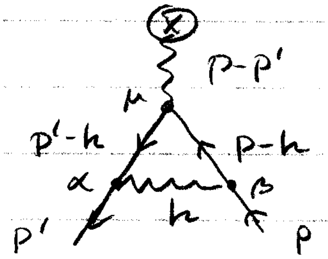
The last divergent 1PI graph is



the matrix element gets the replacement

$$-ie\gamma^\mu \to -ie\gamma^\mu - ie\,i\,(ie)^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\alpha \frac{1}{\not p' - \not k - m}\, \gamma^\mu \frac{1}{\not p - \not k - m}\, \gamma^\beta \frac{g_{\alpha\beta}}{k^2}$$

$$= -ie\gamma^\mu - ie\,\Lambda^\mu(p,p')$$

regularize

$$\Lambda^\mu(p,p') = i\mu^\epsilon \frac{(-ie)^2}{(2\pi)^d} \int \frac{d^dk}{k^2}\, \frac{\gamma^\alpha(\not p' - \not k + m)\gamma^\mu(\not p - \not k + m)\gamma^\beta g_{\alpha\beta}}{[(p'-k)^2 - m][(p-k)^2 - m^2]}$$

which sits between spinors in

$$\bar u(p')\,[-ie\,\Lambda^\mu(p,p')]\,u(p)\,A_\mu^{ext}$$

so it's useful to look at (note: $\not a \gamma^\nu = -\gamma^\nu \not a + 2a$)

$$(\not p - \not k + m)\gamma^\mu u(p) = \gamma^\mu(-\not p + m + \not k)u(p) + 2(p^\mu - k^\mu)u(p)$$

acting on $u(p)$ gives $-m$

A

likewise $\qquad \bar{u}(p')\gamma^\alpha(\not{p}' - \not{k} + m) = \bar{u}(p')[\not{k}\gamma^\alpha + 2(p'^\alpha - k^\alpha)]$

so

$$\Lambda^\mu(p,p') = \frac{-ie^2}{(2\pi)^d}\int \frac{d^dk}{k^2}\, \frac{\begin{aligned}[\,&\not{k}\gamma^\alpha\gamma^\mu\gamma^\beta\not{k}\, g_{\alpha\beta} \\ &+ 2\not{k}\gamma^\alpha\gamma^\mu(p_\alpha - k_\alpha) + 2(p'_\beta - k_\beta)\gamma^\mu\gamma^\beta\not{k} \\ &+ 4\gamma^\mu(p'_\beta - k_\beta)(p^\beta - k^\beta)\end{aligned}}{(k^2 - 2p'\cdot k)(k^2 - 2p\cdot k)}$$

look at terms in numerator...

$$\not{k}\gamma^\alpha\gamma^\mu\gamma^\beta\not{k}\, g_{\alpha\beta} = \not{k}[-\gamma^\mu\gamma^\alpha\gamma^\beta\not{k}\, g_{\alpha\beta} + 2\gamma^\beta\not{k}\, g^{\mu\alpha}g_{\alpha\beta}]$$

$$= \not{k}[-\gamma^\mu\underbrace{\gamma^\alpha\gamma_\alpha}_{d}\not{k} + 2\gamma^\mu\not{k}]$$

$$= \not{k}[-d\,\gamma^\mu\not{k} + 2\gamma^\mu\not{k}] = (2-d)\,\not{k}\gamma^\mu\not{k}$$

$$2\not{k}\gamma^\alpha\gamma^\mu(p_\alpha - k_\alpha) = 2\not{k}(\not{p} - \not{k})\gamma^\mu = 2\not{k}\not{p}\gamma^\mu - 2k^2\gamma^\mu$$

$$= 2\not{k}(-\gamma^\mu\not{p} + 2g^{\alpha\mu}p_\alpha) - 2k^2\gamma^\mu$$

$$2(p'_\beta - k_\beta)\gamma^\mu\gamma^\beta\not{k} = 2\gamma^\mu(\not{p}' - \not{k}')\not{k}$$

$$= 2(-\not{p}'\gamma^\mu + 2p'^\mu)\not{k} - 2k^2\gamma^\mu$$

when we have $\not{p}'$ operating left $\to m$

$\not{p}$ right $\to m$

so these last two terms give

~~$2m(\not{k}\gamma^\mu$~~

$$-2m\not{k}\gamma^\mu + 4\not{k}p^\mu - 2m\gamma^\mu\not{k} + 4p'^\mu\not{k} - 4k^2\gamma^\mu$$

$$= -2m(\underbrace{\not{k}\gamma^\mu + \gamma^\mu\not{k}}_{2k^\mu}) + 4\not{k}(p^\mu + p'^\mu) - 4k^2\gamma^\mu$$

and the whole numerator becomes,

$$(2-d)\not{k}\gamma^\mu\not{k} - 4mk^\mu + 4(p'^\mu + p^\mu)\not{k} - \cancel{4k^2\gamma^\mu}$$
$$+ 4\gamma^\mu(p'\cdot p - k\cdot p - k\cdot p' + \cancel{k^2})$$

$$= (2-d)\not{k}\gamma^\mu\not{k} - 4mk^\mu + 4(p^\mu + p'^\mu)\not{k}$$
$$+ 4\gamma^\mu(p'\cdot p - k\cdot p' - k\cdot p)$$

so

$$\Lambda^\mu(p,p') = \frac{-ie^2}{(2\pi)^d}\int d^dk \frac{\left[(2-d)\not{k}\gamma^\mu\not{k} - 4mk^\mu + 4(p^\mu + p'^\mu)\not{k} + 4\gamma^\mu(p'\cdot p - k\cdot p' - k\cdot p)\right]}{k^2(k^2 - 2p\cdot k)(k^2 - 2p'\cdot k)} \quad \star$$

use a Feynman parameter to do

$$\int_0^1 dx \frac{1}{k^2\left[(k^2 - 2p\cdot k)x + (k^2 - 2p'\cdot k)(1-x)\right]^2}$$

or $$\int_0^1 dy \, \frac{1}{h^2\left[h^2 - 2h\cdot(py + p'(1-y)\right]^2}$$

remember,

$$\frac{1}{ab} = \int \frac{d\cancel{y}\,x}{\left[a\cancel{y}x + b(1-\cancel{y}x)\right]^2}$$

differentiate wrt a,

$$-\frac{1}{a^2 b} = -2\int \frac{\cancel{y}x\,d\cancel{y}x}{\left[a\cancel{y}x + b(1-\cancel{y}x)\right]^3}$$

so $$b \to h^2$$
$$a \to h^2 - 2h\cdot(py + p'(1-y))$$

and we get for the denominator,

$$\cancel{\int_0^1\int_0^1 \frac{2y\,dy\,dx}{[}}$$

$$\int_0^1\int_0^1 \frac{2x\,dx\,dy}{\left[(\cancel{h^2} - 2h\cdot(py + p'(1-y))\right]x + \cancel{h^2}(1-x)\Big]^3}$$

simplify

$$= \int_0^1 \int_0^1 \frac{2x\,dx\,dy}{[h^2 - 2h\cdot P x]^3}$$

where $\quad P^\mu \equiv p^\mu y + p'^\mu (1-y) \quad *$

Define $\quad \ell^\mu \equiv h^\mu - P^\mu x \quad \Rightarrow \quad h^\mu = \ell^\mu + P^\mu x \quad *$

so $\quad h^2 = \ell^2 + x^2 P^2 + 2x\,\ell\cdot P \qquad$ A

and $\quad 2h\cdot P x = 2\ell\cdot P x + 2P^2 x^2$

and denominator becomes,

$$\begin{aligned} h^2 - 2h\cdot P x &= \ell^2 + x^2 P^2 + \cancel{2x\,\ell\cdot P} - \cancel{2\ell\cdot P x} - 2P^2 x^2 \\ &= \ell^2 + x^2 P^2 - 2P^2 x^2 \\ &= \ell^2 - P^2 x^2 \end{aligned}$$

with the ~~sub~~ change of variables, the numerator becomes,

$$(2-d)(\not\ell + x\not P)\gamma^\mu(\not\ell + x\not P) - 4m(\ell^\mu + x P^\mu)$$

$$+ 4(p^\mu + p'^\mu)(\not\ell + x\not P) + 4\gamma^\mu(p\cdot p' - \ell\cdot p' - \ell\cdot p - xP\cdot p - xP\cdot p')$$

$$\begin{aligned} = (2-d)&\left[\not\ell\gamma^\mu\not\ell + \underline{\not\ell\gamma^\mu x\not P} + \underline{x\not P\gamma^\mu\not\ell} + x^2\not P\gamma^\mu\not P\right] \\ &- \underline{4m\ell^\mu} - 4mxP^\mu + 4(p^\mu + p'^\mu)(\underline{\not\ell} + x\not P) \\ &+ 4\gamma^\mu(p\cdot p' - \underline{\ell\cdot p'} - \underline{\ell\cdot p} - xP\cdot p - xP\cdot p') \end{aligned}$$

now we have another symmetric function of l^2 and some odd terms in numerator (underlined) $\rightarrow \emptyset$

so,

$$\Lambda^\mu(p,p') = -\frac{ie^2\mu^\varepsilon}{(2\pi)^d}\int_0^1 dy \int_0^1 2x\,dx \int \frac{d^d l}{(l^2 - P^2x^2)^3}$$

$$\times\Big\{ (2-d)\big[\, \not{l}\gamma^\mu\not{l} + x^2 \not{P}\gamma^\mu\not{P} - 4mx P^\mu$$

$$+ 4(p^\mu + p'^\mu)x\not{P} + 4\gamma^\mu(p\cdot p' + x P\cdot(p+p')\big]\Big\}$$

more Dirac reduction

$$\not{l}\gamma^\mu\not{l} = l_\nu(-\gamma^\mu\gamma^\nu + 2g^{\mu\nu})\not{l}$$

$$= -\gamma^\mu l^2 + 2l^\mu\not{l}$$

$$= -\gamma^\mu l^2 + 2l^\mu l_\nu\gamma^\nu$$

$$= -l^\mu l^2 + \underbrace{2l^\mu l^\eta}\gamma^\nu g_{\eta\nu}$$

replace by $\frac{1}{d}l^2 g^{\mu\eta}$

$$= \left(-1 + \frac{2}{d}\right)l^2\gamma^\mu$$

note

$$P\cdot P = p^2y^2 + p'^2(1-y)^2 + 2p\cdot p'\,y(1-y)$$

$$= (p^2 + p'^2 - 2p\cdot p')\,y^2 + 2p\cdot p'\,y - p'^2(1-2y)$$

Define $\quad q^\mu \equiv p^\mu - p'^\mu$

$$q^2 = p^2 + p'^2 - 2p\cdot p' = 2m^2 - 2p\cdot p'$$

so,

$$\underline{P}^2 = q^2(y^2 - y) + m^2$$

$$\begin{aligned}
\underline{P}\cdot(p+p') &= (py + p'(1-y))\cdot(p+p') \\
&\cancel{= py + p'(1-y)} \\
&= p^2 y + \cancel{p'\cdot p\, y} + p'\cdot p\,(1-\cancel{y}) + p'^2(1-y) \\
&= \cancel{m^2 y} + p'\cdot p + m^2(1-\cancel{y})
\end{aligned}$$

notice that any time $\quad \bar{u}\,\underline{\slashed{P}}\,u = \bar{u}\left[\slashed{p}y + \slashed{p}'(1-y)\right]u$

$$\begin{aligned}
&= \bar{u}\left[my + m(1-y)\right]u \\
&= \bar{u}\,m\,u
\end{aligned}$$

Finally,

$$\begin{aligned}
\underline{\slashed{P}}\,\gamma^\mu\,\underline{\slashed{P}} &= \underline{P}_\nu\left(-\gamma^\mu\gamma^\nu + 2g^{\mu\nu}\right)\underline{\slashed{P}} \\
&= -\gamma^\mu \underline{P}^2 + 2\,\underline{P}^\mu \underbrace{\underline{\slashed{P}}}_{\downarrow\, m} \\
&= \gamma^\mu q^2(y - y^2) - \gamma^\mu m^2 + 2m\,\underline{P}^\mu
\end{aligned}$$

now we have a simple set of coefficients

Proportional to:

$$\gamma^\mu: \quad \frac{\ell^2(2-d)^2}{d} + (2-d)x^2\left[q^2(y-y^2)-m^2\right]+4\left[p\cdot p' - xP\cdot(p+p')\right]$$

$$\equiv \lambda_1$$

A'

$$p^\mu: \quad (2-d)x^2 2my - 4myx + 4mx \equiv \lambda$$

$$p'^\mu: \quad (2-d)x^2 2m(1-y) - 4mx(1-y) + 4mx \equiv \lambda'$$

so we get,

$$\Lambda^\mu(p,p') = \frac{-ie^2}{(2\pi)^d}\mu^\varepsilon \int_0^1 dy \int_0^1 2x\,dx \int \frac{d^d\ell\,[\lambda_1\gamma^\mu + \lambda p^\mu + \lambda' p'^\mu]}{[\ell^2 - m^2x^2 - q^2x^2(y^2-y)]^3}$$

A

notice that $\lambda \leftrightarrow \lambda'$ interchanges by $y \leftrightarrow (1-y)$

add last two terms, and interchange and ½ —

$$p^\mu\left(\frac{\lambda+\lambda'}{2}\right) + p'^\mu\left(\frac{\lambda+\lambda'}{2}\right) = \frac{p^\mu+p'^\mu}{2}(\lambda+\lambda')$$

$$= \frac{p^\mu+p'^\mu}{2}\left[8mx + (2-d)2mx^2 - 4mx\right]$$

$$= \frac{p^\mu+p'^\mu}{2}\left[4mx + 2(2-d)mx^2\right]$$

$$= (p^\mu + p^{\mu\prime}) \left[2 + (2-d)x \right] mx$$

So, replace

$$\Lambda^\mu(p,p') = -\frac{ie^2\mu^\varepsilon}{(2\pi)^d} \int dy \int 2x\,dx \int d^d l$$

$$\frac{\left[\lambda_1 \gamma^\mu + mx\left[2+(2-d)x\right](p_\mu - p'_\mu)\right]}{l^2 + m^2x^2 - q^2x^2(y^2-y)]^3} \qquad A'$$

We can make use of a famous relation --

From $\quad -i\sigma^{\mu\nu} = \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right]$

$$\not{a}\not{b} = a_\mu b_\nu \left\{ \frac{1}{2} \underbrace{(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)}_{2g^{\mu\nu}} + \frac{1}{2} \underbrace{(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)}_{i\sigma^{\mu\nu}} \right]$$

$$= a\cdot b - i\sigma^{\mu\nu} a_\mu b_\nu$$

notice that

$$2iq_\nu\sigma^{\mu\nu} = -(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p_\nu - p'_\nu)$$

$$= -\gamma^\mu(\not{p} - \not{p}') + (\not{p} - \not{p}')\gamma^\mu$$

When this appears between spinors -

$$\bar{u}(p')\left[2iq_\nu\sigma^{\mu\nu}\right]u(p)$$

$$= \bar{u}(p')\left[\underbrace{-\gamma^\mu\not{p}}_{m} + \underbrace{\gamma^\mu\not{p}'}_{-\not{p}'\gamma^\mu + 2p'^\mu} + \underbrace{\not{p}\gamma^\mu}_{-\gamma^\mu\not{p} + 2p^\mu} - \underbrace{\not{p}'\gamma^\mu}_{m} \right] u(p)$$

$$= \bar{u}(p')\big[-m\gamma^\mu - \underset{m}{\underset{\downarrow}{\not p'}}\gamma^\mu + 2p'^\mu - \gamma^\mu \underset{m}{\underset{\downarrow}{\not p}} + 2p^\mu - m\gamma^\mu\big]u(p)$$

$$= \bar{u}(p')\left[-4m\gamma^\mu + 2(p^\mu + p'^\mu)\right]u(p)$$

$$= 2i\,\bar{u}(p')\,\sigma^{\mu\nu}q_\nu\, u(p)$$

This is called the Gordon Decomposition.

So, we can replace,

$$mx\left[2+(2-d)x\right](p^\mu + p'^\mu)$$

$$= 2m^2x\left[2+(2-d)x\right]\gamma^\mu + i\{mx\left[2+(2-d)x\right]\}\sigma^{\mu\nu}q_\nu$$

contributes back to λ_1

$$\Lambda^\mu(p_0 p_0') = -\frac{ie^2\mu^\varepsilon}{(2\pi)^d}\int dy \int 2x\,dx \int d^d\ell \frac{\left\{\left[\lambda_1 + 2mA\right]\gamma^\mu + A\,i\sigma^{\mu\nu}q_\nu\right\}}{\left[\ell^2 - m^2x^2 - q^2x^2(y^2-y)\right]^3}$$

where $A = m\left[2+(2-d)x\right]x$

notice that in 4 dimensions, the term proportional to γ^μ would be divergent — λ_1 has ℓ^2 term

$$\frac{\ell^3 d\ell \cdot \ell^2}{\ell^6} \sim \frac{1}{\ell}\,d\ell$$

the term proportional to $\sigma^{\mu\nu}$ $\quad \dfrac{\ell^3 d\ell}{\ell^6}$ is finite

$$\Lambda^\mu(p,p') = \Lambda^\mu_{\varepsilon_1} + \Lambda^{\mu\,\prime} + \Lambda^\mu_f$$

$$\Lambda^\mu_{\varepsilon_1} = \frac{-ie^2}{(2\pi)^d}\int_0^1 dy \int_0^1 2x\,dx \int d^dl \; \frac{(2-d)^2 l^2 \gamma^\mu}{\left[l^2 - m^2x^2 - x^2q^2(y^2-y)\right]^3}$$

$\Lambda^{\mu\,\prime}$ = everything else from λ, w/out l dependence
→ won't contribute to real electrons ultimately.

Λ^μ_f = term with $\sigma^{\mu\nu}$.

$$\Lambda^\mu_{\varepsilon_1} = \frac{-ie^2}{(2\pi)^4}\int dy \int 2x\,dx \; (2-d)^2 i\pi^2 \frac{\Gamma(\varepsilon/2)}{4} \frac{\gamma^\mu}{\left[x^2q^2(y-y^2) - m^2x^2\right]^{\varepsilon/2}}$$

$$= \frac{e^2}{8\pi^2}\Gamma(\varepsilon/2)\int dy \int x\,dx \left\{1 - \frac{\varepsilon}{2}\ln\left[x^2q^2(y-y^2) - m^2x^2\right]\right\}\gamma^\mu$$

will become finite

$$\Lambda^\mu_{\varepsilon_1} = \frac{\alpha}{4\pi}\Gamma(\varepsilon/2)\gamma^\mu + \text{finite } \gamma^\mu$$

$$\Lambda^\mu_{\varepsilon_1} = \Lambda^{(1)}\gamma^\mu + \text{finite } \gamma^\mu$$

Notice, that again we can assign a renormalization constant to the piece carrying the cutoff... since

$$-ie_0\gamma_\mu \rightarrow -ie_0\gamma_\mu - ie_0\Lambda_c^{(1)}\gamma_\mu + \Theta(p-p')$$

So, from this process we have another renormalization of the charge

$$e = Z_1^{-1} e_0$$

where $$Z_1 \equiv (1+\Lambda_c^{(1)})^{-1} \simeq 1+\Lambda_c^{(1)}$$

Go back and deal with the infrared divergence by inserting a photon mass, λ, and staying in 4 dimensions —

$$\Lambda^\mu(p,p') = \frac{-ie^2}{(2\pi)^4}\int d^4k \frac{\left[2\not k\gamma^\mu\not k - 4mk^\mu + 4(p^\mu+p'^\mu)\not k + 4\gamma^\mu(p'\cdot p - k p' - k\cdot p)\right]}{(k^2-\lambda^2)(k^2-2p\cdot k)(k^2-2p'\cdot k)} \qquad A$$

inserting Feynman parameters

$$\Lambda^\mu = -\frac{ie^2}{(2\pi)^4}\int dy\int 2x\,dx\int \frac{[\quad]}{\left[k^2-2k\cdot(py+p'(1-y))x-\lambda^2(1-x)\right]^3}$$

The infrared divergence will come from the integral when $k\to 0$, so the least order-k numerator piece will dominate,

$$A'$$

$$\Lambda^\mu(p,p') = \frac{-ie^2}{(2\pi)^4}\int dy\int 2x\,dx\int d^4k \frac{4\gamma^\mu p\cdot p'}{\left[(k^2-xP^\mu)^2-x^2P^2-\lambda^2(1-x)\right]^3}$$

change variables

$$l^\mu = k^\mu - xP^\mu \qquad P^\mu = p^\mu y + p'^\mu(1-y) \quad * \text{ same as before}$$

$$= -\frac{ie^2}{(2\pi)^4}\gamma^\mu 4p\cdot p'\int dy\int 2x\,dx\,\underbrace{\int d^4l\frac{1}{\left[l^2-P^2x^2-\lambda^2(1-x)\right]^3}}_{\frac{-i\pi^2}{2(P^2x^2-\lambda^2(1-x))}}$$

$$\Lambda^\mu(pp')_{IR} = \frac{-ie^2}{(2\pi)^4}\gamma^\mu\, 4\, p\cdot p' \left(\frac{-i\pi}{2}\right)\int dy \int 2x\,dx \frac{1}{\left[P^2x^2+\lambda^2(1-x)\right]^2}$$

$$= \frac{-e^2}{4\pi^2}\, p\cdot p' \int dy \frac{1}{P^2}\ln(P^2/\lambda^2)\,\gamma^\mu$$

$$= \gamma^\mu - \frac{\alpha}{\pi}\, p\cdot p' \int dy \frac{1}{P^2}\ln(P^2/\lambda^2)\,\gamma^\mu \equiv \Lambda_2^\mu$$

The finite piece..

$$\Lambda_f^\mu = \frac{-ie^2}{(2\pi)^d}\mu^\varepsilon \int dy \int 2x\,dx \int d^dl \frac{A\, i\sigma^{\mu\nu}q_\nu}{\left[l^2-m^2x^2-q^2x^2y(y-1)\right]^3}$$

just go to 4d....

$$\Lambda_f^\mu = \frac{-ie^2}{(2\pi)^4}\int dy \int 2x\,dx \int d^4l \frac{A\, i\sigma^{\mu\nu}q_\nu}{\left[\qquad\right]^3}$$

$$A = mx\left[2+(2-d)x\right]$$

$$\to 2mx(1-x)$$

$$\Lambda_f^\mu = \frac{-ie^2}{(2\pi)^4}\int dy \int 2x\,dx\, 2mx(1-x) \underbrace{\int \frac{d^4l\, \sigma^{\mu\nu}q_\nu}{\left[l^2-a^2\right]^3}}_{\frac{(-1)^3 i\pi^2}{a^2}\frac{\Gamma(1)}{\Gamma(3)}}$$

$$a^2 = m^2x^2 + q^2x^2y(y-1)$$

$$\Lambda_f^\mu = -\frac{\alpha}{2\pi} m \int dy \frac{1}{m^2+q^2y(y-1)} \int(1-x)\,dx\, \sigma^{\mu\nu}q_\nu.$$

$$\Lambda_f^\mu = \frac{-\alpha m}{4\pi}\int dy \frac{1}{m^2+q^2y(y-1)}\sigma^{\mu\nu}q_\nu$$