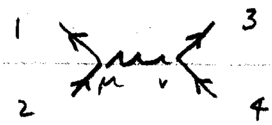
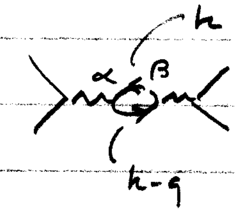
VACUUM POLARIZATION

For a lowest order process like 

$$T = i\, \bar{u}(p_1)(-ie\gamma^\mu)u(p_2)\left(-\frac{ig_{\mu\nu}}{q^2}\right)\bar{u}(p_3)(-ie\gamma^\nu)u(p_4)$$

Adding the vacuum polarization to 1 loop 

makes the total contribution

$$\bar{u}(p_1)(-ie\gamma^\mu)u(p_2)\left[-\frac{ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\alpha}}{q^2}(i\Pi^{\alpha\beta})\frac{-ig_{\beta\nu}}{q^2}\right]\bar{u}(-ie\gamma^\nu)u$$

where

$$i\Pi^{\alpha\beta} = -\mathrm{Tr}\int \frac{d^4k}{(2\pi)^4}(-ie\gamma^\alpha)\frac{i}{\not{k}-m}(-ie\gamma^\beta)\frac{i}{\not{k}-\not{q}-m}$$

$$= -\frac{(ie)^2}{(2\pi)^4}\int d^4k\, \gamma^\alpha\frac{i}{\not{k}-m}\gamma^\beta\frac{i}{\not{k}-\not{q}-m}$$

$$\Pi^{\alpha\beta} = \frac{ie^2}{(2\pi)^4}\mathrm{Tr}\int d^4k\, \frac{\gamma^\alpha}{\not{k}-m}\frac{\gamma^\beta}{\not{k}-\not{q}-m}$$

Remember, we're doing the external field problem, so really we have

so we have

$$\bar{u}(p_1)(-ie\gamma^\mu)u(p_2)\left[\delta_{\mu\nu} + \frac{-ig_{\mu\alpha}}{q^2}\, i\Pi^{\alpha\rho}\, g_{\rho\nu}\right] A^\nu_{ext}$$

$$A_\mu^{ext} = \left(\frac{-ig_{\mu 0}}{q^2}\right)\gamma^0 \frac{2\pi e}{4\pi} = \frac{-i g_{\mu 0}\gamma^0}{q^2}\frac{e}{2}$$

so

$$\bar{u}(p_1)(-ie\gamma^\mu)u(p_2)\left[g_{\mu\nu} + \frac{-i}{q^2}(i\Pi_{\mu\nu})\right] A^\nu_{ext}$$

so V.P. modifies the external field

$$A^\nu_{ext} \longrightarrow \left[g_{\mu\nu} + \frac{1}{q^2}\Pi_{\mu\nu}\right] A^\nu_{ext}$$

where

$$\Pi^{\mu\nu}(q^2) = ie^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\frac{\gamma^\mu}{\not{k}-m}\frac{\gamma^\nu}{\not{k}-\not{q}-m}\right]$$

which we'll write

$$\Pi_{\mu\nu}(q^2) = ie^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\frac{\gamma_\mu}{\not{k}-m}\frac{\gamma_\nu}{\not{k}-\not{q}-m}\right].$$

We can write this generally as

$$\Pi_{\mu\nu}(q^2) = g_{\mu\nu}A(q^2) + q_\mu q_\nu B(q^2)$$

Remember, the definition of the (conserved) EM current.

$$\frac{\partial F^{ext}_{\mu\nu}(x)}{\partial x_\nu} = J^{ext}_\mu(x) \qquad A$$

$$F^{ext}_{\mu\nu} = \frac{\partial A^{ext}_\nu}{\partial x^\mu} - \frac{\partial A^{ext}_\mu}{\partial x^\nu}$$

so $$J^{ext}_\mu(x) = \frac{\partial^2 A^{ext}_\nu}{\partial x_\nu \partial x^\mu} - \frac{\partial^2 A^{ext}_\mu}{\partial x_\nu \partial x^\nu}$$

intimately connected to gauge invariance.

$$\frac{\partial J^{ext}_\mu}{\partial x_\mu} = \frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu \partial x^\mu} - \frac{\partial}{x_\mu}\frac{\partial A_\mu}{\partial x_\nu \partial x^\nu}$$

$$= \frac{\partial}{\partial x_\nu}\Box A_\nu - \frac{\partial}{\partial x_\mu}\Box A_\mu = 0$$

In momentum space

$$\tilde{J}_\mu = q^\nu q_\mu \hat{A}_\nu - q^\nu q_\nu \hat{A}_\mu \qquad A'$$
$$= q^\nu q_\mu \hat{A}_\alpha g^\alpha{}_\nu - q^\nu q_\nu \hat{A}_\alpha g^\alpha{}_\mu$$
$$= (q^\nu q_\mu g^\alpha{}_\nu - q^2 g^\alpha{}_\mu)\hat{A}_\alpha$$

$$\tilde{J}_\mu = (q_\alpha q_\mu - q^2 g_{\alpha\mu})\hat{A}^\alpha$$

So current conservation is obvious.

$$q^\mu \hat{J}_\mu = (q^2 q_\alpha - q^2 q_\alpha)\hat{A}^\alpha = 0$$

So, our general form preserves current conservation

so, $$q^\mu \Pi_{\mu\nu}(q^2) = q_\nu A(q^2) + q^2 q_\nu B(q^2) = 0$$
$$\Rightarrow \quad A(q^2) = -q^2 B(q^2)$$

and our general form becomes,

$$\Pi_{\mu\nu}(q^2) = (-g_{\mu\nu} q^2 + q_\mu q_\nu)\Pi(q^2)$$ – very standard.

The diagram gives

$$\Pi_{\mu\nu}(q^2) = i\mu^{4-d} e^2 \,\mathrm{Tr} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\not k - m} \gamma_\mu \frac{1}{\not k - \not q - m} \gamma_\nu$$

$$= \frac{i\mu^{4-d} e^2}{(2\pi)^d} \mathrm{Tr} \int d^d k \left\{ \frac{(\not k + m)\gamma_\mu [(\not k - \not q) + m]\gamma_\nu}{(k^2 - m^2)[(k-q)^2 - m^2]} \right\}$$

A

$$\mathrm{Tr}\{\,\} = \mathrm{Tr}\{ \gamma_\nu (\not k + m)\gamma_\mu (\not k - \not q + m) \}$$

$$= \mathrm{Tr}\{ \gamma_\nu \not k \gamma_\mu (\not k - \not q - m) + \gamma_\nu m \gamma_\mu (\not k - \not q + m) \}$$

$$= \mathrm{Tr}\{ \gamma_\nu \not k \gamma_\mu \not k - \gamma_\nu \not k \gamma_\mu \not q - \underline{m \gamma_\nu \not k \gamma_\mu} + \underline{m \gamma_\nu \gamma_\mu \not k}$$

$$\cancel{+ m\gamma_\nu \gamma_\mu \not k} - \underline{m \gamma_\nu \gamma_\mu \not q} + m^2 \gamma_\nu \gamma_\mu \}$$

underlined terms have odd # γs $\to 0$

$$= \mathrm{Tr}\{ \gamma_\nu \not k \gamma_\mu (\not k - \not q) + m^2 \gamma_\mu \gamma_\nu \}$$

$$\gamma_\nu \not k \gamma_\mu = -\not k \gamma_\nu \gamma_\mu + 2k_\nu \gamma_\mu$$

$$\mathrm{Tr}\{\,\} = \mathrm{Tr}\{ -\not k \gamma_\nu \gamma_\mu (\not k - \not q) + 2k_\nu \gamma_\mu (\not k - \not q) + m^2 \gamma_\mu \gamma_\nu \}$$

$$= \mathrm{Tr}\{ -k^\alpha k^\beta \gamma_\alpha \gamma_\nu \gamma_\mu \gamma_\beta + k^\alpha q^\beta \gamma_\alpha \gamma_\nu \gamma_\mu \gamma_\beta$$

$$+ 2k_\nu k^\alpha \gamma_\mu \gamma_\alpha - 2k_\nu q^\alpha \gamma_\mu \gamma_\alpha + m^2 \gamma_\mu \gamma_\nu \}$$

$$= \mathrm{Tr}\{ (-k^\alpha k^\beta + k^\alpha q^\beta)(\gamma_\alpha \gamma_\nu \gamma_\mu \gamma_\beta)$$

$$+ (2k_\nu k^\alpha - 2k_\nu q^\alpha)(\gamma_\mu \gamma_\alpha) + m^2 \gamma_\mu \gamma_\nu \}$$

$$= f(d) \cancel{\mathrm{Tr}} \{ (-k^\alpha k^\beta + k^\alpha q^\beta)(g_{\alpha\nu} g_{\mu\beta} - g_{\alpha\mu} g_{\nu\beta} + g_{\alpha\beta} g_{\mu\nu})$$

$$+ (2k_\nu k^\alpha - 2k_\nu q^\alpha)(g_{\mu\alpha}) + m^2 g_{\mu\nu} \}$$

$$= f(d) \cancel{\mathrm{Tr}} \{ -k_\mu k_\nu + k_\nu q_\mu + k_\mu k_\nu - k_\mu q_\nu$$

$$- k^2 g_{\mu\nu} + k\cdot q\, g_{\mu\nu} + 2k_\nu k_\mu - 2k_\nu q_\mu + m^2 g_{\mu\nu} \}$$

$$= f(d) \{ 2k_\mu k_\nu - k_\mu q_\nu - k_\nu q_\mu - k^2 g_{\mu\nu} + k\cdot q\, g_{\mu\nu} + m^2 g_{\mu\nu} \}$$

collecting terms.

A'

$$\Pi_{\mu\nu}(q^2) = \frac{i\mu^{4-d}e^2}{(2\pi)^d}\int d^dk\ f(d)\left\{ \frac{2k_\mu k_\nu - k_\mu q_\nu - k_\nu q_\mu - k^2 g_{\mu\nu} + k\cdot q\, g_{\mu\nu} + m^2 g_{\mu\nu}}{(k^2-m^2)[k^2+q^2-2k\cdot q - m^2]} \right.$$ A

introduce 1 Feynman parameter in the standard way

Denominator becomes, $\cancel{(k^2-m^2)x + [k^2+q^2-2k\cdot q - m^2](1-x)}$

$$\{[k^2+q^2-2k\cdot q-m^2]x + (k^2-m^2)(1-x)\}^2$$
$$= [\cancel{k^2x} + q^2x - 2k\cdot q x - \cancel{m^2x} + k^2 - \cancel{k^2x} - m^2 + \cancel{xm^2}]^2$$
$$= [k^2 - m^2 + x(q^2 - 2k\cdot q)]^2$$

$$\Pi_{\mu\nu}(q^2) = \frac{i\mu^{4-d}e^2}{(2\pi)^d}\int_0^1 dx \int d^dk \frac{\{2k_\mu k_\nu - k_\mu q_\nu - k_\nu q_\mu - k^2 g_{\mu\nu} + k\cdot q\, g_{\mu\nu} + m^2 g_{\mu\nu}\}}{[k^2-m^2+x(q^2-2k\cdot q)]^2}$$ A'

make standard change of variables –

$$k_\mu = l_\mu + x q_\mu$$

A

work on numerator –

$$\text{NUM} = 2(l_\mu + xq_\mu)(l_\nu + xq_\nu) - (l_\mu + xq_\mu)q_\nu - (l_\nu + xq_\nu)q_\mu$$
$$- (l_\mu + xq_\mu)^2 g_{\mu\nu} + (l_\mu + xq_\mu)\cdot q^\mu g_{\mu\nu} + m^2 g_{\mu\nu}$$
$$= 2l_\mu l_\nu + \underline{2x\, l_\mu q_\nu} + \underline{2x\, q_\mu l_\nu} + x^2 q_\mu q_\nu +$$
$$\underline{- l_\mu q_\nu} - x\, q_\mu q_\nu \underline{- l_\nu q_\mu} - x q_\nu q_\mu$$
$$+ \underline{l\cdot q\, g_{\mu\nu}} + x q^2 g_{\mu\nu} + m^2 g_{\mu\nu}$$
$$- (l^2 + x^2 q^2 + \underline{2x\, l\cdot q})\, g_{\mu\nu}$$

work on denominator —

$$\mathrm{DEN} = \left[\ell^2 + x^2 q^2 + 2x\,\ell\cdot q - m^2 + x q^2 - 2x(\ell + xq)\cdot q \right]^2$$

$$= (\ell^2 + x^2 q^2 + 2x\,\ell\cdot q - m^2 + xq^2 - 2x\,\ell\cdot q - 2x^2 q^2)$$

$$= \ell^2 - m^2 + xq^2 - x^2 q^2$$

$$= \ell^2 - m^2 + xq^2(1-x)$$

standard definition: $a^2 \equiv m^2 - xq^2(1-x)$

$$\mathrm{DEN} = \ell^2 - a^2$$

underlined terms in numerator integrate to zero.

$$\Pi_{\mu\nu}(q^2) = \frac{i\mu\, e^2}{(2\pi)^d} \int dx \int d^d\ell\, f(d)$$

$$\times \frac{\left\{ 2\ell_\mu \ell_\nu + x^2 q_\mu q_\nu - 2x q_\mu q_\nu + x q^2 g_{\mu\nu} + m^2 g_{\mu\nu} - x^2 q^2 g_{\mu\nu} - \ell^2 g_{\mu\nu} \right\}}{(\ell^2 - a^2)^2}$$

$$= \frac{i\mu\, e^2}{(2\pi)^d} \int dx \int d^d\ell\, f(d)$$

$$\times \frac{\left\{ 2\ell_\mu \ell_\nu - \ell^2 g_{\mu\nu} - 2(x - x^2) q_\mu q_\nu + (x - x^2) q^2 g_{\mu\nu} + m^2 g_{\mu\nu} \right\}}{(\ell^2 - a^2)^2}$$

A'

Do a Wick Rotation into a Euclidean space and define $Q_{\mu\nu}(q,x) \equiv -2(x-x^2) q_\mu q_\nu + (x-x^2) q^2 g_{\mu\nu} + m^2 g_{\mu\nu}$

$$\Pi_{\mu\nu}(q^2) = \frac{i(-1)^2 i\mu\, e^2}{(2\pi)^d} \int_0^1 dx \int d^d\ell \left\{ \frac{2\ell_\mu \ell_\nu + \ell^2 g_{\mu\nu} + Q_{\mu\nu}}{(\ell^2 + a^2)^2} \right\}$$

This leaves us with 3 sorts of integrals to do:

$$\int d^d l \, \frac{l_\mu l_\nu}{(l^2+a^2)^2} \, , \quad \int d^d l \, \frac{l^2 g_{\mu\nu}}{(l^2+a^2)^2} \, , \quad \int d^d l \, \frac{g_{\mu\nu}}{(l^2+a^2)^2}$$

we have done this.

Generally, $$\int \frac{d^d l}{(l^2+a^2)^m} = \frac{1}{a^{2m-d}} \pi^{d/2} \frac{\Gamma(m-d/2)}{\Gamma(m)} \qquad A$$

To find a recursion-like approach, make a variable substitution —

$$w \equiv l-u \quad \Rightarrow \quad l = w+u \qquad dl = dw$$

$$t^2 \equiv u^2+a^2 \quad \Rightarrow \quad a = (t^2-u^2)^{1/2}$$

Then our integral becomes,

$$\int \frac{d^d w}{[w^2+u^2+2w\cdot u+t^2-u^2]^m} = \int \frac{d^d w}{[w^2+2w\cdot u+t^2]^m}$$

$$= \frac{1}{(t^2-u^2)^{m-d/2}} \pi^{d/2} \frac{\Gamma(m-d/2)}{\Gamma(m)}$$

We can get our needed integrals by differentiating —

go to 729

$$\frac{\partial}{\partial u_\mu} \rightarrow \quad -m\int \frac{d^d w \; 2w_\mu}{[w^2+2w\cdot u+t^2]^{m+1}} = \frac{(-m+d/2)\,\pi^{d/2}\,\Gamma(m-\frac{d}{2})(-2u_\mu)}{(t^2-u^2)^{m-d/2+1}\;\Gamma(m)}$$

remember $x\Gamma(x) = \Gamma(1+x)$ and reorganize.

$$\int \frac{d^d w \; w_\mu}{[w^2+2w\cdot u+t^2]^{m+1}} = -\frac{(m-d/2)\,\Gamma(m-d/2)\,\pi^{d/2}\,u_\mu}{(t^2-u^2)^{m-d/2+1}\; m\,\Gamma(m)}$$

$$= -\frac{\Gamma(m+1-d/2)\,\pi^{d/2}\,u_\mu}{(t^2-u^2)^{m+1-d/2}\,\Gamma(m+1)}$$

rename $m' \equiv m+1$

$$\int \frac{d^d w \; w_\mu}{[w^2+2w\cdot u+t^2]^{m'}} = -\frac{\Gamma(m'-d/2)\,\pi^{d/2}\,u_\mu}{(t^2-u^2)^{m'-d/2}\,\Gamma(m')}$$ ✓

and then, rename back again $m' \to m$.

Go through the same business again and pull up another factor.

$$\int \frac{d^d w \; w_\mu w_\nu}{(w^2+2w\cdot u+t^2)^m} = \frac{\pi^{d/2}}{\Gamma(m)}\,\frac{1}{(t^2-u^2)^{m-d/2}}$$ ✓

$$\times\left\{ u_\mu u_\nu\,\Gamma(m-d/2) + \tfrac{1}{2}\, g_{\mu\nu}\,(t^2-u^2)\,\Gamma(m-1-d/2)\right\}$$

We could contract,

$$\int \frac{d^d l \; w^2}{(w^2+2w\cdot u+\lambda^2)^m} = \frac{\pi^{d/2}}{\Gamma(m)} \frac{1}{(\lambda^2-u^2)^{m-d/2}}$$

$$\times \left\{ u^2\,\Gamma(m-d/2) + \frac{d}{2}(\lambda^2-u^2)\,\Gamma(m-1-d/2) \right\}$$

remembering that in d-dimensions $g^{\mu\nu}g_{\mu\nu} = d$

Transform these two back and set isolated u's $\to 0$

$$\int \frac{d^d l \; l_\mu l_\nu}{(l^2+a^2)^m} = \frac{\pi^{d/2}}{\Gamma(m)} \frac{1}{a^{2m-d}} \frac{1}{2} g_{\mu\nu}\, a^2\, \Gamma(m-1-d/2)$$

$$= \frac{\pi^{d/2}}{\Gamma(m)} \frac{1}{a^{2m-d-2}} \left(\frac{g_{\mu\nu}}{2}\right) \Gamma(m-1-d/2)$$

and.

$$\int \frac{d^d l \; l^2}{(l^2+a^2)^m} = \frac{\pi^{d/2}}{\Gamma(m)} \frac{1}{a^{2m-d}} \frac{d}{2}\, a^2\, \Gamma(m-1-d/2)$$

$$= \frac{\pi^{d/2}}{\Gamma(m)} \frac{1}{a^{2m-d-2}} \left(\frac{d}{2}\right) \Gamma(m-1-d/2)$$

notice that $\displaystyle\int \frac{d^d l \; l_\mu l_\nu}{(l^2+a^2)^m} = \frac{g_{\mu\nu}}{d} \int \frac{d^d l \; l^2}{(l^2+a^2)^m}$

but that they appear with opposite signs in the actual integral, so that we get,

$$\Pi_{\mu\nu}(q^2) = -\frac{f(d)\mu\, e^2}{(2\pi)^d}\int_0^1 dx\left[\int d^d\ell\,\frac{\left(-2\frac{g_{\mu\nu}\ell^2}{d} + \ell^2 g_{\mu\nu} + Q_{\mu\nu}\right)}{(\ell^2+a^2)^2}\right]$$

$$= -\frac{f(d)\mu\, e^2}{(2\pi)^d}\int_0^1 dx\left[\int d^d\ell\left(-\left(\frac{2}{d}-1\right)\ell^2 g_{\mu\nu} + Q_{\mu\nu}\right)\right]$$

using the results of the integral evaluation –

$$= -\frac{f(d)\mu\, e^2}{(2\pi)^d}\int_0^1 dx\left\{-\left(\frac{2}{d}-1\right)\left[g_{\mu\nu}\frac{\pi^{d/2}}{\Gamma(2)}\frac{1}{a^{2-d}}\left(\frac{d}{2}\right)\Gamma(1-d/2)\right]\right.$$
$$\left. + Q_{\mu\nu}\frac{1}{a^{4-d}}\pi^{d/2}\Gamma(2-d/2)\right\} \quad *$$

notice $\left(\frac{2}{d}-1\right)\frac{d}{2} = 1-d/2$ and remembering $x\Gamma(x) = \Gamma(1+x)$, the integrand becomes.

$$\{\} = -g_{\mu\nu}\pi^{d/2}\frac{1}{a^{2-d}}\Gamma(2-d/2) + Q_{\mu\nu}\frac{\pi^{d/2}}{a^{4-d}}\Gamma(2-d/2)$$

Factor out common terms,

A′

$$\Pi_{\mu\nu}(q^2) = -\frac{f(d)\mu\, e^2}{(2\pi)^d}\int_0^1 dx\left[\frac{\pi^{d/2}}{a^{4-d}}\Gamma\left(\frac{4-d}{2}\right)\left(-g_{\mu\nu}a^2 + Q_{\mu\nu}\right)\right]$$

look at () factor.

$$() = -a^2 g_{\mu\nu} + Q_{\mu\nu}$$
$$= -a^2 - (x-x^2)\left[2q_\mu q_\nu - q^2 g_{\mu\nu}\right] + m^2 g_{\mu\nu}$$
$$= -m^2 g_{\mu\nu} + q^2 x(1-x) g_{\mu\nu} - 2x(1-x)q_\mu q_\nu + x(1-x)q^2 g_{\mu\nu}$$
$$+ m^2 g_{\mu\nu}$$

$$= 2x(1-x)\left[q^2 g_{\mu\nu} - q_\mu q_\nu\right]$$

$$\Pi_{\mu\nu}(q^2) = \frac{f(d)\mu\, e^2}{(2\pi)^d}\, 2\,\pi^{d/2}\,\Gamma\!\left(\frac{4-d}{2}\right)\left[q_\mu q_\nu - q^2 g_{\mu\nu}\right]$$

$$\times \int_0^1 dx\; \frac{x(1-x)}{\left[m^2 - q^2 x(1-x)\right]^{4-d}} \qquad A$$

again, defining $4-d \equiv \varepsilon$ and letting $\varepsilon \to 0$ allows us to expand the integrand

$$\lim_{\varepsilon\to 0} \frac{1}{\left[m^2 - q^2 x(1-x)\right]^{\varepsilon}} \simeq 1 - \frac{\varepsilon}{2}\ln\left[m^2 - q^2 x(1-x)\right]$$

$$\lim_{\varepsilon\to 0} f(d) \to 4$$

$$\Pi_{\mu\nu}(q^2) = \frac{4e^2}{\underset{16\pi^4}{(2\pi)^4}}\, 2\pi^2\, \Gamma(\varepsilon/2)\left[q_\mu q_\nu - q^2 g_{\mu\nu}\right]$$

$$\times \int dx\; x(1-x)\left[1 - \frac{\varepsilon}{2}\ln\left[m^2 - q^2 x(1-x)\right]\right]$$

$$\int dx\; x(1-x) = 1/6 \qquad \Gamma(\varepsilon/2) \to \frac{2}{\varepsilon} - \gamma \sim \frac{2}{\varepsilon}$$

$$x - x^2 = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6}$$

$$\Pi_{\mu\nu}(q^2) = \frac{e^2}{12\pi^2}\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\left\{\left(\frac{2}{\varepsilon}\right)\right. \qquad A'$$

$$\left. - \frac{e^2}{2\pi^2}\underbrace{\int dx\,(1-x)\ln\left[m^2 - q^2 x(1-x)\right]}_{\text{no } \varepsilon \text{ dependence.}}\right\}\left[q_\mu q_\nu - q^2 g_{\mu\nu}\right]$$

2α

The divergent terms is: $\Pi_{\mu\nu}(q^2) = \frac{\alpha}{3\pi}\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\left(\frac{2}{\varepsilon}\right)$

$$\Pi_{\mu\nu}(q^2) = \frac{\alpha}{3\pi}\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\left(\frac{2}{\varepsilon}\right) + \Pi_c$$

Go back and look at the "constant term"

$$\Pi_c \;\cancel{\text{le}}\; = -\frac{e^2}{2\pi^2}\left(q_\mu q_\nu - g_{\mu\nu} q^2\right)\int dx\, x(1-x) \ln\left[m^2 - q^2 x(1-x)\right]$$

let's look at the $\underline{\text{low}}$ q^2 limit

$$\ln\left[m^2 - q^2 x(1-x)\right] = \ln m^2 + \ln\left[1 - \frac{q^2}{m^2} x(1-x)\right]$$

$$\simeq \ln m^2 - \frac{q^2}{m^2} x(1-x)$$

so that the leading term in q^2 is

$$\Pi_c(q^2)_{\mu\nu} = +\frac{2\alpha}{\pi}\left(q_\mu q_\nu - g_{\mu\nu} q^2\right)\int dx \left[x(1-x)\right]^2 \frac{q^2}{m^2}$$

$$= \frac{2\alpha}{\pi}\left(q_\mu q_\nu - g_{\mu\nu} q^2\right)\frac{q^2}{m^2}\underbrace{\int_0^1 dx\, x^2(1-x)^2}_{\frac{1}{30}}$$

$$= \frac{\alpha}{15\pi}\left(q_\mu q_\nu - g_{\mu\nu} q^2\right)\frac{q^2}{m^2} + \text{other finite terms.}$$

this finite term leads to
a measurable effect

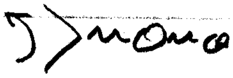
write generally as $\Pi(q^2) = \frac{\alpha}{3\pi}\left(\frac{2}{\varepsilon}\right) + \frac{\alpha}{15\pi}\frac{q^2}{m^2} = \Pi_\varepsilon(q) + \Pi_f(q)$

$$\Pi_{\mu\nu}(q^2) = (q_\mu q_\nu - q^2 g_{\mu\nu})\,\Pi(q)$$

Remember how A^{ext} was modified,

$$g_{\mu\nu} A^\nu_{ext} \longrightarrow \left[g_{\mu\nu} + \frac{1}{q^2}\Pi_{\mu\nu}(q)\right] A^\nu_{ext}$$

This is always contracted with a conserved current

J 

inside $\Pi_{\mu\nu}$.

So the $q_\mu q_\nu$ term will never contribute — forget it.

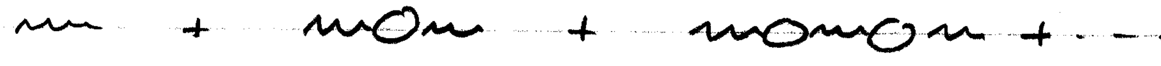
$$g_{\mu\nu} A^\nu_{ext} \longrightarrow \left[g_{\mu\nu} - g_{\mu\nu}\Pi(q)\right] A^\nu_{ext}.$$

$$= \left[1 - \Pi_\varepsilon - \Pi_f(q)\right] A_\mu^{ext} \qquad \text{to order } \alpha$$

$$= \underbrace{\left[1 - \frac{\alpha}{3\pi}\left(\frac{2}{\varepsilon}\right)\right]}_{\equiv Z_3}\left[1 - \frac{\alpha}{15\pi}\frac{q^2}{m^2}\right] A_\mu^{ext} \qquad \text{okay.}$$

$$= Z_3\left[1 - \Pi_f\right] A_\mu^{ext}$$

This figures into the full propagator line for self-energy



$$iD'_{\mu\nu} = \frac{-ig_{\mu\nu}}{q^2} + \left(\frac{-ig_{\mu\alpha}}{q^2}\right)\left(i\Pi^{\alpha\beta}\right)\left(-ig_{\beta\nu}\right) + \cdots +$$

$$= \frac{-ig_{\mu\nu}}{q^2} + \left(\frac{-ig_{\mu\alpha}}{q^2}\right) i\left(q^\alpha q^\beta - q^2 g^{\alpha\beta}\right)\Pi(q)\left(\frac{-ig_{\beta\nu}}{q^2}\right) + \cdots +$$

forget again

$$\cancel{= \frac{-ig_{\mu\nu}}{q^2} - \left(\frac{-ig_{\mu\alpha}}{q^2}\right) g^\alpha_\nu \Pi(q) + \cdots} = \frac{-ig_{\mu\nu}}{q^2} + (-ig_{\mu\nu})\frac{1}{q^2} i q^2 \Pi \left(\frac{-i}{q^2}\right)$$

$$\cancel{= \frac{-ig_{\mu\nu}}{q^2} + \frac{ig_{\mu\nu}}{q^4}\Pi(q) + \cdots} = -ig_{\mu\nu}\left(\frac{1}{q^2} + \frac{1}{q^2} q^2 \Pi \frac{1}{q^2} + \cdots\right)$$

series, again

$$iD'_{\mu\nu} = \frac{-ig_{\mu\nu}}{q^2 - q^2\Pi(q)} = \frac{-ig_{\mu\nu}}{q^2\left[1 - \Pi(q)\right]}$$

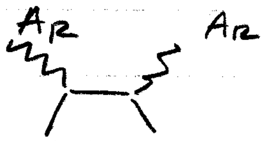
$$= \frac{iD^0_{\mu\nu}}{\left[1 - \Pi(q)\right]} = \frac{iD^0_{\mu\nu}}{Z_3\left[1 - \Pi_f\right]}$$

and the propagator gets renormalized via Z_3 with finite corrections which go like:

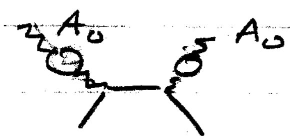
$$\frac{q^2}{m^2}\frac{\alpha}{15\pi} \longrightarrow q^2 = 0 \text{ for real photons}$$

so for real photons, the only effect is the Z_3 renormalization which can be thought of as $A^\mu_{ext_R} = \sqrt{Z_3}\, A^\mu_{ext_0}$

so, in external lines of Feynman diagrams, we can use A_R and never deal with, eg Compton



don't need



For internal lines its useful to think of the following

$$\bar{u}(-ie\gamma^\mu)u\left(\frac{-i}{q^2}\right)\left[g_{\mu\nu} + g_{\mu\alpha}(i\Pi^{\alpha\beta})\left(-\frac{ig_{\beta\nu}}{q^2}\right)\right]\bar{u}(-ie\gamma^\nu)u$$

$$= \bar{u}(-ie\gamma^\mu)u\left(-\frac{i}{q^2}\right)\left[g_{\mu\nu} + ig_{\mu\alpha}(-g^{\alpha\beta}q^2 + \underbrace{q^\alpha q^\beta}_{\text{go away}})\Pi(q)\left(-\frac{ig_{\beta\nu}}{q^2}\right)\right]\bar{u}(-ie\gamma^\nu)u$$

$$= \bar{u}(-ie\gamma^\mu)u\left(\frac{-i}{q^2}\right)\left[g_{\mu\nu} + g_\mu^{\ \beta}\, q^2\Pi(q)\left(-\frac{ig_{\beta\nu}}{q^2}\right)\right]\bar{u}(-ie\gamma^\nu)u$$

$$= \bar{u}(-ie\gamma^\mu)\left(\frac{-i}{q^2}\right)\left[g_{\mu\nu} + g_{\mu\nu}\,\Pi(q)\right]\bar{u}(-ie\gamma^\nu)u \qquad \star$$

$$= \bar{u}(-ie_0\gamma_\nu)\left(\frac{-i}{q^2}\right)\left[Z_3(1+\Pi_f)\right]u(-ie_0\gamma^\nu)u$$

again, in a scattering experiment in the limit q^2 small, one has the amplitude to 2^{nd} order in powers of

$$e_0^2 Z_3 \longrightarrow e_R^2$$, the renormalized electric charge.

$$e_R = \sqrt{Z_3}\, e_0$$

charge renormalization from a different source from before.

Back to the potential problem. — $A_\mu \to \frac{e_0}{q^2}\delta_{\mu 0}$

$$i\frac{\bar{u}\gamma^0 u}{q^2}\,\alpha_0\left\{1-\Pi_2-\alpha_0\frac{q^2}{15\pi m^2}\right]$$

$$= i\frac{\bar{u}\gamma^0 u}{q^2}\left\{\alpha_0(1-\Pi_2)-\alpha_0\frac{\alpha_0 q^2}{15\pi m^2}\right]$$

$$= i\frac{\bar{u}\gamma^0 u}{q^2}\left\{\alpha-\frac{\alpha^2}{(1-\Pi_2)^2}\frac{q^2}{15\pi m^2}\right] \quad \text{to order } \alpha^2$$

$$= i\frac{\bar{u}\gamma^0 u}{q^2}\left\{\alpha-\frac{\alpha^2 q^2}{15\pi m^2}+O(\alpha^4)\right\}$$

in configuration space, as a bound problem, the potential which was.

$$V(r) = -\frac{Ze_0^2}{4\pi r} \longrightarrow \left(1-\frac{\alpha}{15\pi}\frac{\nabla^2}{m^2}\right)\frac{Z\alpha}{4\pi r}$$

$$= -\frac{Z\alpha}{4\pi r} \underbrace{-\frac{\alpha}{15\pi m^2}\,Z\alpha\,\delta(\vec{r})}_{\text{a splitting.}}$$

$$\Delta E_{nj} = \frac{-Ze^2}{15\pi m^2}\,|\psi_{nj}(0)|^2 \quad \text{for hydrogen-like atoms:}$$

$$= -\left(\tfrac{1}{2}Z^2\alpha^2 m\right)\frac{8Z^2\alpha^3}{15\pi n^2}\,\delta_{j0}$$

$\nu = \dfrac{\Delta E}{h} = -27\text{ MHz}$, which lowers the S level

– calculated in 1935 by Uehling (called the "Uehling Term")

This is a contribution to the Lamb shift of $\underline{\underline{+1057.9\text{ MHz}}}$ Measured in 1947

$2P_{3/2}$

$2S_{1/2}$ —— deg w/ $2P_{1/2}$ - - - - - - -

$2P_{1/2}$

Dirac Theory, Lamb shift due to 

Lamb found his "shift" actually searching for the Uehling term – Bethe unraveled the situation

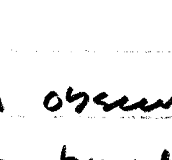
 &  don't contribute (only mass/charge ren.)

to free electron observables – but <u>do</u> contribute to bound electron.

That's what Bethe evaluated in 1947, w/ found $E(2S_{1/2}) - E(2P_{1/2}) = 1040\text{ MHz}$

more recent $\simeq 1052.1\text{ MHz}$

The Uehling term effects only the $2S_{1/2}$ level, down by 27 MHz



screened charge

However, the 1s electrons penetrate and see more +

now look at large q^2 — go back to the definition of Π

$$\Pi(q) = \frac{\alpha}{3\pi}\left(\frac{2}{\varepsilon}\right) - \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x) \ln\left[1 - \frac{q^2 x(1-x)}{m^2}\right]$$

now when $q^2 \gg m^2$

$$\Pi(q) \to \frac{\alpha}{3\pi}\left(\frac{2}{\varepsilon}\right) - \frac{2\alpha}{\pi}\int dx\, x(1-x)\ln\left(-\frac{q^2}{m^2}\right) + C$$

$$= \frac{\alpha}{3\pi}\left(\frac{2}{\varepsilon}\right) - \frac{2\alpha}{\pi}\ln\left(-q^2/m^2\right)\frac{1}{6} + C$$

$$= \frac{\alpha}{3\pi}\left[\frac{2}{\varepsilon} - \ln\left(-q^2/m^2\right)\right] + C$$

We can recast this in terms of a cutoff — remember

$2/\varepsilon \sim \ln\left(\Lambda^2/m^2\right)$

$$\Pi(q) = \frac{\alpha_0}{3\pi}\left[\ln \Lambda^2/m^2 - \ln\left(-q^2/m^2\right)\right]$$

$$= \frac{\alpha_0}{3\pi}\left[\ln\left(\Lambda^2/-q^2\right)\right]$$

choose some arbitrary scale, $-q^2 = \mu^2$

$$\Pi(\mu) = \frac{\alpha_0}{3\pi}\ln\left(\Lambda^2/\mu^2\right)$$

and form the difference

$$\Pi(q) - \Pi(\mu) = \frac{\alpha_0}{3\pi}\left[\ln(\Lambda^2/-q^2) - \ln(\Lambda^2/\mu^2)\right]$$

$$= \frac{\alpha_0}{3\pi}\left[\ln(\mu^2/-q^2)\right]$$

no cutoff.. at the expense of an arbitrary scale.

Now, we can isolate the bare coupling (from ✱)

$$\alpha_R(q) = [1-\Pi(q)]\alpha_0$$

$$\alpha_R(\mu) = [1-\Pi(\mu)]\alpha_0$$

so
$$\frac{\alpha_R(q)}{1-\Pi(q)} = \frac{\alpha_R(\mu)}{1-\Pi(\mu)}$$

and
$$\alpha_R(q) = \alpha_R(\mu)\left(\frac{1-\Pi(q)}{1-\Pi(\mu)}\right)$$

$$\simeq \alpha_R(\mu)\left[1-\Pi(q)+\Pi(\mu)\right]$$

$$= \alpha_R(\mu)\left[1+\frac{\alpha_0}{3\pi}\ln(-q^2/\mu^2)\right]$$

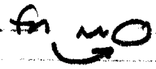
do it again,
$$\alpha_R(q) = \alpha_R(\mu)\left[1+\frac{\alpha_R(\mu)}{3\pi}\ln(-q^2/\mu^2)\right] + \mathcal{O}(\alpha^4)$$

the electric charge increases logarithmically with increasing q^2.

The addition of 
amounts to,

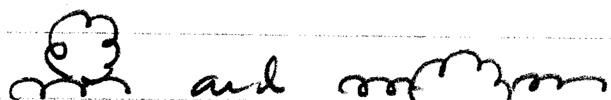
$$\alpha_R(q^2) = \alpha_R(\mu)\left\{1 + \frac{\alpha(\mu)}{3\pi}\ln(-q^2/\mu^2) + \left[\frac{\alpha(\mu)}{3\pi}\ln\left(\frac{-q^2}{\mu^2}\right)\right]^2 + \cdots\right\}$$

$$= \frac{\alpha_R(\mu)}{1 - \frac{\alpha_R(\mu)}{3\pi}\ln(-q^2/\mu^2)}$$

$\alpha(M_Z) \approx \frac{1}{128}$ when $\mu = m_e$ and all quark channels are included for 

In QCD, the vacuum polarization graph

contributes in the same way... however also there are



for n_f = # quark flavors.
N = # color dof.

$$\alpha_s(q^2) = \alpha_s(\mu^2)\left[1 + (2n_f - 11N)\frac{\alpha_s(\mu)}{12\pi}\ln(-q^2/\mu^2) \cdots\right]$$

For $n_f = 6$, $N = 3$,

$$\alpha_s(q^2) = \alpha_s(\mu^2)\left[1 - \frac{21}{12\pi}\ln(-q^2/\mu^2) \cdots\right]$$

negative – strong charge decreases with increasing q^2 (small r)

or increases with decreasing q^2 (large r)!