Chapter 5

Collisions

Smashing Things Together

Christiaan Huygens by Caspar Netscher, 1671.

Christiaan Huygens, 1629-1695

"There are many degrees of Probable, some nearer Truth than others, in the determining of which lies the chief exercise of our Judgment." *Cosmotheoros (1695)*

Isaac Newton wasn't the only smart guy around. Although he had respect for only a few contemporaries, Christiaan Huygens, a Dutch gentleman of means—and oh, by the way, a genius astronomer, inventor, and mathematician of such esteem that he was a elected foreign member of the British Royal Society—was one of them. The other, much to his consternation, was his rival for the invention of calculus, Gottfried Wilhelm von Leibniz. It's interesting that neither of these two had academic day jobs. Huygens did what he pleased and Leibniz was a diplomat for the House of Hanover for much of his life.

Figure 5.1: The surface of Saturn's moon, Titan as captured by the ESA space probe, Huygens.

Figure 5.2: Read about Christiaan Huygens in the The MacTutor History of Mathematics archive.

5.1 Goals of this chapter:

- Understand:
  - The meaning of Momentum Conservation
  - How to use the momentum conservation equation in one dimension
  - How to draw the Feynman Diagram for collisions of two objects
- Appreciate:
  - How momentum conservation works graphically in two dimensional collisions
- Be familiar with:
  - The history of understanding collisions
  - Huygens' life

5.2 A Little Bit of Huygens

Christiaan Huygens grew up in a privileged household in the Hague where his father, Constantijn Huygens, was a diplomat and an advisor to two princes of Orange. The Huygens' home was unusual. Guests included Descartes and Rembrandt, whom Constantijn helped support. Constantijn was friends with Galileo, a poet and musician and was even knighted in both Britain and France and so perhaps it was logical that he would provide for Christiaan's home-schooled.

When ready, he was sent to Leiden University to study law two years after Newton was born, whereupon Christiaan largely discovered professional mathematics. A pattern we've seen before, although unlike Galileo's case, Christiaan's mathematical education had included encouragement from Descartes when he as a child. So his father was much more understanding...and the need for a "job" was never an issue in this family.

Huygens became interested in astronomy and learned to grind and polish lenses in a new way which led to the development of a telescope of unparalleled quality. Among his first discoveries the large moon of Saturn, named Titan and then the resolution of Saturn's rings, which to Galileo had been only a confusing, unresolved bulge. The space probe "Huygens" was designed by the European Space Agency and landed on Titan on January 14, 2005. From Christiaan Huygens' first glimpse of Titan, to the photograph of its surface (Figure 5.1) is a nice story.

His astronomy led him to a need for accurate time, whereupon he invented the first pendulum clock (think "Grandfather") —by inventing a pivot that made the pendulum swing in the pattern of a cycloid, rather than strictly circular motion of an unhindered pendulum. Galileo had shown that the period of a pendulum was independent of the amplitude of the swing ("isochronous") but this is only approximately true when the amplitude is small. Huygens showed mathematically and then by construction that if the bob can be made to take the path of a cycloid, that the motion would be isochronous even for large swings...and hence useful in a clock. He also carefully considered the forces on an object in circular motion and the derivation of the centripetal force in Chapter 4 was actually obtained first by Huygens for a circle. However, he was confused by the tendency of an object to move away from the center of a circle and called the force *out* "centrifugal force." Newton also was initially confused by this but figured out that gravity, for example would pull *in* and hence coined the name "centripetal" to contrast it to Huygen's idea.[1] Newton's analysis was general and included circular, elliptical, parabolic, and hyperbolic orbits, so we tend to credit him with the correct understanding of curved motion.

[1] It should be noted that Huygens was also a committed Cartesian and interpreted this centrifugal force as the force of a fluid on an object, the source fo that fluid for an astronomical object being something similar to Descartes' little balls in vortices.

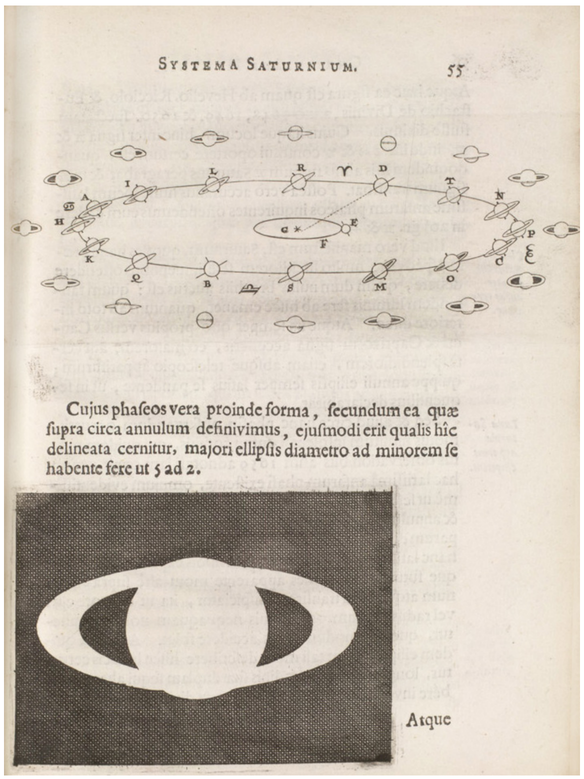


Figure 5.3: huygenssaturn

Huygens traveled widely and spent considerable time in Paris in multiple long stays. He visited Britain many times as well, and when he thought he was dying (he was often in frail health) bequeathed his notes to the British Royal Society. The British Royal Ambassador wrote, "...he fell into a discourse concerning the Royal Society in England which he said was an assembly of the choicest wits in Christendom...he said he chose to deposit those little labours...in their hands sooner than any else..." His scientific circles were very broad and he counted as among his friends, most of the intellectuals of the day, including Boyle, Hooke, Pascal, and indeed, Newton whom he visited shortly after *Principia* was published.[2] Christiaan Huygens died at the age of 66 four years after his visit with Isaac Newton, who referred to him as one of the three "great geometers of our time."

[2] Leibniz met Huygens in Paris and credited him with mentoring his mathematical development as a young man.

For our purposes, it was Huygens' consideration of collisions that is his legacy. Descartes had considered the problem of colliding two objects together and he applied his notion that all motion in the universe is conserved to such problems. But he was confused about just what was conserved and didn't have an appreciation of vectors or momentum. Putting Huygens' work together with Newton's gives us our modern ideas about how the total momentum of a system of colliding objects is preserved and that's our concern in this chapter.

In EPP we're all about collisions. We make huge facilities to do nothing but crash things together and we still use the same ideas and language first invented in the 17th and 18th centuries, albeit fancied up for modern applications.

5.3 Early Ideas About Collisions

One of the treasured concepts for physicists is the idea of the Conservation of some quantity. We'll make use of that idea over and over.

While we have a sophisticated justification for this affection for conserved things, even before Conservation Laws were a notion at all, natural scientists had an intuitive sense that Nature seemed to preserve some qualities. The first such serious assertion became known as the "Conservation of Momentum." Descartes started it all when he declared that the total "amount of motion" is unchanged, just shared among all of the various bodies in the cosmos.

Some of the quantities which appear to be absolutely conserved in Nature are:

- momentum
- energy
- angular momentum
- electric charge

As we'll see, energy and momentum are actually a single quantity which is conserved.

His model of the universe assumed that it was originally kick-started with all of the material bits set into initial motion and all of this primordial motion shared among all objects forever. Add those individual bits of motion up at any time and you get the amount of motion you started with. "Bits of motion" for him meant: speed. This eventually led him to his Big Idea that outer space was filled with various sized balls which were originally rotating together in a great "vortex," and in that way dragging the planets along with them. Those balls constituted his choice as material cause of the orbits of the planets

Definition: Conserved Quantity.
A conserved quantity in physics is one that is unchanged during a time interval—typically a "before" and "after" some event. These statements are referred to as "Conservation Laws."

Wait. *What balls?*

Glad you asked. *I'm sorry? Oh, you mean how did he come up with this idea? As a materialist he was forced to postulate some contact force between objects to make anything move, including the planets. Descartes' philosophy influenced his science: it was top-down. Postulate a cause and then work it out. But don't postulate motion without first setting up the mechanism. His commitment meant: I see the planets moving, so something has to be pushing them. Balls sharing the original primordial motion is as good as anything else.*

While Newton dismantled Descartes' vortices and after considerable massaging the preservation of the total motion was an idea that was still around after Newton was finished. Certainly, the rotations of the planets (and the Moon) about their centers is a modern reflection of the original rotations of the matter that under gravity slowly coalesced into their solid masses. Still orbiting, after all these years.

5.3.1 Modern Ideas About Collisions

Let's think about how Descartes might have come to his conclusion and how he was wrong. Figures 5.4, 5.5, and 5.6 picture three familiar kinds of collisions.

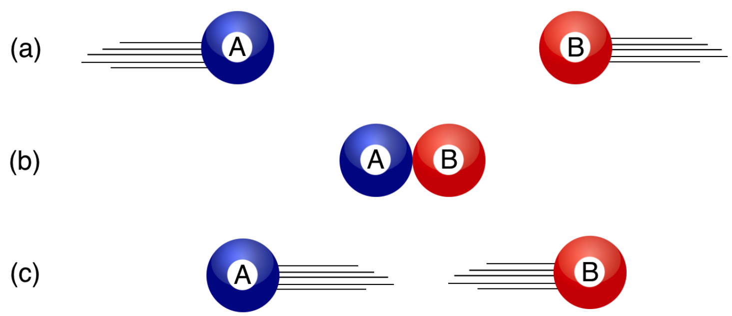


Figure 5.4: Two identical billiard balls are moving towards one another with the same speeds in (a), collide at (b), and recoil from one another at (c).

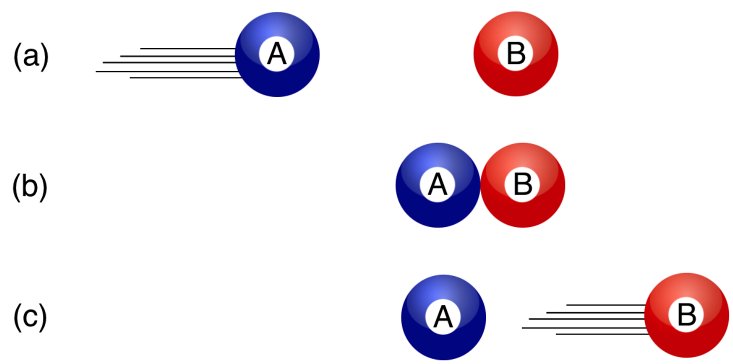


Figure 5.5: One billiard ball B is sitting still and another ball A is headed right for it in (a), they collide at (b), and then B shoots off to the right, leaving A stopped where they hit.

From your own experience you can guess at the outcomes of each. Look at Fig. 5.4: two billiard balls (each with the same mass) are in a head-on collision. B comes from the right at speed v and A from the left, also at speed v. What happens? Descartes would have said that the total speed at before they collide

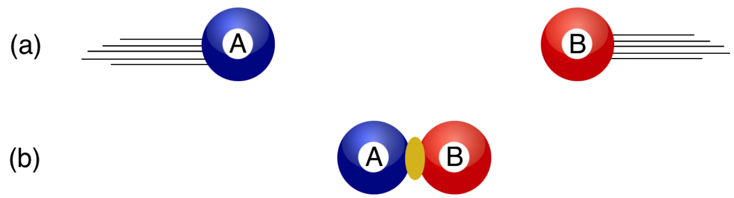


Figure 5.6: Two identical billiard balls are moving towards one another with the same speeds in (a), collide at (b), and stick together (that's gum between them).

is $2v$ and so the total speed after they collide would also be $2v$. And he would be right. In this collision, both balls would recoil from one another and just reverse their original motions.

In Fig. 5.5, now B is sitting still minding its own business and A comes along with speed v and strikes it. This is a familiar "dead ball shot" in billiards and I'll bet you've done it with pool balls or something similar. The outcome is that A stops in its tracks and B shoots out with the same speed that A originally had, so Descartes would still be right.

But now look at Fig. 5.6. It's the same as Fig. 5.4 except now the balls stick together when they collide. The result is that they stop dead—much like two cars crashing together or a running back and fullback colliding at the line of scrimmage.[3]

Here, Descartes would get it wrong: he would have said that the original motion was $2v$, but we know that the final "motion" is 0.

Obviously he didn't appreciate the importance of the *directions* of the *velocities* in the addition to their magnitudes. That is, he didn't know about vectors. In the sticking-together collision we instinctively know that the result is: $v_A - v_B = 0$ so that they stop. *Speed* is not the conserved quantity. Newton's *momentum* was the key, defined as the vector quantity, $\vec{p} = m\vec{v}$, magnitude and direction. He had the beginnings of an appreciation for the direction of momentum, filling a concept-gap that neither Descartes nor Galileo had come to on their own. But Huygens got it right: he understood the idea of vectors.[4]

Collisions were a fascinating study for those working in the 17th century. Everyone understood that friction confused the real picture, so people relied on colliding pendula where these effects were reduced. One has visions of everyone having many "executive toy" contraptions in their workrooms, changing out the bobs and causing clacking collisions with careful measurements of the outcomes. Huygens made use of his home-town canals in Amsterdam as way to collide masses in a controlled way.

Amsterdam's canals provided a near-frictionless racetrack for accelerating particles. He would station a colleague on a canal-boat with a pendulum which would collide with one held by someone on the shore. As the boat went by, nearly frictionless collisions were created and, he was able to get study the collision from the point of view of a "fixed" coordinate system (say, the shore-guy when the boat-guy went by) and

[3] where maybe "dead" is an unfortunate choice of phrase!

[4] Actual vector notation wasn't invented until the late 19th century.

Figure 5.7: A woodcut from Huygens' work illustrating his little 17th century particle accelerator. One person with a pendulum is in a boat on an Amsterdam canal and the other is on the pavement with a pendulum as well. Back and forth they went, with different masses dangling from their ropes. It must have been a sight.

the “moving” coordinate system (the boat-guy). His geometrical explanation was very complicated, but essentially correct. I’ll describe it here in modern language.

I want to concentrate on the collision of Fig. 5.5. We’d better be able to predict the outcome of this mundane example to believe Newton’s mechanics.[5]

[5] Notice, that Descartes is happy and applauding in the background since the total amount of motion before and after the collision is unchanged. But we already know that his theory is busted.

5.4 Impulse and Momentum Conservation

Let’s develop the simple machinery from Newton’s ideas. Remember from Eq. 4.6 that the the momentum change of an object is equal to the force that alters its motion times the time through which that force acts.

$$\mathbf{F}\Delta t = \Delta \mathbf{p} \tag{5.1}$$

Think about what happens when object A collides with object B. Let’s imagine that A is your left hand and B is your right hand. Now give Huygens a big hand (shall we?) and clap them together. You can make two simple, but important observations from this:

- When they just start to make contact your left hand begins to exert a force on your right hand and your right hand exerts a force on your left. Is there any difference? Does one hand fly away from the other? No...from Newton’s 3rd law, these forces are equal (and opposite). So: $\mathbf{F}_{\text{left}} = \mathbf{F}_{\text{right}}$.
- Nothing is perfectly stiff, so there is some elasticity or crumpling or bending and so the total force that each exerts is spread out over the same times as they continue to press against one another. Is the time your left hand is in contact with your right hand any different from the time that your right hand is in contact with your left? No, of course not. So: $\Delta t_{\text{left}} = \Delta t_{\text{right}}$.

So let’s look at the impulse experienced by any two colliding objects A and B. Here are those relations:

Pencil 5.1. ✎

$$\mathbf{F}_A\Delta t_A = \Delta \mathbf{p}_A \tag{5.2}$$

$$\mathbf{F}_B\Delta t_B = \Delta \mathbf{p}_B \tag{5.3}$$

So as we learned “by hand,” both, $\mathbf{F}_A = \mathbf{F}_B$ and $\Delta t_A = \Delta t_B$ so:

$$\Delta \mathbf{p}_A = \Delta \mathbf{p}_B. \tag{5.4}$$

That is, the change of momentum that A feels is the same as the change in momentum that B feels. Let's pretend that their collision is in one dimension and remember the convention for "change of" and that we'll use the subscript "0" to indicate the initial quantities:

$$\Delta p_A = p_A - p_{A,0}$$
$$\Delta p_B = p_B - p_{B,0}$$

Further, we'll presume that A was initially moving in the positive x direction so that its force and velocity are initially in that direction, and so from that designation, B is moving in all of the opposite directions. We then find:

$$\begin{aligned} \Delta p_A &= -\Delta p_B \\ p_A - p_{A,0} &= -(p_B - p_{B,0}) \qquad \text{rearrange these terms...} \\ p_{A,0} + p_{B,0} &= p_A + p_B \end{aligned} \tag{5.5}$$

Equation: Momentum conservation for two bodies.
$p_{A,0} + p_{B,0} = p_A + p_B$

Equation 5.5 is a really important relation! It says in words that the total momentum (of A plus B) at the beginning, before the collision, is equal to the total momentum of both objects at the end. This is the statement of our first serious "Conservation Law." In this case, Momentum Conservation.

Wait. *You used the L word.*

Glad you asked. *You're right. These conservation rules are so important—Nature is really telling us something when we discover such a rule–that it's one of those times when history and custom forces us using the Law designation.*

Definition: Momentum Conservation.
The total momentum at the beginning of any process is equal to the total momentum at the end of that process.

You can take it to the bank:

Momentum is always conserved. *Key Concept 12*

5.5 Examples of Collisions

Armed with the idea of momentum conservation as summarized by Eq. 5.5 we will analyze some example collisions. In EPP this is an essential part of our story since in order to peer inside of our elementary particles we have to probe them by shooting other elementary particles into them.

For simplicity's sake, our collisions will be between particles that have no physical size—"point particles"—and do not deform upon impact.

Wait. *There you go again. You've decided to describe an unrealistic circumstance. How is that helpful?*

Glad you asked. *As is usually the case, when we build a model we do so in a way that emphasizes the dominant physics at the expense of less significant, but realistic effects, that would get in the way of a simple description. In this case it's also a pretty good approximation for many real situations. Superballs are very resilient and behave like almost perfect objects. Also if we shot one electron toward another election, the electrostatic repulsion is responsible for the recoil and it's almost perfect. So this is a pretty good model for us!*

Some of the discrimination among them involves energy, which we'll cover in the next chapter, but I think this will make sense even in an everyday sense.

We'll consider four different examples of collisions in one dimension, and one in two dimensions. I'll do this simple one in this chapter, and the other four in the Diagrammatica 7 chapter that follows. I'll label them like a little formula, or chemical reaction. Here they are:

1. $\boldsymbol{A}+\boldsymbol{B} \to \boldsymbol{A}+\boldsymbol{B}$ An elastic collision of two equal mass objects called A and B.
2. $\boldsymbol{A}+\boldsymbol{b} \to \boldsymbol{A}+\boldsymbol{b}$ The elastic collision of two objects of different masses (A more massive than $\boldsymbol{b}$, a slightly different version of the above).
3. $\boldsymbol{A} \to \boldsymbol{B}+\boldsymbol{C}$ The decay of a large object, A into two different smaller objects, B and $\boldsymbol{C}$.
4. $\boldsymbol{A}+\boldsymbol{B} \to \boldsymbol{C}$ The collision of two different objects, sticking together to make a bigger, third object, $\boldsymbol{C}$.
5. $\boldsymbol{A}+\boldsymbol{A} \to \boldsymbol{b}+\boldsymbol{c}$ The head-on collision of two identical objects that produce two other objects. (This we'll consider after we've been through some quantum mechanics.)

Definition: **Event.**
In a physical process...and event is when "something happens"! The makeup of the systems before and after an event may be very different.

Let's also establish some terminology, some of which is repurposing regular words into specialized physics terms:

Definition: **Initial State, Final State.**
The configuration of a system before a collision (or decay) and after a collision (or decay).

- I'll use the words *collision* and *scatter* interchangeably.
- We'll often call the whole scattering process—objects approaching, colliding, and separating—an *event*.

- The *state* of an object is its particular values of position and momentum. These six quantities are enough to completely characterize an object's motion.
- Let's treat the moment of collision as a special time in the event serving to separate the *initial state* (before) from the *final state* (after). So an event consists of the initial state, the collision, and the final state.
- For our purposes, we'll reserve the phrase *elastic collision* to mean that the two objects that scatter at the beginning are the same two objects after they collide. They have different states, but they are still the same objects in an elastic collision.
- A *decay* is an atomic or nuclear physics idea, but it has everyday analogues. A firecracker that explodes into a few pieces is like the "decay" of a firecracker. My analogy below is more imaginative!

Notice that in Fig. 5.4 and 5.5 I separated out these three states of the event. The initial state is in each (a) and the final state is (c). The collision itself is the intermediate, (b). All three together constitute the story of an event.

And, an aside on units: The units of momentum are associated with the units of $p = mv$, so mass times velocity. So a kilogram-meter-per-second would be a perfectly good unit for a momentum. In the English system, so would slug-ft-per-second, which is fun to say, but it's not usually used.

You Do It 5.1. kgm/s-hour

Rather than kg m/s, what would a unit of momentum be if you read your speedometer (in Canada!) in kilometers per hour? Assume the mass is again in kg.

or copy the solution

But I recognize that the units are unfamiliar and could get in the way of grasping the important stuff, so let's just invent our own unit-less arbitrary measure. If I say that a particular momentum is "5," we'll interpret that to be in arbitrary "momentumunits." Let's look at a variety of different processes with this in mind and repurpose some words:

- **Beam**. Any of our objects that are moving toward a collision, we'll call a beam or beam particle.[6]
- **Target**. A beam is aimed at an object which we'll call it a "target." Just what is the target and what is the beam is a matter of your point of view, but it will be clear in our contexts. You'll see.

A particular collision is when the target is sitting still and the beam hits it—like our billiards or ice hockey example. This situation is called a Fixed Target Collision.

Let's play pool.

Definition: Beam.
When a collision happens between two objects and one of them is aimed at the other, it's called the Beam or the Projectile.

Definition: Target.
When a collision happens between two objects and one of them is struck by the beam, it's called the Target.

[6] We'll use the ideal circumstance of our colliding objects having no extent: that is, everything will be a point-sized "particle." Once an object has a finite size, then hitting on an edge will start rotations of the colliding constituents and that complicates things beyond where we need to be.

5.6 Elastic Scattering: $A+B \rightarrow A+B$

In Fig. 5.5 we had the "simplest collision of all" in which two identical things collide and bounce off from one another:

$$A+B \rightarrow A+B \tag{5.6}$$

Let's analyze this event in detail, and then...one more time after this! Remember that B is sitting still, minding its own business, when it's struck by A, so the target is B and the beam is A.

Definition: Fixed Target Collision.
A collision in which the beam strikes a target which is at rest.

5.6.1 Two Body Scattering in Everyday Life

Sports are the easiest places to imagine elastic scattering in everyday life. For example, a bat hitting a pitched baseball is such a collision. Here we would probably identify the target as the bat and the beam as the ball. In any case, unless the bat splinters, the initial state is a bat and a ball and so is the final state. Another such collision is a golf club striking a golf ball. This would be a fixed target scattering event since the golf ball is sitting still.[7] Similarly you could think of a football kick-off; the iconic and rarely performed, full sliding mug of beer into your hand at the end of the bar; and of course a car accident in which the cars bounce off from one another (don't couple together).

In particle physics, many experiments are performed with a stationary target, like a liquid hydrogen container—which is then a bucket of protons—and a beam that could be any from among the zoo of particles. Currently at the Fermi National Accelerator Laboratory, beams of neutrinos are emerging from such targets.

[7] A much harder—and more fun to watch—version of this game would be one in which the ball was in motion, I guess.

5.6.2 A Particular Elastic Scattering Example: #1

Let's stick with the venerable pool ball collision example. Ball B is sitting still and the cue ball A strikes it directly in its center so that there is no sideways motion after the collision. Can't get any simpler.

Pencil 5.2.

In our fake "momentumunits" let's invent an example and follow it through:
Here's what we know in this example:

- The initial momentum of the cue ball is 12, $p_0(A) = 12$.[8]
- The initial speed of the B ball is zero, $v_0(B) = 0$.
- The mass of each ball is 6, $m_A = m_B \equiv m = 6$.
- The velocity of the outgoing B ball is 2, $v(B) = 2$

[8] Remember the subscript "0" means "initial."

Questions:

1. What is the initial momentum of the B ball?[9]
2. What is the *total* momentum of the entire initial state?
3. What is the initial speed of A, $v_0(A)$?
4. Using your experience, what is the final momentum of B, $p(B)$?
5. Using your experience, what is the final velocity of b, $v(B)$?

[9] A "who's buried in Grant's Tomb question.

5.6.3 Momentum Conservation

Before looking at momentum conservation, we can deal with the first three questions easily:

1. If the B ball is stationary, then its velocity is 0 and so its momentum is 0, $p_0(B) = 0$.
2. The total momentum of the intial state is the sum of all of the individual momenta of any objects in the initial state. So in this case, $p_0 = p_0(A) + p_0(B) = 12 + 0 = 12$.
3. The momentum is $p = mv$. Since the mass of A is $m_A = 6$ and the momentum is $p_0(A) = 12$, then $v_0(A) = 12/6 = 2$.

Now let's conserve momentum and solve the event. Our particular situation in symbols is:

$$A + B = A + B$$

cue ball + stationary ball, B → now, stationary cue ball + now, moving ball B

Let's turn this into a momentum conservation equation:

$$\vec{p}_0(\mathrm{A}) + \vec{p}_0(\mathrm{B}) = \vec{p}(\mathrm{A}) + \vec{p}(\mathrm{B}) \tag{5.7}$$

Since we are dealing in only one dimension, we can stop using the vector notation and let the algebraic sign indicate direction (+ to the right and − to the left). So we have the simpler:

$$p_0(\mathrm{A}) + p_0(\mathrm{B}) = p(\mathrm{A}) + p(\mathrm{B}) \tag{5.8}$$

From playing pool, you know what happens, but let's do some momentum-accounting by filling in tables like Table 5.1.

	Before (initial state)	After (final state)
A	12	a
B	0	b
total	12	c

Table 5.1: Momenta in arbitrary units for the billiard ball collision with some blanks to fill in.

In the before column of our momentum-accounting in Table 5.1 we've listed what we know. In the after column, we know that momentum conservation insures that c has got to be the same as the initial state's total, or $c = 12$. Our experience tells us that when we strike B with the A, that the A suddenly stops dead and B jumps forward. Pool balls are especially rigid and, apart from the effects of rolling, their collisions are pretty good examples of elastic scattering. So our *experience* would tell us to put in $a = 0$. So, here's the big question (drumroll): What is $b =$? Of course, it's 12 from momentum conservation and that's the answer to question #4.

The last question asks about the speed of B after the scattering event. Since the mass is 6, $m_B = 6$ and the momentum is $p(B) = 12$, then we can see easily that $v(B) = 12/6 = 2$ and B goes scooting away with the same speed as A had before the collision. So Table 5.2 completes our understanding of this collision, using experience as our guide.

	Before	After
A	(6)(2)	0
B	0	(6)(2)
total	12	12

Table 5.2: Momenta in arbitrary units for the billiard ball collision, but indicating the masses and the velocities in their own arbitrary units.

But wait. There's actually a problem.

We used our experience to determine the final state. But what if we let Newton and Huygens explain this result? Put on your seatbelt.

Since our two balls are identical, their masses are the same and we'll use the single m for that common mass:

$$m_A = m_B \equiv m.$$

Momentum conservation says:

$$p_0(A) + p_0(B) = p(A) + p(B)$$
$$mv_0(A) + mv_0(B) = mv(A) + mv(B)$$

where I've put in for the common definition of mass and unique velocity for each ball.

Since m is the same everywhere, we can cancel them all on the left and right sides. And, for this particular example, B was sitting still at the beginning... so... what's its velocity at the end?

Of course, we know that B is stationary, $v_0(B) = 0$. So, we get:

$$v_0(A) = v(A) + v(B). \tag{5.9}$$
$$v(B) = v_0(A) - v(A)$$

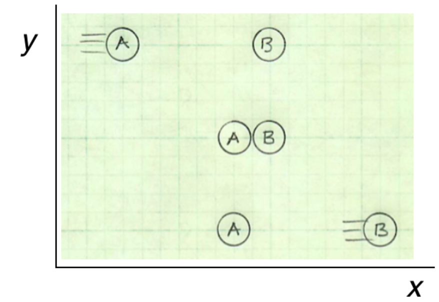
Equation 5.9 is Descartes' idea again: the total "motion" of the first object at the beginning is not lost but shared between the motions of all of the objects after the collision. But we're not happy. No, not at all.

Equation 5.9 is a simple, but terrible result! The beam object should stop dead so that $v(B) = v_0(A)$ but that's not what results from momentum conservation alone. Equation 5.9 doesn't *exclude* that result, but our balls don't just *sometimes* behave the normal way, **they always do**! Yet using Newton's and Huygens' mechanics the result we got is wishy-washy. Equation 5.9 allows the final speeds to be anything for either ball as long as they add up to $v_0(A)$. We expected to see specifically $v(B) = v_0(A)$, but we didn't get that.

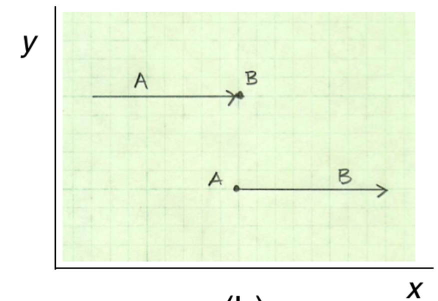
Wait. *You mean that Newton and Huygens were wrong?*

Glad you asked. *Looks that way, but it's not quite that bad. We need more physics.*

We need to introduce the big subject of kinetic energy which is the subject of Chapter 6.



(a)



(b)

5.7 Diagrams. Lots of Diagrams

I want us to become familiar with three kinds of diagrams for collisions: space diagrams, spacetime diagrams, and momentum diagrams.

5.7.1 Space Diagram

I think that the Space Diagram comes to mind easiest. Figure 5.8 is an abstraction of the collision in (a) looking down on the table. The bottom figure (b) shows the trajectories of the final state balls. Just like a map...tracing the paths. The dot indicates that a ball is stationary. The coordinates are space distances in both dimensions and time is implied, from the earliest (top) to the latest (bottom) as if snapshots were taken from above the table.

5.7.2 Spacetime Diagram

Figure 5.9 is the Spacetime Diagram for this collision. First, since time is one of the axes, "before" and "after" come for free on one drawing. Second, since this collision happens in one dimension, the vertical x axis represents all of the action. The B ball is just sitting still in space at position x_0 but it's moving in time. Finally, the collision happens at a particular time that's indicated to be t_C. So B's spacetime representation is a horizontal line at x_0 and extending from before t_C until exactly $t = t_C$.

Meanwhile, A is moving with a positive velocity (the x distance is increasing in time in a positive sense) and so its speed is represented as a positively sloped line in spacetime. Until t_C. Then A stops and B continues with the same velocity that A enjoyed before it collided with B.

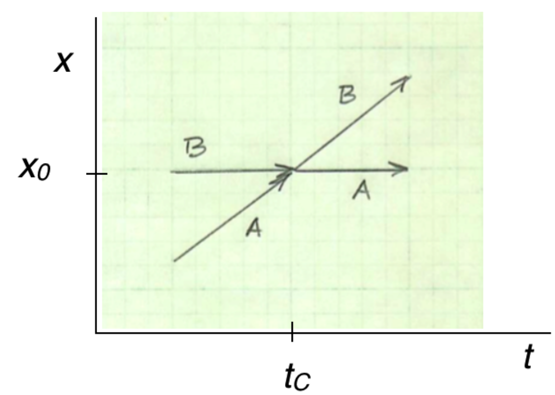


Figure 5.9: collisionspacetime

Momentum Diagram

It's also useful to include another diagram...one that represents the collision in a "mathematical space" that's sort of regular space but an overlay on regular space. The action happens in space, but what's drawn are vectors that represent momenta. Since a momentum vector points in the space (x, y, z) direction of the velocity, we can do this. But now the *length* of a momentum vector will be the value of the mass times

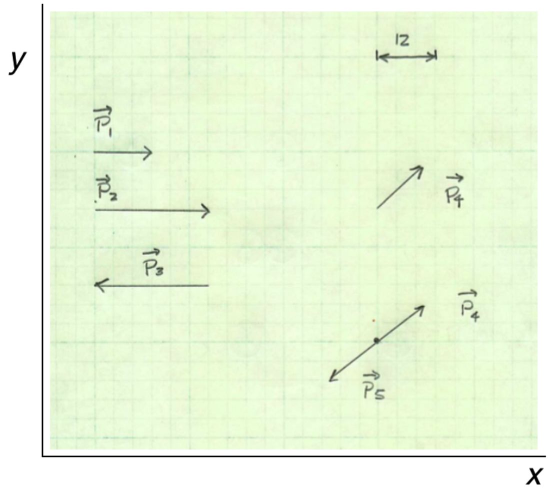


Figure 5.10: momentumcollection

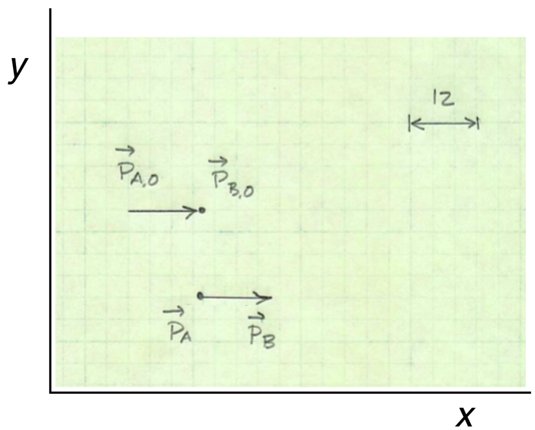


Figure 5.11: simplestonemomentum

the velocity. Figure 5.10 is a collection of momentum vectors using our fake units. The "space" is regular coordinate space, but the length of the vectors is in momentum units. The key at the top shows how long a momentum vector of 12 would be.

Momentum p_1 is a momentum pointing to the right (positive x direction) and it has value (length) of 12, according to the key. Notice that none of the momenta in the collection are identical. p_2 and p_3 both have lengths of 24, but different directions, so they are different momenta. p_4 is also a length of 12, but points in a direction that's a little of x and a little of y. Just below it is another vector, p_5 which points in the opposite direction from p_4. So we'd say that these momenta are **balanced** and that $p_4 = -p_5$.

In fact, this pair of momentum vectors is precisely what you would expect to see on a momentum diagram for a conserved momentum situation!

So for our simplest collision, we can refer back to the development of the table and draw our momenta to scale. Figure **??** is the momentum diagram for this collision. I've explicitly labeled the momenta as $\vec{p}_{A,0}$, etc. But in order to simplify notation, in the future I might just call such a vector $\vec{A}_0$ or even A_0 if the direction is trivial.

Let's summarize all three diagrams in Fig. 5.12. This Collection O'Diagrams...is repeated over and over in the Diagrammatica 7 chapter where all of the collisions we might see are described.

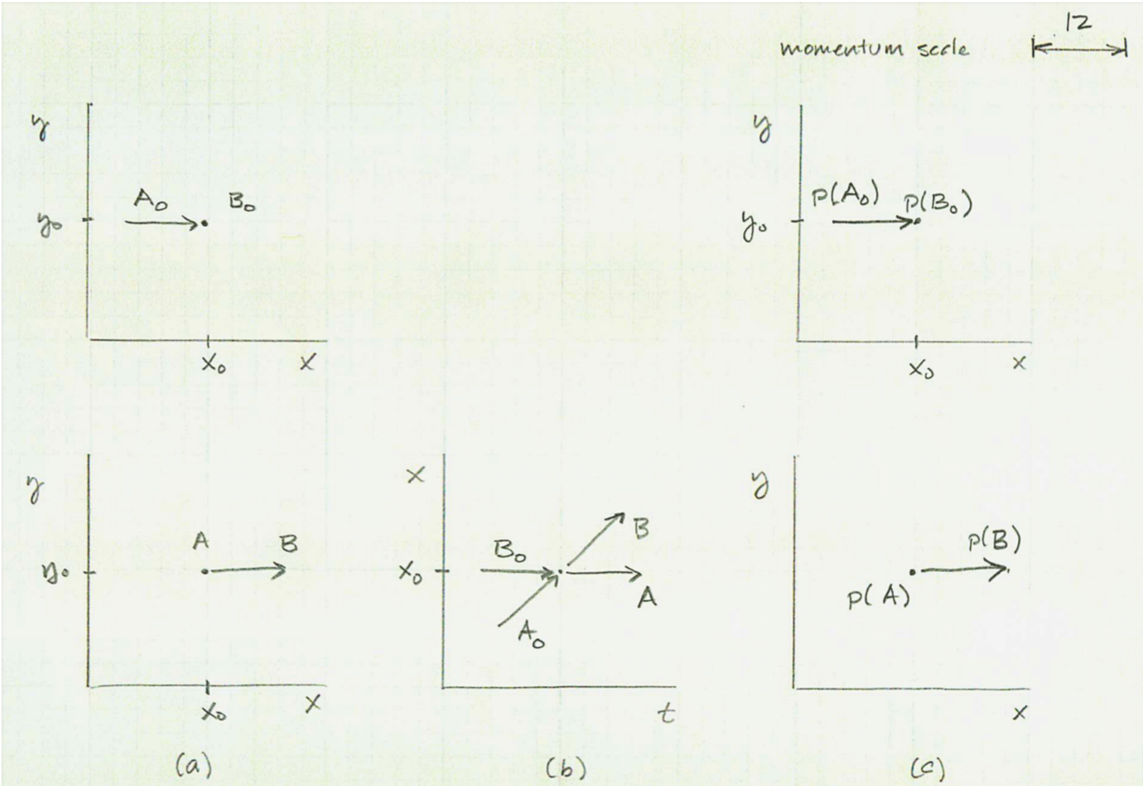


Figure 5.12: One one figure, all three diagrams are shown for our "most simple" collision. The Space Diagram on the left, the Space-time Diagram in the middle, and the Momentum Diagram on the right. In the next Diagrammatica chapter, these diagrams are all developed and cataloged for the various collisions we'll consider.

Box 5.1 Three Important Kinds of Diagrams Summary

We now have three kinds of diagrams that are useful in thinking about any reaction (like a decay) or collision. In fact—any physical process that takes place in time. Two of these we saw in Chapter 3 and Diagrammatica #1, Chapter **??**, and momentum conservation motivates the third.

Space Diagrams. This is the map picture, where in a space of distances in both axes (thinking in two dimensions), a trajectory is traced out as a contour. Time is implied as the trajectory takes you from a starting to an ending place with each point in between having an implied time stamp. For our earlier examples, we had only one object that moved...our car in Michigan, the bicycle, and so on. In particle physics, we have multiple objects and often one thing turning into another thing or things. So we should have a space trajectory for each object and it can get messy. For simplicity, we'll represent separate Space Diagrams for the before picture and the after picture...the initial state and the final state...where something will have happened in between. You'll see. For the pool balls example, the Space Diagram is shown in Fig. 5.12a.

Spacetime Diagrams. We've seen examples of Spacetime Diagrams—Feynman Diagrams, we'll call them—and they will figure prominently as we go forward. Here a representative space dimension is pictured along the vertical axis (the circumstances govern which one is chosen) and the horizontal axis represents time. With this arrangement, the slope of any Spacetime trajectory is its velocity, space divided by time. The Spacetime Diagram for the pool balls is in Fig. 5.12b. In fact, we've seen our first interesting Feynman Diagram and one that, with a little quantum mechanical tweaking, is highlighted on Richard Feynman's famous van show in Fig. 5.13. There's a story there.

Momentum Diagrams. Momentum Diagrams make the conservation of momentum visually apparent. In fact, they can be used to make predictions, as we'll see below. Time figures into Momentum Diagrams in the same way as our Space Diagrams by drawing one diagram for "before" and another diagram for "after." The point of them is that whatever the total vector sum of momentum is in the *before* picture has to be the vector sum in the *after* picture. The Momentum Diagram for the pool balls is in Fig. 5.12c.

Figure 5.12 shows the three diagrams side by side for emphasis. I need for you to be comfortable with all three.

Figure 5.13: I'll talk a lot more about Richard Feynman later, but he was famous for many things including his "out there" personality. This is a photograph of his van that I took a couple of years ago, which he'd covered in... Feynman Diagrams. The one right over the California license plate corresponds exactly to our "simplest" collision, when quantum effects are included. Stay tuned!

5.7.3 All of our collisions

We will consider a small number of different collisions and we can categorize them all by their diagrams. Our Second Diagrammatica Chapter 7, looks at each one and presents the diagrams for each and we'll refer back to it often.

5.8 Two and More Dimensions

We will not explicitly calculate in more than one dimension in QS&BB, but the sorts of collisions that you're most familiar with happen in more than one space dimension. Looking down on the pool table?

The balls move in x and y. In Demolition Derby, the cars scatter all over the infield. A pitch that's precisely horizontal as it passes home plate, is lofted into the air in collision with the bat. We could calculate the consequences of all of these kinds of two and three dimensional collisions, but we won't. We'll draw pictures.

In our previous considerations of apples and billiard balls colliding, we assumed that they had no size—point-like. But striking a billiard ball precisely on its center line with a cue ball is tricky. More likely is that they would strike just slightly off-center like in Fig. 5.14. In that case, if we define our x axis along the direction of the beam-ball, the target and the beam will both scatter into the y direction. Momentum conservation helps to determine the outcome.

The most general statement of the momentum conservation rule[10] is:

$$\mathbf{p}_{A,0} + \mathbf{p}_{B,0} = \mathbf{p}_A + \mathbf{p}_B \tag{5.10}$$

Equation 5.10 is really a symbolic statement—a *mathematical paragraph*, if you will—and not an equation that you can actually solve. Embedded in it are two (or three, if three dimensional space) "real" equations that you can actually manipulate...one for the momentum along the x axis and another for momentum along the y axis. Momentum conservation has to hold separately in each of them but we'll not actually solve the equations themselves. Rather, we can construct the Momentum Diagram and draw conclusions without doing any algebra. Let's put together a momentum diagram for this situation using our arbitrary units with the following parameters to start with. Pool balls? That's for children. We'll throw bowling balls at one another. I'm going to use the language of "beam" and "target" here, as that's more to our EPP liking. (In our previous collisions, A would have been the beam and B would have been the target.) So, now B will mean beam and T will mean target. Sorry.

- mass of the beam ball, B: 7 kg
- mass of the target ball, T: 7 kg
- speed of the beam ball: v(B) = 10 m/s

The coordinate system that we set up in Fig. 5.15a indicates the directions of motion in space, and so also, the directions of the momenta. In our previous Momentum Diagrams we were casual about the scale in our fake momentumunits, but let's do this one more precisely. We'll set the scale of the diagram as "1 inch equals 50 kg-m/s" and you can see that as the bottom line in Fig. 5.15. Now our ruler is calibrated in momentum units.

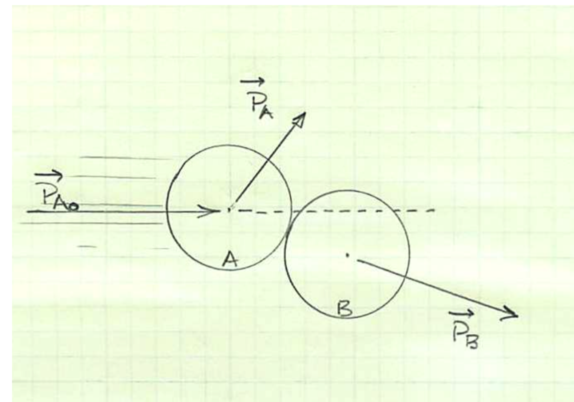


Figure 5.14: offcenter

[10] Remember? The total momentum of a system at the beginning of a collision is going to equal the total momentum of that system at the end

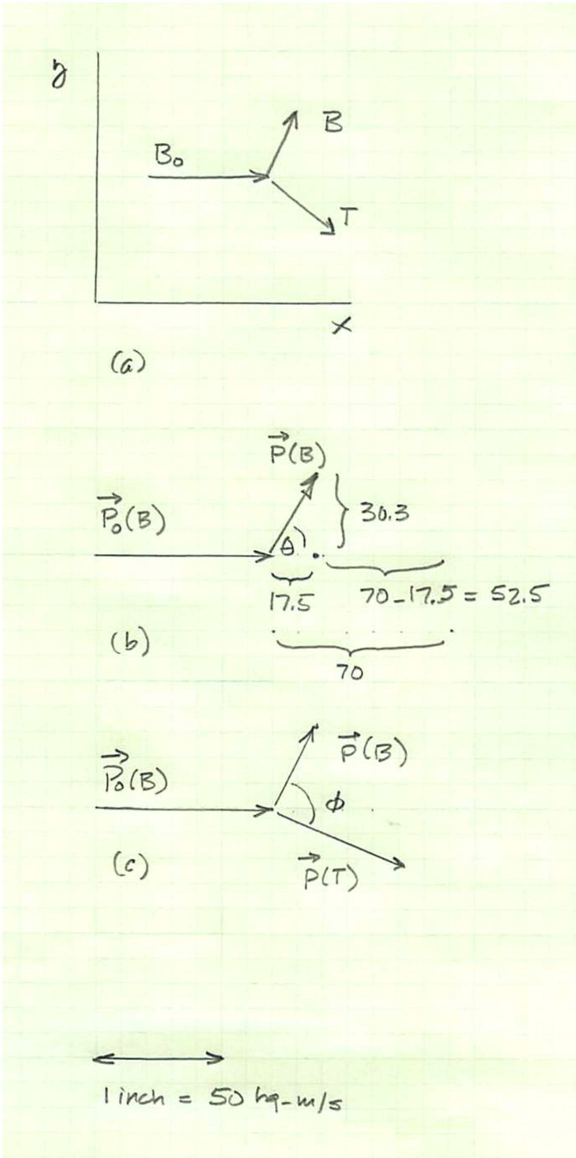


Figure 5.15: twobodyp

So, again, what do we know? We know the initial momentum of the beam. (Notice I've decorated the momentum symbol twice in the subscript. The 0 as the first of the pair indicates that this is an initial value like usual. The second, x means that this is in the x direction.)

$$p_{0,x}(\mathrm{B}) = m(B)v(\mathrm{B}) = (7)(10) = 70 \text{ kg-m/s}.$$

We also know the initial momentum of the target ball:

$$p_0(\mathrm{T}) = 0 \text{ kg-m/s}.$$

Let's suppose that the degree to which the centers are off is such that the beam ball bounces quite a bit off the target and shoots out in the $+y$ direction at an angle of 60° relative to the horizontal. Without doing the trigonometry, I'll just tell you that this means that the outgoing horizontal momentum of the beam is

$$p_x(\mathrm{B}) = 17.5 \text{ kg-m/s}$$

and that the vertical component is

$$p_y(\mathrm{B}) = 30.3 \text{ kg-m/s}.$$

This is all the information that we need to predict the motion of the target ball! Had Huygens and Newton been bowling buddies, they could have worked this out.

In Fig. 5.15b I've shown some distances and drawn in a vector momentum. The distance 17.5 is the x component of the final B momentum and the distance 30.3 is the y momentum. The arrow is the combination of the two components.

The target ball was just sitting still, minding its own business when it was hit by the beam ball. We can precisely determine what happens to it from the information we've gathered. We simply conserve momentum vertically and horizontally...by "eye." That is, by drawing in the **T** momentum vector according to momentum conservation.

- All of the x-directed momentum of the initial state system (all do to the beam) was 70 kg-m/s and all directed along x. So the sum of all momenta in the x direction of the final state system has to also add up to 70 kg-m/s.
- All of the y-directed momentum of the initial state system? Zero. There was no motion in the y direction. So the sum of the vertical components of final state momenta have to balance exactly, to sum to zero.

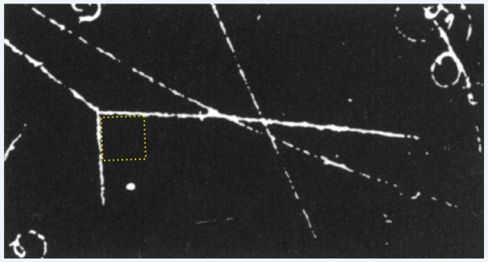
I've indicated the initial momentum of 70 in the final state as a bracketed length in Fig. 5.15b. That's so that I can use momentum conservation to find the x momentum of the target, it's just the amount left after the 17.5 from the beam which I've shown as 52.5. The vertical component of the target's momentum after the collisions? That's easy. It just balances the 30.3 value from the scattered beam.

So piece these together graphically, using the scale, I can sketch in the recoiling target's momentum vector and it's in Fig. 5.15c. By eye, the easiest thing to see is that the up and down momenta of the two final objects balance. You'd need your ruler to see that the sums of the horizontal components of each of the final state objects add to about the total of the original beam's momentum vector. We'll just do this by eye when we need to, but you get the idea!

Example 5.1

Resolving Elastic Scattering of the Target

Question:



Referring to our bowling ball example in Fig. 5.15:

a) What are the x and y components of the target ball's momentum?

b) Draw the total momentum vector of the target ball along with that of the beam.

Solution:

a) The total horizontal momentum has to be 70 kg-m/s, and we know that 17.5 of it is taken up by the beam ball's contribution to the final state motion. So the mome ntum of the target ball—in the x direction—must be $70 - 17.5 = 52.5$ kg-m/s. Now the initial and final state motions are balanced, horizontally.

The vertical component of momentum has to balance to zero and since the target's contribution is 30.3 kg-m/s, "up" $(+y)$, then the target's contribution must be -30.3 kg-m/s, "down" $(-y)$.

BTW...a fun fact. The angle that the target is ejected towards is a special one for this situation...in fact, it's $30°$ so that the sum of the two outgoing angles is $60° + 30° = 90°$. This is always true when the target and the beam have the same masses and the collision is elastic. The outgoing particles themselves are separated by $90°$. Look at the picture in Figure 5.1 which shows a bubble chamber photograph of a proton that enters the picture from the upper left and hits a proton in the liquid and scatters it in one direction while itself recoiling in another. The angle between the outgoing protons can be seen to be about $90°$.

b) The whole thing is drawn on Fig. 5.15c.

What about the Feynman diagram of the whole "interaction"? Well, that's complicated. Let's use the x direction as our vertical space coordinate and of course time is always our horizontal coordinate. The above example shows us the relative values of the x momenta, and since the masses are the same, also the relative speeds. The relative speeds of the balls are then as follows: $v_{0,x}(\mathrm{B}) > v(\mathrm{T}) > v(\mathrm{B})$ so if we draw this diagram for just the x space coordinate versus time, it would look something like the sketch in Fig. 5.16.

The examples in the next Diagrammatica chapter, Section 7.4 will illustrate some two dimensional scattering, along with some real-life collisions.

5.8.1 Two Body Scattering in Two Dimensions

Finally, for EPP a relevant situation is one in which two balls, A and B, are head-on but also have a finite size and their centers are slightly shifted so that the collision starts off in one dimension, but the scattering is into two dimensions. Figure 5.17 shows this. Here the X marks where the collision happens.

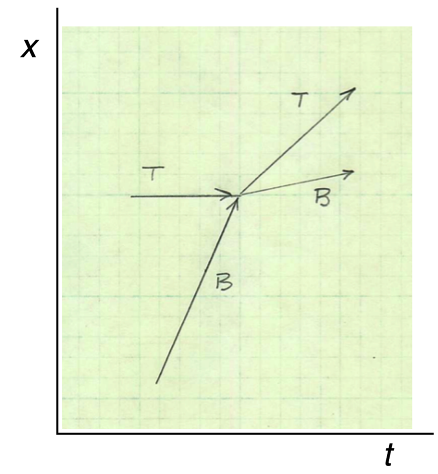


Figure 5.16: twoballsspacetime

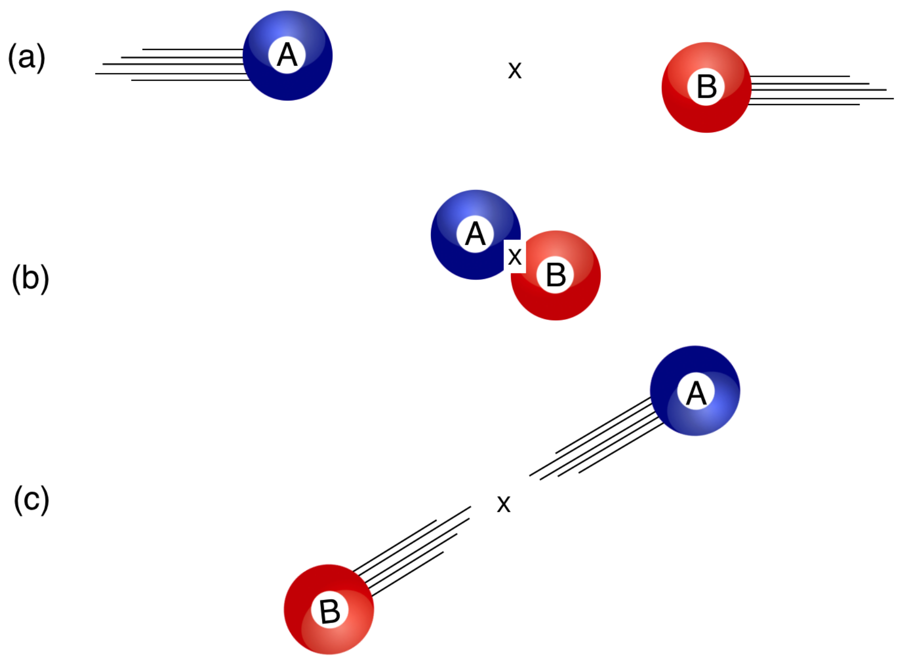


Figure 5.17: A more realistic collision. Again, looking down from the top two balls strike one another, but just off center so that they scatter in directions different from the original, oppositely converging directions. (a) and (c) are the initial and final states and (b) is the point of collision, where they bang together and momentarily stop. The X just indicates the point in space where they collide.

You Do It 5.2. Beams

Refer to Fig. 5.17 and draw (a) the Space Diagram and, if we assume that the balls are both the same mass and each have the same initial speeds, draw (b) the momentum diagram.

or copy the solution

5.8.2 Collisions At the Large Hadron Collider

Now let's look at some actual collisions in our experiment at CERN. Figure 5.18 shows a side view-slice of our "ATLAS" detector at the Large Hadron Collider at the CERN laboratory in Geneva, Switzerland. We'll learn what the various colors symbolize but we know enough now to understand what's happening.

A collision has happened and is one in which two protons collide head-on right in the center—one beam from the right and the other from the left (like the billiard-balls above and exactly like Fig. 5.17)—and produce, in this case, two electrons that emerge and leave their traces in our detector as the two

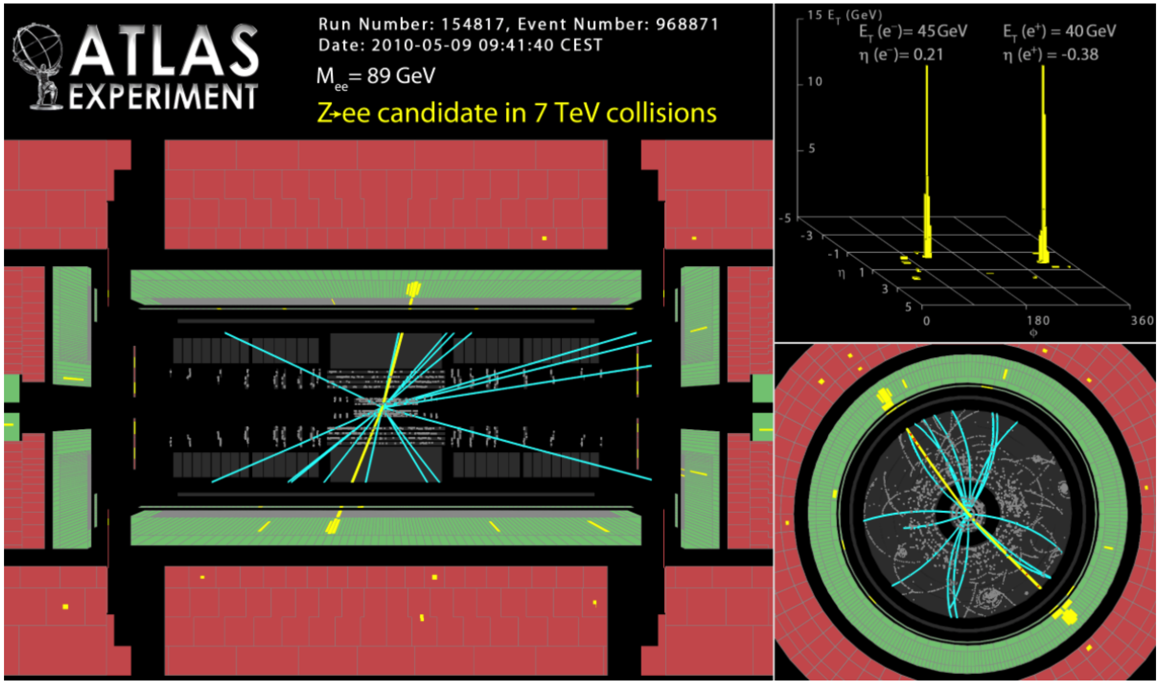


Figure 5.18: This is a computer reconstruction of the consequences of colliding two protons with one another in the ATLAS detector at the Large Hadron Collider at CERN. The left-hand view is a vertical slice through the detector with up being toward the surface (the device is 300 feet underground). The two beams of (identical) protons enter from the left and the right and collide in the center.

diagonal blotches of yellow color. You can think of the yellow lines and little blobs as the momentum vectors for the outgoing particles from the collision.

Momentum is being conserved in this simple reaction as you can almost see by eye. The initial state momentum in all directions is zero, since both beam particles are identical protons, so the momenta of the two electrons each have to be conserved in the horizontal and vertical directions. The amount of color in the "blobs" is a good measure of the magnitude of the momentum in each electron, and again, by eye you can just about see that they are equal and opposite.

Figure 5.19 is a completely different situation! Again, the collision is two protons, head-on. Again, in the final state there is an electron...but it looks like momentum is not conserved since there appears to be nothing emerging from the collision on the other side! That's our clue that a particularly elusive particle called a neutrino was produced with the electron, and we would even say, "Look, there's a neutrino in that event."

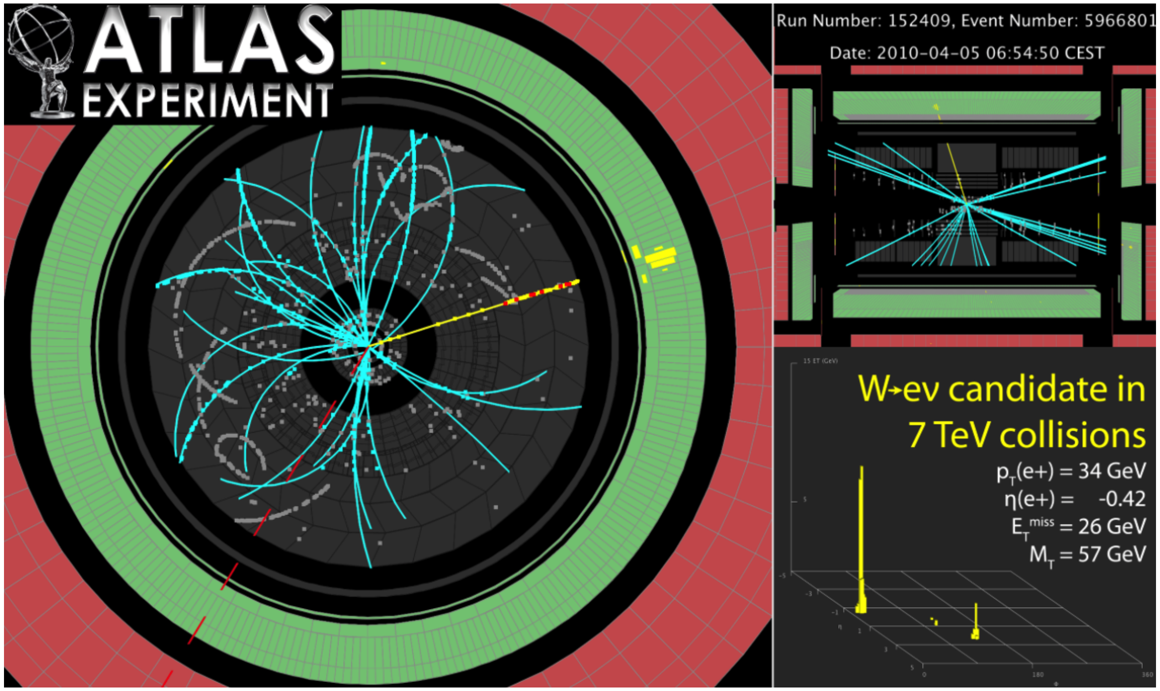


Figure 5.19: This is another event from the ATLAS detector, but now notice that the final state has only one yellow blob—one electron. Notice that there is nothing on the other side! Is momentum conservation violated?

Wait. *You mean that you observe this neutrino-particle by. . . seeing nothing?*

Glad you asked. *Yup. We believe in momentum conservation so strongly, that we are certain that some particle emerged that left no trace in the detector. Neutrinos are such a particle that interact with matter so rarely, that they essentially never show their presence in our detectors except by being the cause of an apparent momentum imbalance. That's our clue. Notice that this is really not the detection of a neutrino on an event-by-event basis. No, there are other hypothetical particles that might also leave no trace after they're produced and we are searching for exactly those kinds of particles.*

Wait. *Sorry to bother you again. But how do you know the difference then?*

Glad you asked. *No problem. We have to make such discoveries on a statistical basis. We know the rules for producing the "normal physics" with neutrinos and if there are other hypothetical particles that are produced, then we'd have to see them behave differently in many, many collisions. A statistically significant anomalous behavior.*

Have you got enough energy to learn about energy?