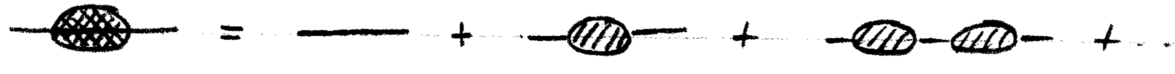


reducible

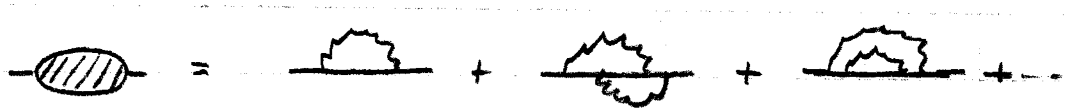


1PI

A completely general graph for a fermion line — to arbitrary order — can be drawn:



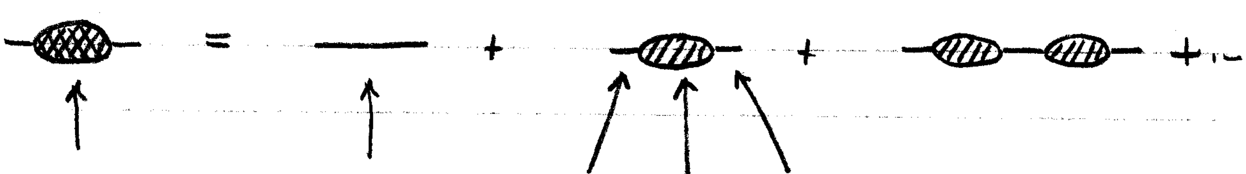
where  denotes 1PI

ie, 

(excluding "external" lines)

Define $-i\Sigma(p) \equiv$  1PI

and $iS_F'(p) \equiv$  the whole thing



$S_F'(p) \qquad iS_F(p) + iS_F(p)\left[-i\Sigma(p)\right]iS_F(p) + iS_F(-i\Sigma)iS_F(-i\Sigma)iS_F$

$+ \ldots$

"bare" propagator

$\dfrac{i}{\not{p} - m_0}$ ← "bare mass"