renormalizability

Renormalizability –

The study of divergences is well developed and it was discovered a while ago that it depends critically on the number of spacetime dimensions. This is most easily seen in a toy theory like φ^4 – "phi-fourth theory".

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{1}{2}m^2\varphi - \underbrace{\frac{g}{4!}}_{\text{quartic self interaction}}\varphi^4$$

$$[\mathcal{L}] = \frac{\text{energy}}{\text{volume}} = \text{energy}\cdot\text{energy}^3 = E^4$$

use notation $\qquad [\mathcal{L}] = [p^4] = 4 \;$ = # spacetime dimensions, d, generally.

then, the units of φ: $\qquad [m^2\varphi^2] = 2[\varphi^2] = 4$

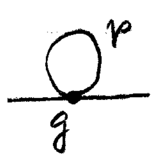
$$[\varphi] = 1$$

so, units of φ are "energy"

This power counting is useful. – Notice that

$[g] = 0$ in this theory $\Rightarrow$ dimensionless.

Now look at a loop

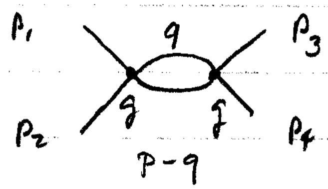


self energy $\qquad g\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-m^2}$

so the degree of divergence. $\int dp \frac{p^3}{p^2} \sim \int p\,dp$

or quadratic

or



$$g^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2-m^2)} \frac{1}{[(p-q)^2-m^2]}$$

$$\sim \int \frac{q^3 dq}{q^4} \quad \text{log divergent.}$$

For L loops and I internal lines, the degree of divergence is

$$D = \underbrace{4}_{\equiv d,\ \#\text{ spacetime dimensions}}L - 2I \qquad = 4(1) - 2(1) = 2$$

$$= 4(1) - 2(2) = 0 \quad (\text{means "log"})$$

$$= dL - 2I$$

If $\exists$ I internal momenta, then $\exists$:

i) momentum conservation @ each vertex

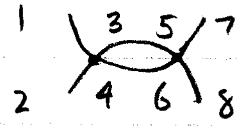
ii) n vertices

iii) overall momentum-conservation $\to$ $n-1$ relations among momenta.

$\Rightarrow$ # loops $= L = I - (n-1) =$ # independent momenta.

Count legs: E external

I internal



4 legs @ each vertex (each vertex counts twice)

each vertex = 4 legs.

total # legs = $4n = E + 2I \Rightarrow I = \frac{4n-E}{2}$

internal ones count twice, connected to 2 vertices.

putting together, eliminating I, L...

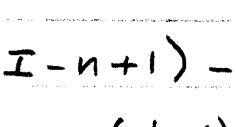
$$D = d(I - n + 1) - (4n - E)$$

$$D = d + n(d-4) - \left(\tfrac{d}{2} - 1\right)E$$

for $d = 4$
$$D = 4 + n(0) - (2-1)E$$
$$D = 4 - E = \left.\begin{matrix} 2 \\ 0 \end{matrix}\right\}\text{ for our two}$$

$\Rightarrow$ independent of n in $4d$, all diagrams with more than 4 external legs will converge, $D \leq -1$ —

so



would be convergent.

Our two graphs renormalize the mass and coupling -- all infinities are used up and more loops do no damage.

For electrodynamics, things are slightly different. The analogous formulae are

$$D = dL - 2P_i - E_i$$

internal γ lines   internal electron lines

$$L = E_i + P_i - n + 1$$

so

$$D = d + n\left(\frac{d}{2} - 2\right) - \left(\frac{d-1}{2}\right)E_e - \left(\frac{d-2}{2}\right)P_e$$

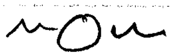
external

For $d = 4$

$$D = 4 - \tfrac{3}{2}E_e - P_e$$

independent of number of vertices $\Rightarrow$ absolutely renormalizable to all orders $\rightarrow$ so you can always soak up Z's to renormalize bare quantities.

$$D = 4 - \tfrac{3}{2}(2) - 0 = 1$$



$$= 4 - \tfrac{3}{2}(0) - 2 = 2$$

$$= 4 - \tfrac{3}{2}(2) - 1 = 0$$

This renormalizability condition is often used as a criterion for acceptable theories.

The coupling requires some thought

$$\mathcal{L}_I = -e A^\mu \bar{\psi} \gamma_\mu \psi$$

$$[\mathcal{L}] = d = [m\bar{\psi}\psi] = 1 + [\psi^2] = 1 + 2[\psi]$$

so

$$[\psi] = \frac{d-1}{2}$$

and $$d = [\partial_\mu A^\mu]^2 = 2(1) + 2[A]$$

$$[A] = \frac{d}{2} - 1$$

so

$$[A\psi\psi] = d = [e] + [A] + 2[\psi]$$

$$= [e] + \frac{d}{2} - 1 + d - 1$$

$$[e] = -\frac{d}{2} + 2$$

ie for 4 dimensions e is dimensionless (an important part of QED's renormalizability) what about weak interactions?

$$\mathcal{L} = \frac{G}{\sqrt{2}} \bar{\psi}\gamma\psi\bar{\psi}\gamma\psi$$

$$d = [G] + 4[\psi]$$

$$[\psi] = \frac{d-1}{2}$$

$$[G] = d - 4\left(\frac{d-1}{2}\right)$$
$$= d - 2d + 2$$
$$[G] = 2-d$$

So $4d \Rightarrow [G] = -2$ ie, $M^{-2} \Rightarrow$ not dimensionless — a problem. solved by the Weinberg Salam — t'hooft - Veltman approach.

The program is to exploit the bad behavior's correlation with the spacetime dimensionality → we can avoid some mathematical difficulties by working in an arbitrary number of dimensions.

This is called <u>Dimensional Regularization</u> – an alternative to the cut-offs, which is called <u>Pauli</u>-<u>Villars Regularization</u>.

We'll need a variety of new tools.

Recall, $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ w/ $(+---)$ signature for g.

then
$$\delta^\mu_\mu = 4$$
$$\gamma^\mu \gamma_\mu = 4$$

$\downarrow$ d dimensions

$$\delta^\mu_\mu = d$$
$$\gamma^\mu \gamma_\mu = d$$
$$\gamma^\mu \gamma_\nu \gamma_\mu = (2-d)\gamma_\nu$$

4 in 4d

Still: $\mathrm{Tr}[\text{odd } \# \gamma] = 0$

$\mathrm{Tr}[\text{other combinations}] = f(d) \times$ usual trace

Note for the dimensionless coupling e, in d-dimensions it needs to appear with $\mu^{2-d/2}$ factor, where μ is some dimensionful quantity, unimportant in the end.

Let's try it out on what we just laboriously finished.

The self energy factor in d dimensions is

$$\Sigma(p) = -ie^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\,\gamma_\mu \frac{1}{\not p-\not k-m}\,\gamma_\nu\,\frac{g^{\mu\nu}}{k^2}$$

$$= -ie^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\,\frac{\gamma_\mu(\not p-\not k+m)\gamma^\mu}{[(p-k)^2-m^2]k^2}$$

— fewer Feynman parameters required and less technical difficulty.

Introduce one Feynman Parameter.

$$= -i\mu^{4-d}e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\,\frac{\gamma_\mu(\not p-\not k+m)\gamma^\mu}{[(p-k)^2x-m^2x+k^2(1-x)]^2}$$

manipulate the denominator as before.

$$= -i\mu^{4-d}e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\,\frac{\gamma_\mu(\not p-\not k+m)\gamma^\mu}{[(k-xp)^2-a^2]^2}$$

$$a^2 = xm^2 - x(1-x)p^2$$

change variables $\quad k \to \ell = k - xp$

$$= -i\mu^{4-d}\frac{e^2}{(2\pi)^d}\int_0^1 dx\int d^d\ell\,\frac{\gamma^\mu[\not p(1-x)+m]\gamma_\mu}{[(k-xp)^2-a^2]^2} + \cancelto{0}{\int d^d\ell\,\frac{\not\ell}{[\;\;]}}$$

numerator $= \gamma^\mu \not{p}(1-x)\gamma_\mu + m\gamma^\mu\gamma_\mu$

$= \gamma^\mu \not{p}\gamma_\mu - x\gamma^\mu\not{p}\gamma_\mu + m\gamma^\mu\gamma_\mu$

$= (2-d)\not{p} - x(2-d)\not{p} + dm$

$= (2-d)\not{p}(1-x) + dm$

so,

$$= -i\mu^{4-d} e^2 \int_0^1 dx \int d^d l \left\{ \frac{(2-d)\not{p}(1-x) + dm}{(l^2-a^2)^2} \right\}$$

As before, perform a Wick Rotation (pick up i) and the Euclideanized integral becomes,

$$= -i\mu^{4-d} e^2 \int_0^1 dx \int d^d l \frac{\{\ \}}{(l^2+a^2)^2}(i)$$

look at general integral

$$(-i)^m i \int \frac{d^d l}{(l^2+a^2)^m}$$ ... need an intermediate step to do this

The volume element: $d^d l\, f(l^2) = \Omega_d\ l^{d-1} dl\ f(l^2)$ A

solid angle of d-dimensional hypersphere

trick: let a test $f(l^2) = e^{-l^2} = e^{-l_1^2} e^{-l_2^2} e^{-l_3^2} \cdots e^{-l_d^2}$

when $l^2 = l_1^2 + l_2^2 + \cdots l_d^2$

Then, this is one we can calculate...

then, $\int_{-\infty}^{\infty} dl_1 e^{-l_1^2} \int_{-\infty}^{\infty} dl_2 e^{-l_2^2} \int \cdots \int_{-\infty}^{\infty} dl_d e^{-l_d^2} = \Omega_d \int l^{d-1} dl\, e^{-l^2}$

Since we know $\underbrace{2\int_0^\infty e^{-l_i^2} dl_i}_{\frac{\sqrt{\pi}}{2}} = \int_{-\infty}^{\infty} e^{-l_i^2} dl_i = \Gamma(1/2) = \sqrt{\pi}$ so

one can figure out: $\Omega_d \int_0^\infty l^{d-1} dl\, e^{-l^2} = \pi^{d/2}$

let $t = l^2$

$$\int_0^\infty \frac{dt}{2} l^{-1} l^{d-1} e^{-t} = \int_0^\infty \frac{dt}{2} l^{d-2} e^{-t}$$

$$= \int_0^\infty \frac{dt}{2} t^{\frac{1}{2}(d-2)} e^{-t}$$

$$= \frac{1}{2}\int_0^\infty dt\, t^{d/2-1} e^{-t} = \frac{1}{2}\Gamma(d/2)$$

so,

$$\Omega_d \frac{1}{2}\Gamma(d/2) = \pi^{d/2} \quad \text{and} \quad \boxed{\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}}$$

and then going all the way back we can use it

$$\int \frac{d^d l}{(l^2+a^2)^m} = \Omega_d \int_0^\infty \frac{l^{d-1} dl}{(l^2+a^2)^m} \qquad A'$$

we've done this and just did this

$$= a^{d-4} \pi^{d/2} \Gamma\left(\frac{4-d}{2}\right)$$

For our integral, $m=2$

$$= a^{d-4} \pi^{d/2} \Gamma\left(\frac{4-d}{2}\right) \qquad A$$

So,

$$\Sigma(p) = -i\mu^{4-d}\frac{e^2}{(2\pi)^d}(i)(-1)^2\int_0^1 dx\,\{(2-d)\not p(1-x)+dm\}\pi^{d/2}a^{d-4}\Gamma\left(\frac{4-d}{2}\right)$$

$$= \mu^{4-d}\frac{e^2\pi^{d/2}}{(2\pi)^d}\Gamma\left(\frac{4-d}{2}\right)\int_0^1 dx\,\left\{\frac{(2-d)\not p(1-x)+dm}{[p^2x^2-(p^2-m^2)x]^{\frac{4-d}{2}}}\right\}$$

standard notation $\qquad \varepsilon = 4-d$

A'

$$\Sigma(p) = \mu^{\varepsilon}e^2\frac{\pi^{2-\varepsilon/2}}{(2\pi)^d}\Gamma(\varepsilon/2)\int_0^1\frac{[(2-d)\not p(1-x)+dm]\,dx}{[p^2x^2-(p^2-m^2)x]^{\varepsilon/2}}$$

ε is close to zero, so the denominator can be expanded

$$(A)^{-\varepsilon/2} \longrightarrow e^{-\varepsilon/2\ln A} = 1-\frac{\varepsilon}{2}\ln A$$

$$\left[\frac{1}{\ }\right]^{\varepsilon/2} = 1-\varepsilon/2\ln\left[p^2x^2-(p^2-m^2)x\right]$$

So, the integral can be written in 2 pieces,

$$\int_0^1 dx\,\{(2-d)\not p(1-x)+dm\} - \int_0^1 dx\,\{(2-d)\not p(1-x)+dm\}\frac{\varepsilon}{2}$$
$$\times\ln[p^2x^2-(p^2-m^2)x]$$

can let $d\to 4$ here

$\parallel$

$$\int_0^1 dx\,\{-2\not p(1-x)+4m\}$$

$$= -2\not p(1-1/2)+4m$$

$$= -(\not p-m)+3m$$

$$\Sigma(p) = \mu^{\varepsilon} \frac{e^2}{2^d} \pi^{2-\varepsilon/2-d} \Gamma(\varepsilon/2) \{ 3m - (\not{p} - m)$$

$$- \frac{\varepsilon}{2} \int_0^1 dx \{ (2-d) \not{p} (1-x) + dm \} \ln [p^2 x^2 - (p^2 - m^2) x \}$$

$$\Gamma(\varepsilon/2) \simeq \frac{2}{3} \Gamma(1+\varepsilon/2) = \frac{2}{\varepsilon} - \gamma + O(\varepsilon/2)$$

γ ← Euler's constant

so, as $\varepsilon \to 0$, the leading term becomes $2/\varepsilon$.

Let that happen --

$$\Sigma(p) = \frac{e^2}{16} \pi^{-2} \left(\frac{2}{\varepsilon}\right) \{ 3m - (\not{p} - m) \}$$

$$- \frac{e^2}{16} \pi^{-2} \int_0^1 dx \{ 2\not{p}(x-1) + 4m \} \ln [p^2 x^2 - (p^2 - m^2) x]$$

$$= \frac{3\alpha m}{4\pi} \left(\frac{2}{\varepsilon}\right) - \frac{\alpha}{4\pi} (\not{p} - m) \left(\frac{2}{\varepsilon}\right) + \Sigma_f'$$

so, comparing, we see that $2/\varepsilon$ plays the role of $\ln(\Lambda^2/m^2)$