Chapter 3

Motion

Getting Around

Galileo Galilei, *circa*1624.

Galileo Galilei, 1564-1642

"I do not feel obliged to believe that the same God who has endowed us with sense, reason, and intellect has intended us to forgo their use." *Letter to the Grand Duchess Christina*

As he got older, his mouth got him in more and more trouble until he was imprisoned in his own home for the rest of his life. But by then, he'd created physics and defined diverging paths for religion and science. Galileo's life can be segmented into four distinct periods: his young life, education, and university employment at Pisa; his second job at the University of Padua in the Venetian Republic; his return to Florence as the Chief Mathematician and Philosopher to the Grand Duke of Florence; and then his house arrest at his villa outside the city gates. We'll follow his scientific path from falling bodies, to astronomy, to his method of doing science. When we're done, physics will have been born.

3.1 Goals of this chapter:

- Understand:
  - How to calculate distance, time, and speed for uniform and constantly accelerated, linear motion
  - That falling objects all have the same acceleration near the Earth.
  - How to graph simple motion parameters
  - How to read graphs of realistic motion parameters
- Appreciate:
  - The algebraic narratives in the development of the formulas
  - The shape of the trajectory of a projectile
  - That a projectile's motion is made of two components with different accelerations
- Be familiar with:
  - Ideas of motion before Galileo
  - Galileo's life
  - Galileo's experiments with motion

3.2 A Little Bit of Galileo

The original Florentine Galileo was a 15th century medical doctor and civil leader of the family Bonaiuti. So significant was this elder Galileo that the subsequently middle-class family renamed itself Galilei and our Galileo Galilei's two identical names was a subtle parental reminder that he was expected to do great things.[1]

Figure 3.1: The tomb of Galileo Galilei in the Basilica of Santa Croce

[1] So venerated, the original Galileo was buried in the Basilica of Santa Croce, eventually the resting place of Machiavelli, Michelangelo, and a pantheon of Renaissance personalities. . . and eventually, our Galileo as well as shown in Fig. 3.1

Galileo was born in Pisa within a year of the death of Michelangelo and educated at the newly restored University of Pisa. He always considered himself a citizen of Florence, although he lived there only briefly in his early years. His father was a musician—of necessity, a wool merchant in his wife's family business—and determined that his son would be a medical doctor. But as a student, he was disrespectful of his professors as an innate skeptic regarding the natural philosophy taught which conformed to the European standard: Aristotle. What Aristotle said about motion had not made sense to anyone for centuries, but his authority was almost absolute. Galileo—like his father—didn't "do" authority.

While a medical student, Galileo accidentally discovered that he had an aptitude for mathematics that led him to an intense but clandestine program of the study of Euclid with Ostilio Ricci, the Court Mathematician to the Grand Duke of Tuscany. He eventually abandoned medicine, leaving the university a year short of his degree. By this time he was doing original mathematical research (in geometry) and had caught the attention of scholars in Pisa and Rome. He lived at home for three years and gave private mathematics lessons in Florence and Sienna[2] while he cultivated patrons for help in finding a university position. After some rejections, he succeeded...back at the University of Pisa as a lecturer of mathematics. He wasn't altogether welcomed by his former teachers.

[2] He became a passionate follower of Archimedes' mathematics and invented a more precise way to measure the density of metals, following in his hero's wet footsteps. He also gave invited lectures at the Florentine Academy on the geometry of Hell from Dante's Inferno

His reputation as an original mathematician was growing when his father died and he inherited the responsibility of a significant dowry for one sister and responsibility to provide for the other's. Galileo spent the rest of his life in search of a higher salary, which as a lowly mathematician at Pisa was a factor of four smaller than that of a philosophy professor. He got his break when he was offered the position of Professor of Mathematics at the University of Padua, among the most prestigious universities in Europe and safely in the progressive Republic of Venice. It was at Padua where the magic happened.

3.3 From Here to There

This is important:

> ***Almost everything in physics boils down to: motion.***[3] *Key Concept 5*

[3] ...even "boiling"!

Whether it's runners on a track, the cosmic rays piercing us all the time, orbiting planets, electrons in a wire, electromagnetic waves, quark wavefunctions inside of a proton, electrons and holes in a semiconductor, or the stretching of spacetime itself. Everything is about motion.

These first chapters on the physics of my grandparents' generation will establish our language and tools that we'll need in order to pursue the more exotic forms of motion and we'll become skilled at manipulating concepts (and their attendant symbols) like velocity, kinetic energy, mass, momentum, and force. Each of these terms has a 16th to 19th century origin, but each has managed to keep up with the times as layer upon layer of subtlety is discovered about each of them as we dig deeper and discover more.

But at its most basic, it's all about how to get from here to there or from then to now, and to be able to explain how that happened.

3.3.1 A Greek Version of Here to There

The correct understanding of everyday motion was long incoming. Really long. Classical works had been out of reach of Europe until Greek philosophy and science essentially fell into their laps in the form of hundreds of conflicting Arabic translations in the 1300s. Aristotle—eventually referred to as "The Philosopher" —had invented formal logic that taught people how to evaluate arguments. But while the Philosopher's methods were refreshing, his ideas about motion were confused at best—but nonetheless became firmly stuck in the academic and religious communities where they were protected as *philosophy*, not as *science*.[4] Not until the end of the 16th century did Galileo shed Aristotle and lay the groundwork for the first systematic understanding of what it means for something to move. It took three centuries!

For Aristotle, motion[5] was of two sorts: natural and unnatural. Natural motion near the Earth was in a *straight line*, either *down* to the center of the Universe (which he located at the center of the Earth, proportional to the amount of "earthy" composition of the object, and hence its weight) or *up* (proportional to its lightness).[6]

Natural motion beyond the orbit of the Moon was to be *circular* with every extraterrestrial body attached to its own rotating crystalline sphere.

Wait. *Why would they insist on circles for the stars and planets?*

Glad you asked. *If you go out on a dark night and watch the motions of the stars in the north...you'll convince yourself that they are moving in circles around Polaris. You'd be wrong about the North Star's involvement, but you'd be pretty sure: circles. So were they.*

These spheres were all nested with common centers and rotated around the Earth to account for the apparently circular orbits that we see from the Earth. They included all of the known planets, the Sun, and the Moon...and even the stars in the outermost shell. Natural motion just happened...naturally, but unnatural motion required a pusher...an active force that was in contact with the object. Therein lay one of the most obvious flaws in his model.

Make no mistake, translation of Aristotle's *Physics* from original Greek, to Arabic, and then to Latin did not make his ideas any less confused than they originally were. Where he got himself into big trouble was with projectiles, like a thrown spear. Since for Aristotle the philosophy came before observation, he had to do an embarrassing dance to explain that when a rock was thrown, the continuous "push" that Aristotle insisted was needed came from the displaced air rushing around behind the rock and pushing it forwards.[7] Everyone knew that this was nonsense, but his authority reigned, and organization of the first medieval universities with Philosophers and Theologians at the top guaranteed that natural science was taught by them and not by the mathematicians and astronomers, whose roles were aimed at more mundane activities like casting horoscopes and designing military weaponry and fortifications.

Galileo was the one person who changed the landscape and uncovered the modern notion of how things move, ridding the intellectual community of Aristotle's baggage. He began his revolution while he was at Pisa where he wrote an unpublished manuscript, *de Motu* ("On Motion"). While he was unsatisfied, one of his conclusions was right on: all objects fall at the same rate, contrary to Aristotle's insistence that

[4] While Aristotle stumbled with physics and astronomy, he really was amazing. He practically inventing biology, zoology, anatomy, psychology, logic,ethics... the list goes on and on.

[5] To Aristotle, objects in nature "moved" according to causes and one had to beat one's common sense into submission in order to allow his explanations of everyday events into the mainstream. Motion for him was a very general thing: anything that changed in time, like when an seed grows into a sapling and then into an oak tree... is "motion." The kind of motion that we think about was "locomotion."

[6] Actually, he classified four kinds of elements: earth, water, air, and fire. Each had its natural place and substances went to that natural place according to the mixture of the qualities of the elements.

[7] Yes. He actually suggested that.

heavier objects fell faster. His data? Not the Leaning Tower of Pisa. That's a myth. He just looked around him. And saw things differently.

3.4 Speed in Modern Terms

To us, motion and its measure—speed—is a simple matter. Our cars and even devices on our wrists readily tell us how far we go, how long it takes us to get there, and the rate at which we do it. In fact, we can be penalized for traveling on our roads at rates that are...too enthusiastic. Speed, or its more sophisticated word-cousin, velocity, is so familiar to us that we hardly pay any attention to just how fundamental this concept is. Let's start this very slowly since some of our more sophisticated physics later will build on a strong underpinning of a few basic concepts. Speed is one such concept because it's a blend of two even more fundamental concepts of space and time.

Just wait until Albert Einstein gets his hands on it.

Speed is a rate. For example, 60 mph describes the change in our spatial position and how long it took to make that change. Like all rates, it's a ratio with respect to time:

$$\text{speed} = \text{"the change of distance divided by the change in time"} = \frac{\text{change of distance in space}}{\text{change of time}} \tag{3.1}$$

Wait. *Everyone knows this. We all drive and we can calculate how long it takes to get home. So why the fuss?*

Glad you asked. *Apart from the fact that trying to understand motion took almost 1500 years, we continue to misunderstand it over and over, as you'll see. We need to start gently with ideas that will seem trivial. But hold on to your hat, since it will get weird. I'll remind you that you thought that this definition was silly.*

Let's make this more compact by inserting customary symbols to get rid of the English words. Here are the rules of the use of motion in QS&BB:

Pencil 3.1.

- We'll limit ourselves almost exclusively to motion in one dimension in space.
- We'll use the symbol v for speed (because customarily, we'll speak of "velocity"... more about this below).
- We'll use the symbol x for distance in one dimension, regardless of which direction it points.
- We'll use the symbol t for time and almost always presume that we set our clocks so that the beginning time of any interval is $t_0 = 0$.
- Oh, and we'll use the subscript $_0$ to indicate the beginning of some time interval "t_0" or location "x_0" in a sequence of events.
- We'll use the Greek symbol Delta, Δ to mean "change of"... this will come up a lot.

Definition: Δ.
means "change of."

So our formula for calculating speed changes from the English sentence in Eq. 3.1 to a mathematical statement becomes:

$$v = \frac{\Delta x}{\Delta t} \tag{3.2}$$

Definition: Velocity..
Velocity (or speed) is the rate of change of distance.

Equation: Velocity.
$v = \Delta x / \Delta t$

Definition: initial quantity.
We will always put a little subscript 0 to indicate that some quantity is the "initial state" of a process. So, x_0 would be the initial position, E_0 might be the initial electric field, and so on.

It's important to think about how we would measure any quantity. For speed, we'd need something that functions like a ruler in order to measure a distance in space and something that indicates time intervals—a clock. So if we're on the interstate, we could imagine the mile-markers along a highway that tell us miles or better, small fractions of miles. We could arbitrarily designate a starting point as $x = 0$ and then, after speeding up, travel—without acceleration—until some pre-determined time, say 2 hours, had elapsed. Of course we would measure time with a clock in the car that begins ticking when the car passes our starting point. Then after precisely two hours had elapsed the clock would cause a camera to take a picture of the mileage sign that was closest to being opposite the car. Let's suppose that the sign read "100 miles." Without putting pencil to paper (this time!) you could quickly calculate the average speed that the car traveled in that time as 50 miles per hour.

Likewise, if I asked you how long it would take to travel north from Detroit 300 miles to the Mackinac Bridge at an average 50 mph, I'll bet you could tell me. You have done this sort of calculation a thousand times and so you would calculate:

You Do It 3.1. Travel60mph

or copy the solution

Calculate the time it would take in hours to travel 120 miles at an average speed of 60 mph

Did you get two hours? Your brain is already doing physics. Let's go "up north."

3.4.1 Calculating a Speed

"Change" and "change-of" always means the difference between where you *are* as compared with where you *were*. Suppose I start out with \$100 and my wife gives me \$50. The change in my net worth is \$50, right? But we can represent this simple transaction as

$$\Delta(\text{my wealth}) = \text{where I ended up} - \text{where I started} = \$150 - \$100 = \$50.$$

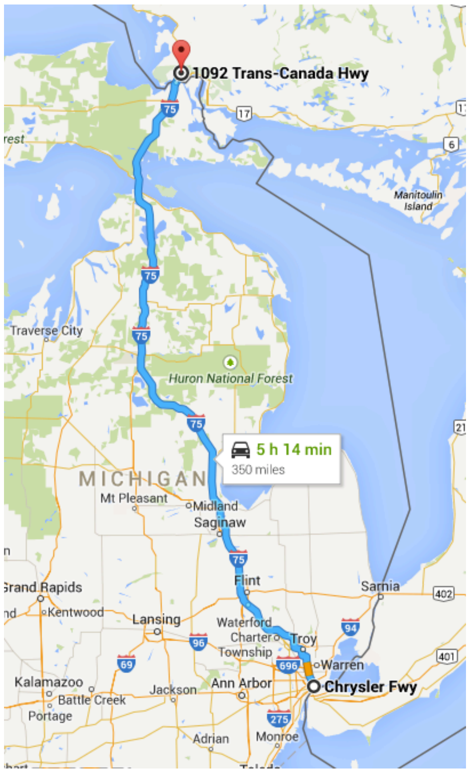


Figure 3.2: A trip from Detroit to Canada.

Remember that we're using the standard notation in which the "initial state" of any quantity will be decorated with a little "0" subscript, like x_0 here. The "final state" will have no subscript and just be x.

In our trip from the bottom to the top of Michigan, we will stay on I-75 as in Fig. 3.2, beginning at the nearest Comerica Park exit in Detroit which is near mile marker 50 at Grand River Avenue. Then we'll go all the way up, across the Mackinaw Bridge to Newberry, Michigan in the Northern Peninsula which we'll say is marker 350. We'll go fast.

So the change in my displacement is

$$\Delta x = 350 - 50 = 300 \text{ miles}.$$

Now, we can write the real velocity relationship:

$$v = \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - t_0}.$$

where I snuck in a "initial" decoration for the beginning time as well. If I start my clock ($t_0 = 0$) at mile 50 and stop it at mile 350, our average speed is:

$$v = \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - t_0} = \frac{350 - 50}{5 - 0} = \frac{300}{5} = 60 \text{ mph}.$$

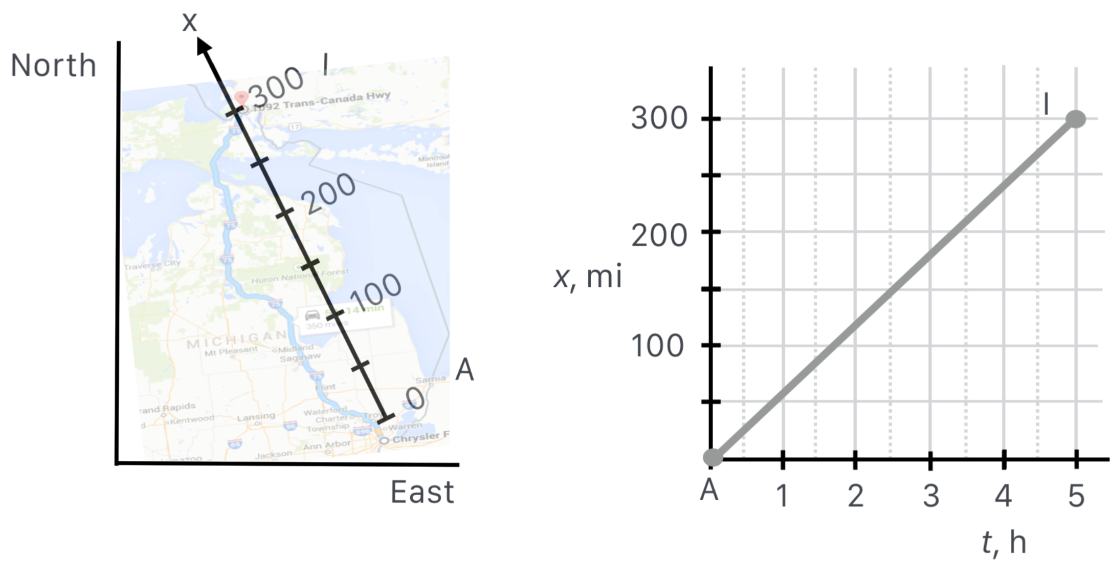
3.4.2 Diagramming Motion

We will need a variety of graphical ways of representing motion...from here to there. A perfect example of such a representation is when you draw a route on a map. The map has "space axes" of east-west (x) and north-south (y) and when you go from one town to another you might draw a along your route in space, recognizing that each mark corresponds to a different time as you move along the road images. So time is represented implicitly on such a graph.

3.4.3 Space Space Graphs

Let's take a realistic trip. A regular map, like in Fig. 3.2, or the display on a GPS system, is a familiar way of looking at travel. Let's make an approximation to that curvy map trip by straighteining out all of the road's curves and bends so that it looks like the approximate straight line in Fig. 3.3 a. Time is still implicit and the single coordinate is a space direction. Notice that I've labeled the axis along our straightened-out route the x axis, increasing from Detroit ($x = 0$, called A) to Newberry $x = 300$ miles, called I).

as that famous crow flies?



(a)

(b)

Figure 3.3: Two approximate views of the trip in space (a) and spacetime (b).

3.4.4 Spacetime Graphs

Now let's represent the trip in a different way using axes that aren't space "and space" but space and time. In this trip I drove and you fell asleep when we left Detroit and woke up five hours later when we arrived—you should go to bed earlier. You looked at your watch and saw that 5 hours had elapsed and looked at

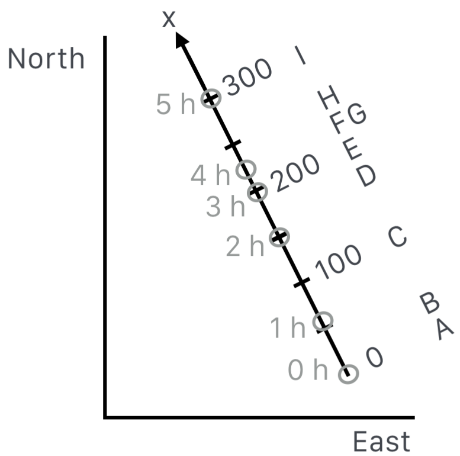


Figure 3.4: A trip from Detroit to Canada "on a napkin." Another, more life-like view of the trip. The open circles are drawn every hour and are not evenly spaced, indicating that the velocity changes. So the Fig. 3.3 b picture is an approximation and simply a global average. The labels refer to places where the speed changed as described later in the narrative.

the odometer and saw that 300 miles had been traversed and you calculated that your trip's speed was $\frac{300}{5} = 60$ mph. You even drew the spacetime diagram for your idea of the trip in Fig. 3.3 b.

First, notice that your simple speed calculation corresponds to the finding the slope of this line. Also notice that you shifted the beginning point to define our distance origin, where $x = 0$. In this case:

$$v = \frac{\Delta x_{AI}}{\Delta t_{AI}} = \frac{300 - 0}{5 - 0} = 60$$

which is a fine thing to have done.

At any point during your nap, the slope of that space-time trajectory would the same as at any other point. So any region in which you calculate the speed by evaluating

$$v = \frac{\Delta x}{\Delta t}$$

over and over gives you the same value. Let's rearrange things slightly and get a little equation...a predictive *model* of our motion:

$$\Delta x = v \Delta t. \tag{3.3}$$

You give me a time, and using the model, I'll tell you where you are. It's a little physics machine.

Remember the equation of a straight line with a slope of m and which passes through the y axis at b (the "intercept")? Sure you do. It's

$$x = mt + b.$$

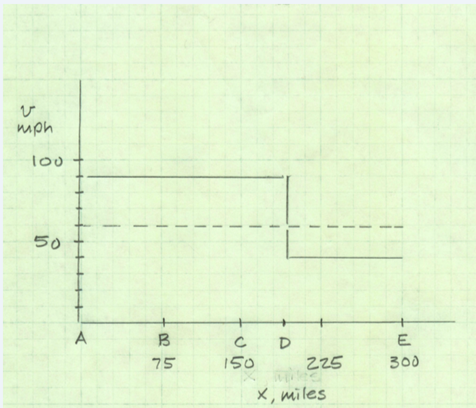
Our spacetime trip plot fits this form with a zero intercept and a slope of v.

Gotta stop to eat once in a while.

Example 3.1

Racing to the bridge.

Question:



Let's suppose that we are able to get out of town early in the morning when the roads are empty. Not that I'd ever do this, but we'll start our trip at a steady speed of 90 mph which we can keep up for 2 hours before traffic slows us down. Then we travel at a slower speed and find that we still arrive 5 hours after we began.

- How far were we able to go before we had to slow down?
- If the overall trip took 5 hours what was our average speed in the second, 3 hour segment?
- What was the overall average speed?

Solution: The first segment, 90 mph for 2 hours, means that we went 180 miles before hitting the brakes. If we traveled for 5 hours in total, then the second segment took 3 hours and the distance left was $300 - 180 = 120$ miles. So the average speed in the second segment is

$$v = \frac{120}{3} = 40 \text{ mph.}$$

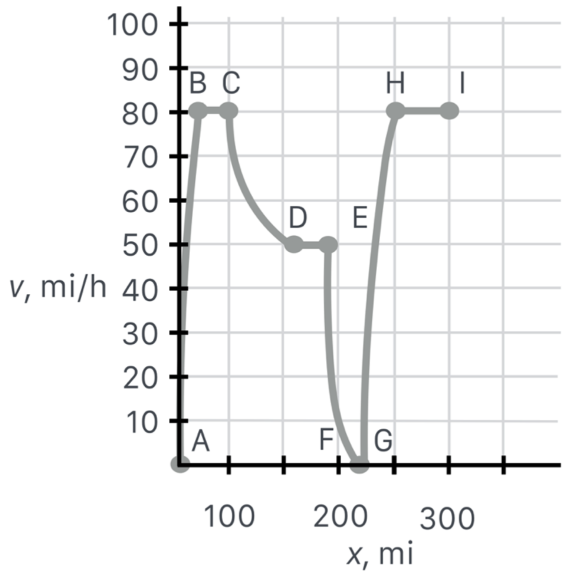
Finally, the overall average speed is still the total distance divided by the time that it took. Still 60 mph. Figure 3.1 shows the speed profile as the solid pair of curves. Notice that this is not the average of the averages. There is more distance covered by the fast trip segment than the slower trip segment.

3.5 Acceleration

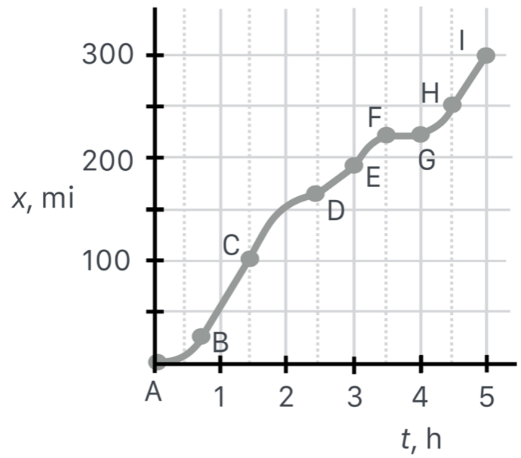
Figure 3.3 b showed that in one interpretation of our trip we never deviated from 60 mph. Can you drive like that? I can't and I didn't! While you were asleep, I sped up, slowed down, and stopped for sushi. You calculated an *average* velocity for the whole trip, which doesn't care what happened between the beginning and the end.

Nothing beats gas station sushi.

Let's be a little more realistic and invent a trip profile shown in Fig. 3.4. . It's like the previous version but in the open circles I've added where we actually are at each hour...if you'd been awake, you'd have realized that we went pretty fast between about hour 1 and hour 2. You were sound asleep when I stopped for that meal at the 4 hour mark. And so on—the constant-time interval hour marks are closely spaced (meaning little ground covered, and so slow) and spaced more apart (meaning lots of ground covered, and so a high speed).



(a)



(b)

Figure 3.5: The trip is further diagrammed showing how the velocity changed as a function of distance (a) and how the distance traveled changed as a function of time (b). The curved lines indicate where acceleration has taken place and will be described below: for a constant acceleration, the velocity changes like the square root of the distance and distance varies as the square of the time interval.

Figure 3.5 tells the whole story. With a little bit of artistic license, Fig. 3.5 a shows the speed at each point along the road:

- We started at A and accelerated to B, from 0 to 80 mph.
- We drove at that speed from B to C for a while and then started to reduce speed—must have seen a highway patrol car.
- So from C to D we slowly reduced our speed for traffic to 50 mph and held it steady until E when I got hungry.
- So at E we started to slow for an exit and stopped for snack at point F.
- When we got back on the road I accelerated back to 80 mph, to H and stayed at that speed until we got to our destination at I and you woke up.

Figure 3.5 b shows that same trip, but now plotting the space time representation. You can find each of the segments and compare them. From B to C, for example we're traveling fast (the $x-t$ slope is steep) while from D to E, we're traveling slower (the slope is less). From F to G, we didn't change our position at all, but time elapsed while I enjoyed my raw fish.

Finally, for completion, Fig. 3.6 shows how our speed changed as a function of time.

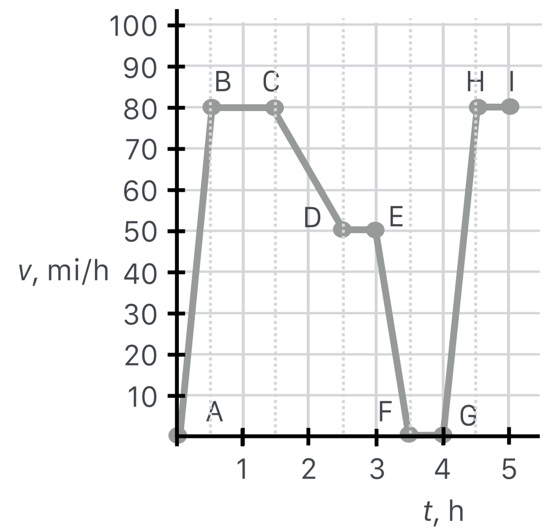


Figure 3.6: A more realistic speed profile for our trip.

Any time a speed changes is an **acceleration**. And just as velocity is the rate at which distance changes, acceleration is the rate at which *velocity* changes. So it's defined similarly:

Definition: Acceleration..
Acceleration is the rate at which velocity changes. If it gets higher, it's called acceleration and if it gets lower, it's called deceleration.

$$a = \frac{\Delta v}{\Delta t}. \tag{3.4}$$

Equation: Acceleration.
$a = \Delta v / \Delta t$

Let's look at our trip before and after the sushi break and concentrate on how the velocity changes in time in Fig. 3.5. Putting the meaning of Δ back in explicitly, let's look at G-H, when we're refreshed from our sushi break:

$$a = \frac{\Delta v}{\Delta t} = \frac{v_H - v_G}{t_H - t_G} = \frac{80-0}{4.5-4} = \frac{80}{0.5} = 160 \text{ mph per hour.}$$

Where the speed was constant, like B to C, the slope of the $v-t$ graph is zero, so there's no acceleration at all, or more correctly, $a = 0$.

How about the interval C-D? First, notice from the graph that it's going to have a negative slope, the opposite of the slope we just calculated. Explicitly, we can evaluate:

$$a = \frac{\Delta v}{\Delta t} = \frac{v_D - v_C}{t_D - t_C} = \frac{50-80}{2.5-1.5} = \frac{-30}{1} = -30 \text{ mph per hour.}$$

What's the significance of the negative sign? Well, for one thing whenever an acceleration is negative it means the object is slowing down. In vector-world, it means that the direction of the acceleration vector is the opposite from the direction of the motion.

The units of speed are easy to remember because we use them every day. They're units of distance divided by units of time, or *per* unit time: miles per hour, feet per second, meters per second (the standard in physics), or kilometers per hour (if you drive in Canada).

But the units of acceleration, while simple to figure out, are a little unusual since we don't use them in everyday life. From Equation 3.4 they would of course be units of *speed* divided by units of time, but since the units of speed are distance per time, then the units of acceleration would be distance per time squared: "meters per second per second"[8] is what we might say out loud, meters/seconds^2 or m/s^2 is

[8] or feet per second per second, or miles per hour per hour

what we'd write. That explains the units from our two intervals as mph per hour.

At any point along the distance-time plot we could determine the "instantaneous speed" by carefully drawing a line that's tangent should the velocity change be varying (not a straight line).[9] Your speedometer is not doing that but rather it's measuring the average speed in time increments that are so small, that to you it looks like you're receiving a report of your speed "right now," but it's not instantaneous in reality.

[9] Remember that a line that just touches a curve perpendicular to it at a point is "tangent."

Let's take a bike ride.

Graphical Kinematics

Let's work out an extended example with realistic numbers. Follow along with me because if you become comfortable with this, what follows will be smooth sailing.[10]

[10] . . . er. . . biking.

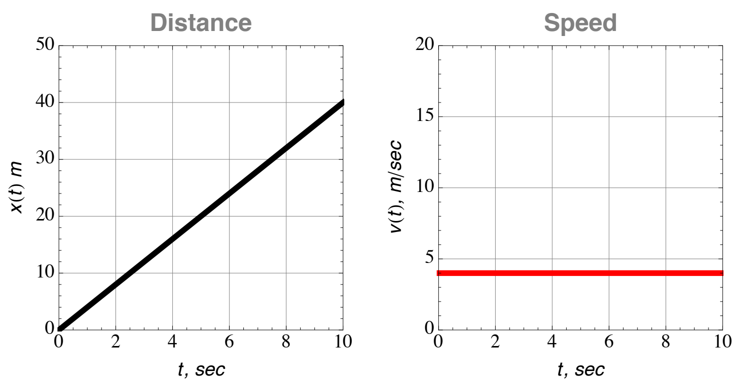


Figure 3.7: The right hand curve shows a constant speed of 4 m/s, holding steady for 10 s. The left hand curve shows the distance that an object will travel at that constant speed as a function of time.

Let's think about a cyclist moving at a constant speed. These two pictures in Fig. 3.7 describe all that you need to know about an object moving at a constant 4 meters per second (about 9 mph—a sprinter's speed, or a relatively slow cyclist's speed). Here we take the initial position to be zero (it could be our house). And, we take the initial time as zero (it could be when we leave). So, to find the distance traveled after 6 seconds by an object moving at that speed we would look at the graph at $t = 6$ s and read about 24 meters. Or, equivalently, we could use the formula and calculate:

$$v = \frac{x}{t}$$

$$x = vt$$

$$x = (4\text{m/s})(6\text{s}) = 24\text{m}.$$

Equation 3.3 has that geometrical meaning we spoke of. Referring to Fig. 3.7, in the first second, in the right plot I see that my speed is 4 m/s and that I traveled about 4 m (left plot). In the next second, my speed is *still* 4 m/s (right plot) and I travel another 4 m (left plot). The *slope* of the quantity x plotted as a function of time t is the constant $\Delta x/\Delta t$ which I pointed out is the *slope* of the curve. That's easy since the speed is constant.

Now, suppose I speed up to pass the cyclist in front of me. My speed increases and I cover ground at a faster rate: during each time interval, I travel more than the time before. Of course now I'm accelerating if I'm going faster during each interval ("decelerating," if I'm going slower). In either case, the state of my motion *is* changing from my previously steady speed. As we'll see, if the state of my motion is changing, there's a force: which I've applied by pumping my legs faster and faster, which is transmitted as a push on the road (Earth) through the rubber tires.

Figure 3.8 shows the predictive power of kinematics when applied to *constantly* accelerated motion. The curve on the right shows a *fixed acceleration* of 2 m/s^2. If this were a car, it's like your foot is on the accelerator in order to maintain a constant force and hence, a constant acceleration on your bike, you need to overcome friction and wind, so maintaining a constant force on the pavement is hard work. This constant acceleration means that the velocity is changing at a constant rate—proportional to time—and that's shown in the middle plot. So, at 6 seconds, your acceleration is 2 m/s^2, and your velocity or speed has increased to 12 m/s from the start. In the mean time the distance you traveled is going up faster: it increases proportionally to the square of the time and this is shown in the left hand plot. If you started

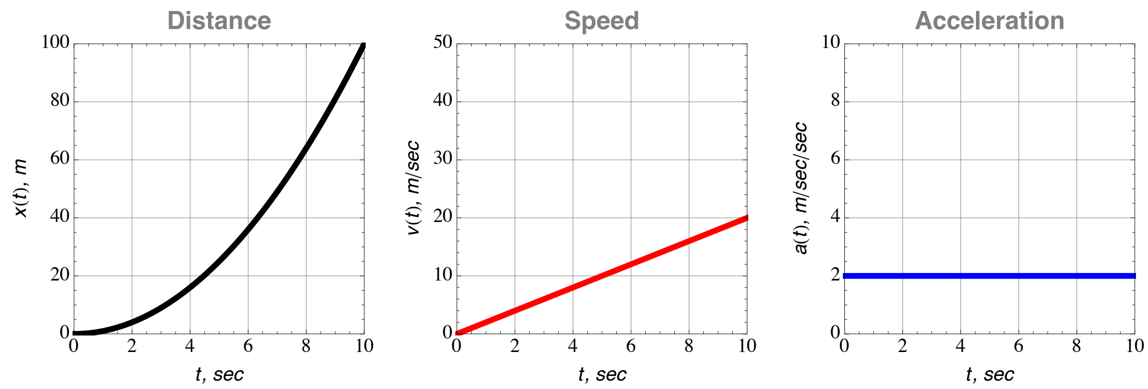


Figure 3.8: This shows the result of a constant acceleration of 2 m/s^2. Each plot is as a function of time: the acceleration, is independent of time—a straight line. The velocity is then a constantly increasing function of time, and the distance increases as the square of time.

from rest, at $x = 0$, by the time 3 seconds have passed, you've traveled 9 meters, and by a total of 6 seconds, you've gone 36 meters, so your distance intervals are increasing with each increment of time.[11]

Finally, thought of geometrically, the slope of the middle plot is constant—the change of speed with respect to time increases at a steady rate, the value of the constant acceleration. But, now the slope of the left hand plot of distance changing in time is *not* constant—at each successive time interval, it's steeper and that's reflected in the fact that the middle plot value changes at each time and so *instantaneously* (the tangent, if you want to be fancy), the speed is different and so the distance increases.[12]

So for the constant acceleration (right hand plot, a = constant), which results in a steadily (linear, $v \propto t$) increasing speed (middle plot), the changing distance covered per unit time increases a lot: the shape of that curve (right plot) is a parabola, $x \propto t^2$:

$$x = \frac{1}{2}at^2. \tag{3.5}$$

Observing this parabolic, or quadratic, increase in distance with respect to time is the smoking gun for constant accelerated motion.

Now go back to our trip up north and in particular, Fig. **??**. We can see that those constantly accelerated regions are parabolas, turned down for slowing down between B and C, and turned up for constantly accelerating between D and E. Again, that deceleration at the end is not a parabola, so it's not a constant deceleration as can be seen in Fig. **??**

3.5.1 Special Kinds of Acceleration

We can categorize acceleration into three separate categories with what we now know.

[11] That's moving right along for a bike—about 25 mph.

[12] Notice that the maximum distance in the right hand figure is 100 m, reached in about 10 seconds. Usain Bolt's world record is 9.58 s.

Definition: *propto*.
In an equation, *propto* means "approximately equal to."

Equation: **Distance for constant acceleration.**
$x = \frac{1}{2}at^2$

Zero Acceleration

No acceleration indicates a constant velocity.

Positive and Negative Acceleration

Positive and Negative accelerations mean very different things here and we'll illustrate this with a simple pair of examples. When you leave the city limits, your speed goes from 35 mph to 55 mph, so the numerator[13] in the definition of acceleration in Eq. 3.4

$$\Delta v = v - v_0 = 55 - 35 = +20 \text{ mph}$$

and this would be a positive acceleration (when you divide by the time it takes), or just "accelerating." But when you come into town from the highway, your speed goes from 55 mph to 35 mph and the numerator in the acceleration equation

$$\Delta v = 35 - 50 = -20 \text{ mph}$$

would be a *negative* acceleration, which is called a *deceleration*. So acceleration is negative or positive by virtue of whether the object is slowing down or speeding up.

[13] Of course the time difference is always positive.

Definition: deceleration.
A negative acceleration which indicates that the speed is getting smaller: slowing down.

Varying Acceleration

One more time for Fig. **??**. At any particular point in time, the value of this graph gives the *instantaneous* speed while the slope at any point gives the instantaneous acceleration. Instantaneous quantities were unimaginable before Isaac Newton, so this would have been a confusing thing even for Galileo. Let's think about what it means to take an average...a simple one, of two quantities.

Suppose we want the average height of two people, one of whom is 5 ft tall and the other, 6 ft tall. You'd calculate:

$$\text{average height } = \frac{5+6}{2} = 5.5 \text{ ft.}$$

The same thinking applies to motion—if, and only if the acceleration is constant. So, in our trip, if at some point we were traveling at 80 mph when we saw the state police car (point B) and 50 mph when we slowed down (point C), what was our average speed during that time? Just like the average of heights, it would be

$$\text{average speed} = \bar{v} = \frac{1}{2}(v + v_0) = \frac{50+80}{2} = 65 \text{ mph,}$$

Definition: **average symbol .**
We represent an average quantity with a bar over the top, like $\bar{A}$

where I've introduced another notation that we'll sometimes use: an average of a quantity is represented with a bar over the top.

The average deceleration during that slow-down from Gaylord, MI at point F is over a long time interval, and you'd watch your speedometer steadily reporting a lower and lower speed. It's really not reporting an instantaneous speed, since the electronics reporting what you read on the speedometer is really calculating an average speed over very small digitally-computed time windows.

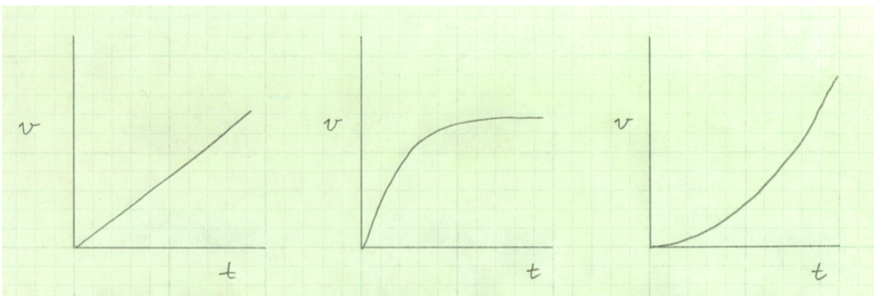
We could calculate over a smaller time interval, and then a smaller one and of course are we really even going precisely at an absolutely constant speed? That is, we could imagine decreasing the time interval all the way to zero, at which point we're doing calculus. We don't need to do that, but we should remember that for any finite time interval, all of the displacement formulas are really averages. But the fact that the slope of the velocity curve in the F to G segment of Fig. **??** is of all different values shows that this is not constantly accelerated or constantly decelerated motion.

Cars are not the only place where we encounter non-constant accelerated motion. How about running? A sprinter comes out of the blocks with a very high change of speed, and so a very high acceleration. As she gets into her stride and become more erect, her effort must fight against wind resistance and besides, her internal quick expenditure of energy would be hard to maintain, so there's a *reduction* of acceleration. Note this doesn't mean a reduction in velocity, just that the velocity increases at a slower rate as the race goes on. Nor does it mean a deceleration...she's still going faster, just at a lesser rate than before.

You Do It 3.2. Sprinter Graph

Which graph of velocity versus time for a sprinter matches the description of a runner from the text?

or copy the solution



(a) (b) (c)

The figure that you just chose shows the increase of speed of a sprinter: start fast and then level off in almost constant speeds. Figure 3.9 shows how the designers in *Madden Football '11* modeled the rate of speed of one of the players in the game.

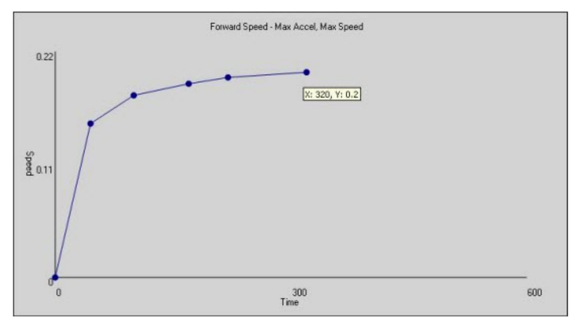


Figure 3.9: Modeling a running back in Madden NFL11 (`http://www.operationsports.com/ncaa/utopia/topic/69752-madden-nfl-11-locomotion/`

Constant Acceleration

From the above figures and our experience it's plain that everyday motion would likely involve accelerations that vary in time—like the slope of the realistic sprinter graph above— changes almost throughout the run, starting out very high and becoming smaller. But historically and practically there is a special, constant, acceleration.

3.6 Galileo and Free Fall

Galileo is revered in physics for essentially creating two modern sciences (physics and astronomy), and for enunciating the principles of how science should work that are still relevant today. Very early on he recognized that the unnatural/natural motion notions[14] of Aristotle were wrong. For example, Aristotle insisted that two objects made of the same material but of different weights would not hit the ground at the same time if dropped. The heavier one would be more "eager" to go to its natural place (the Earth's center) than the light one. Or an object that was more earthy (like a rock) would fall faster than an object that's less earthy (say a stick) even if they weighed the same...again, the more earthy the object, the more anxiously it tended towards the center of the Earth which was...well, totally earthy. Galileo simply watched that hail stones of differing sizes seemed to hit the ground at nearly the same time which was not consistent with the Aristotelian ideas, and that caused him to "think different."

[14] I actually wrote "motion notions"?

Wait. *This is trivial! You mean nobody else noticed this in nearly 2000 years?*

Glad you asked. *You're right. People did question Aristotle's concepts about motion but not seriously until the beginnings of they medieval university system in Europe. Everyone knew that there were problems with Aristotelian science but the overbearing importance of Aristotle's whole philosophical package and its merging with Catholic theology carried out by St. Thomas Aquinas in the 13th century, meant that questioning it was a hard sell. Indeed, to pick away at just the motion part of his philosophy would weaken other pillars of his system—like the famous Four Causes—and that would have been hard to bear.*

It had "been in the air" for a century that the speeds of falling objects increase either in proportion to increasing *distance* or increasing *time*. All one had to do was watch water drips from a roof...the distances that they travel in a given time are much longer near the ground than when they first leave the edge. So they are accelerating, but which way? Long before Galileo, smart philosophers at Paris and Oxford realized that if, during equal time intervals, the distance of a falling object increased like the squares of the time interval...then the speed must increase proportionally with time, This was pretty sophisticated geometrical reasoning for people without algebra, graphs, or graphical representation of functions. Remember, from what we know from the above discussion, this quadratic time dependence told him that gravity produced a constant acceleration.

Galileo first started to think differently about motion when he was a professor at Pisa. That's where he was supposed to have dropped a wooden and iron ball from the Leaning Tower, although we know that he couldn't have done it the way the story went. His unpublished pamphlet on motion was a mixture of old language and his attempts at a new one, and so was not quite there yet.

3.6.1 Pisa To Padua

His salary was abysmally low and he was clearly ready for a more lucrative position. So after three years in Pisa, when he was offered a job at the University of Padua he took it and stayed for 18 years until 1610. Not only was his salary higher, the Venetian Republic[15] was a much more independent region of Italy. Indeed, unlike Florence in Tuscany where connections to Rome and the Papacy were long-standing, Venice was often in the Papal Doghouse for its independent ways.[16] Here Galileo was a popular professor, he created tools for military use that he was able to sell, including a manual. He settled into a common-law relationship with a woman with whom he had three children.[17] But money was always problematic and they even took in students for tutoring and rent into their crowded home.

While Galileo was in Padua, he resumed some of his earlier experiments with moving objects. His original ideas were unformed but he reworked them as he shed his Aristotelian influences. There he had a workshop and eventually hired a toolmaker.

[16] Popes railed against Venice, sometimes excommunicating everyone, sometimes even going to war, but Venice was the most powerful naval power facing the troublesome East and so Rome needed Venice periodically.

[17] For a professor to remain single was not unusual and Galileo was a frequent visitor to Venice and its literary and art community. This was a sophisticated society with thousands of highly literate courtesans. While little is known about the early relationship between Galileo and Marina di Gamba, some have speculated that she might have been such a professional companion. Galileo stayed with her for almost a decade which might suggest that they were intellectually compatible—indeed, given his personality, it would have been surprising were he to spend time with someone not up to his conversational standard. In any case, Gamba moved to Padua to live with him where they had three children, two girls and a boy. (Birth records do not refer to a father.) When he went back to Florence in 1610, he took Livia and Virginia with him (aged 9 and 10) but left four year old Vincenzio with Gamba who remained in Padua and eventually married. He installed the girls in convents, where they would cost no dowry, and brought his son to Florence. Eventually, Vincenzo became a musician and was partially supported by the Pope, as a deference to Galileo. Virginia, later Sister Maria Celeste, relied on him for his influence in supporting her convent. She died in 1633, the year he entered house arrest as an old man. He was devastated.

3.6.2 The Pendulum

A pendulum is a trivial toy, yet it encompasses many key physics ideas about motion and force. Galileo, Huygens, and Newton each made many pendulums and did extensive experiments with them. Because of its immediate simplicity, quantitative measurements are easily done with modest tools. It will come up over and over! Figure 3.10 is a picture of a simple pendulum which I'll refer to periodically over the next few chapters.

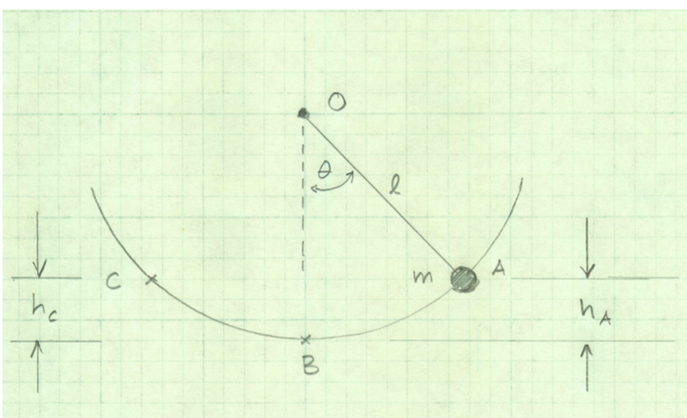
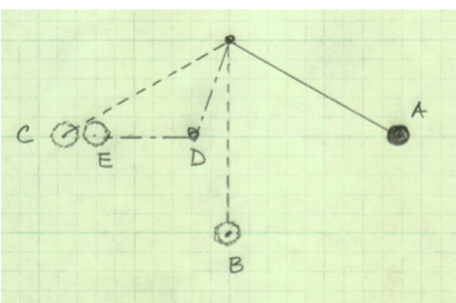


Figure 3.10: A simple pendulum which oscillates in a vertical plane (that is, gravity acts down). There are three points indicated on the circular path that it makes: A is the point of release, a height h_A from the lowest point, B. C is the point of the highest elevation which it obtains on the other side and that height is labeled as h_C. The pendulum bob has a mass of m and the length of the string connecting it to its point of suspension, O, is ℓ. The angle that the string makes with the vertical is θ.

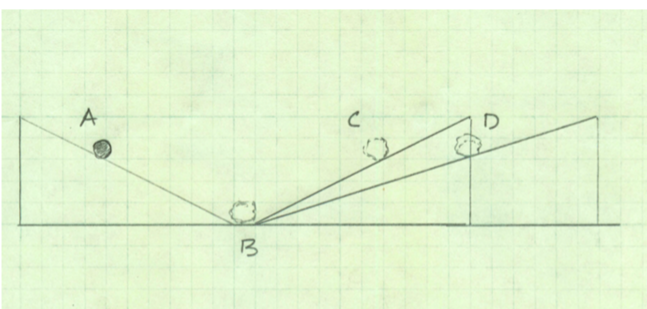
Among the surprising facts about a pendulum, no matter where you release the bob, it takes the same amount of time to make a complete back and forth. That is in Fig. 3.10, $h_C = h_A$. Except that's not true. Only for very small values of θ (which is the same as making h_A relatively small) does this approximately work. For Galileo, the differences were small enough that he presumed that the pendulum's motion is isochronous, that is, that it swung back and forth in equal times.

Galileo first studied the pendulum in Pisa. The initial striking result that he noted was that the bob will make a complete arc, following a circle, sweeping through point B and come to rest to essentially the same height from which it started, at C. He pushed against this result by putting a peg in the way at D and the bob still returned to the same height, now at E as in Fig. 3.11a. The second thing he noticed was that the time that it takes to make one compete oscillation from the higher point *is the same* as when it was started from the lower position. In fact, he found that the *period of oscillation*—how long for a back and forth trip—only depends on the length of the cord. Not the mass of the bob and not the material of the bob.

This led him to make the leap from pendulums to motion in general and in Padua he began to experiment with inclined planes. Suppose you have two inclined planes opposite one another, like in Fig. 3.11b and you start a ball rolling from the top of the left-hand inclined plane at A, it will fall to B and then rise to pretty nearly the same height on the right-hand one at C. Like the pendulum.



(a)



(b)

Figure 3.11: The right-hand figure shows various positions on a pendulum bob which is released from rest at point A. At B it's moving to the left and comes to rest at point C. Point D is a peg that's inserted in the path of the string and causes a sharp change in the trajectory...but it sill comes back to the same height, now at E. The left-hand figure shows a set of inclined planes and positions of a ball released from rest at point A, which passes through B on its way back up the first plane, rising to the level at which it was released at point C. If the slope of the plane is decreased, the ball still rises to the same height, now at D...but further along the incline in order to get back to that height.

Now, make the angle of the right hand plane smaller so that the ball has further to go to reach the same height, now at D. He reasoned, it still would and showed that. Now, here's the genius part: suppose you lower the right-hand plane more and more until it's flat. What does the ball do now? Galileo said it would roll forever. He then packaged that reasoning into a statement that Newton appropriated nearly 60 years later: that an object that's started in motion on a flat plane will continue forever, unless something stops it. No pushing involved. Once it's started, it goes. Notice how far behind Aristotle looks in Galileo's rear view mirror! He's not worried about natural or unnatural motion and he's not concerned himself with *why* the motion continues. Rather he's asking different questions...*How* questions rather than *Why* questions. Aristotle's students would start with a philosophical prejudice and interpret what they saw in accordance with that philosophical system. Galileo threw all of that aside and observed nature without bias or preconceived notions. So if you release the ball in Fig. 3.11 *where would it go?* is his question. Not, *why* would it go. And he'd set up the circumstance and observe it—no Aristotelian would do an experiment. They would think about it in the specific context of the philosophy and passively observe.

The other important thing is that Galileo was fully aware that there were a whole host of impediments to a pendulum going back and forth forever, or even coming back to the original point after one swing. In fact—*and this is important*—he imagined that the real rules of pendulum motion were those of the zero friction limit and that our actual pendula are incremental modifications to the true situation through successive addition of friction, air resistance, stretching of the string, etc. Likewise, with the rolling ball struggling up the right hand incline there are overlays of real impediments to the otherwise perfect motion. The bob nor the balls really, exactly came back to their starting points...just really close. Plus nearly every event would be slightly different. But rather than dwell on the differences, Galileo was the first to decide that there was *something that was the same in every event.*

3.6.3 Free Fall

The big question for Galileo (and many others) was to explain the behavior of falling objects. What rules govern how fast an object falls. Aristotle said that an objects "nature" was the cause and that its determined how fast. Clocks didn't yet exist, and so even Galileo couldn't drop something and measure the time for it to fall, even from very high places.[18]

[18] His observations of the timing of pendulums' periods led to Christan Huygens' invention of the pendulum clock a generation later.

What Did He Do?

But he invented a trick...a way to dilute gravity so that he could make timing measurements using 16th century tools. Suppose instead of dropping a ball, you rolled it down an incline from the same height. He hypothesized that whatever pull a ball felt in free-fall would still be there in the slow descent down an incline. If he made the slope so shallow that the ball would roll ("fall") slowly enough, he could measure it and sneak up up on the rate of falling directly. He found that inclines of only two or three degrees were ideal. Figure 3.12 shows the setup.

Three different distances are indicated for the ball. The ball starts from point A and rolls down the plane to point D along the path ℓ taking some time to get there. We can ask about the separate "motions" in the horizontal and vertical directions since the path is in the plane. So along the x axis, the ball goes from point A to point B and it "falls" in the vertical direction along the y axis from point A to point C. A little bit of trigonometry shows that the rate that the ball travels along ℓ is the same as the rate to fall along the y axis. So all he needed to do was measure how far the ball rolls along ℓ in each increment of time. That same rate would be the free-fall rate. Pretty clever.

Galileo was good with his hands, and armed with the understanding that he needed to reduce the effects of friction and other extra impediments to determining the real rules, he built inclines with finely machined grooves to minimize the effects of friction[19] and then made elaborate schemes to measure the time that it took for an object to "fall" as it rolled slowly down the incline. He used his pulse. When that wasn't sufficiently regular, he hired a string quartet to play at an even meter. He then built a "water clock" that slowly dripped at a regular rate and it was with that, that he was able to see how far a ball would roll during each "tick" of his water clock.

[19] I'm being a little coy here. "Rolling" only happens when friction is at work between the ball and the surface. Think about it. If there were no friction, the ball would slip and it would not translate. What he reduced was irregularities in the slot that the ball rolled in.

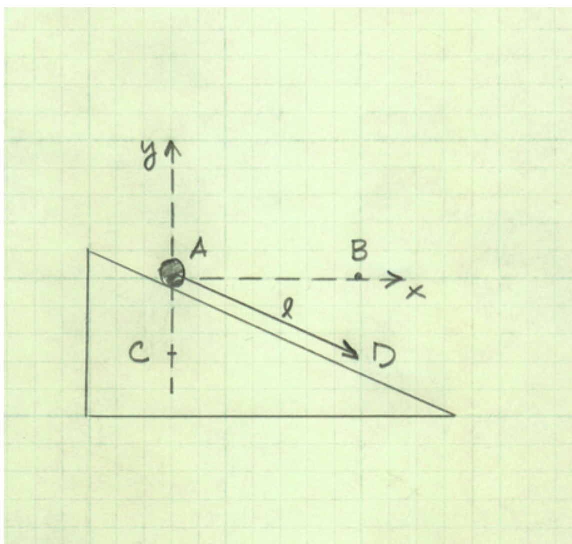


Figure 3.12: Various positions of a ball released at A measured along the incline and separately along the horizontal and vertical directions. See the text for the story.

Galileo kept careful records of his experiments and these have been preserved. Fig. 3.13 shows the aha! moment in one of his pages where he's taking his measurements, scribbling around the sheet, and keeping his records in the upper left hand corner. The late Stillman Drake from the University of Minnesota, the preeminent Galileo scholar, has studied these notebooks and on this particular page, he noticed that the ink used in the column with the squares was different from the ink in the actual data-taking. In fact that ink was consistent with pages that came from entries many days later. Clearly Galileo had taken the data, then pondered the results and realized later that his measurements fit the square-time rule and came back to the page to add them in. Imagine what his emotional state must have been when he realized what he'd done!

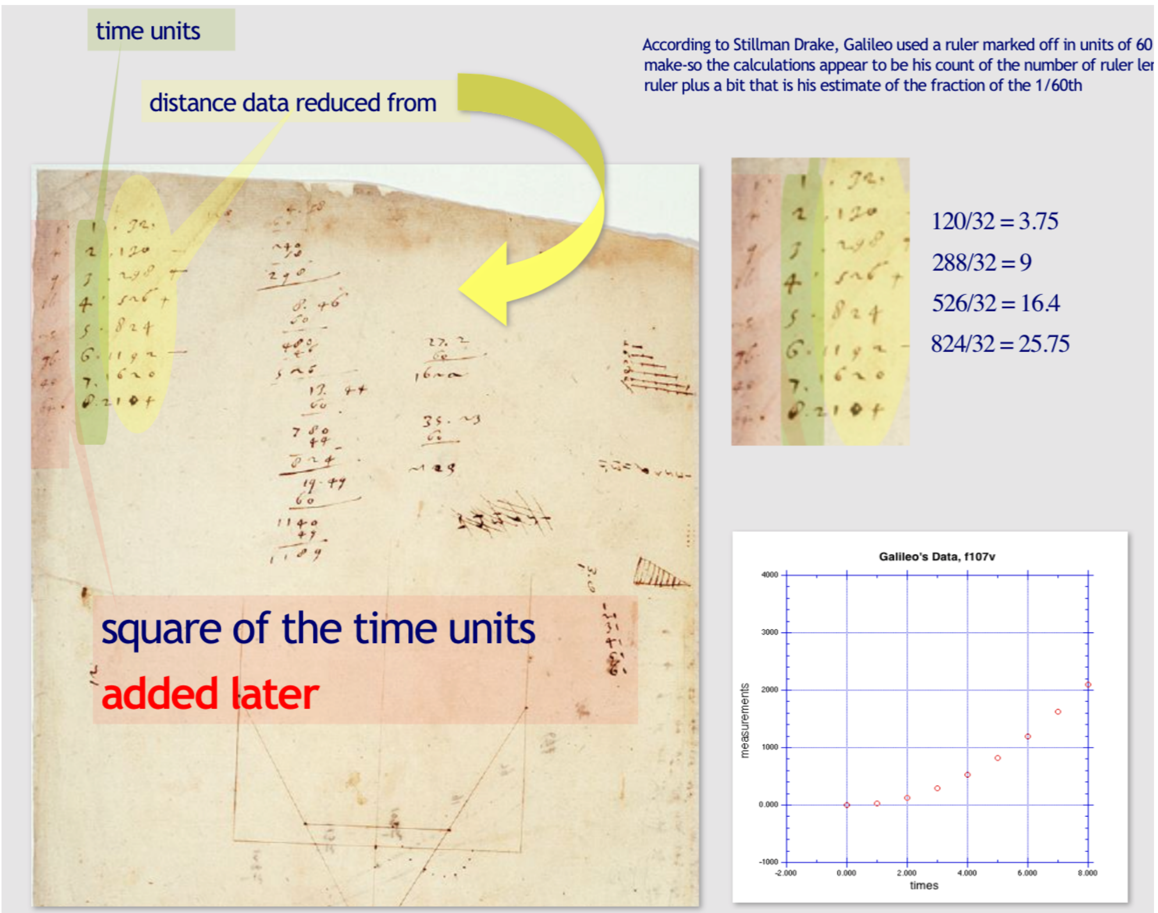


Figure 3.13: Galileo's notes of his inclined plane data, including his "aha moment" where he returns to it after a few days and adds the time-squared calculation.

Little g

What he was discovering was the shape of the left graph in Fig. 3.8 (plotted for his data in Fig. 3.13 as the white inset) that the distance is quadratic in the time traveled, like we saw in Eq. 3.5. Quadratic? He knew that meant a constant acceleration is at work for his rolling objects. He then went one step further. By measuring the results with balls of differing materials and weights he found that they all showed the same result.

Now the big leap...so to speak. He reasoned that the same would be the result of dropping objects vertically. Casting aside 1,500 years of Aristotle's insistence that the heavier, most "earthy" objects would fall much faster, Galileo insisted that all objects would fall with the same acceleration and that the distances would increase like the square of the time.

Today we write:

$$x = \frac{1}{2} g t^2 \tag{3.6}$$

where we give this special, constant acceleration a name, the "acceleration of gravity," g, or "little g"[20] which is roughly 32 ft/sec/sec, or 9.8 m/s^2 which was first measured by Isaac Newton a generation later...using pendulums.

[20] Is it called "g" because of "Galileo"? What do you think?

Objects near the Earth's surface fall with essentially constant acceleration. *Key Observation 2*

Constant of nature: Acceleration due to Earth's gravity at its surface.
$g = 9.8$ m/s^2

He then made yet another important leap: In practice, a heavier object might hit the ground slightly before a lighter one, but he correctly reasoned that this was because the effect of the resistance of the air would be a larger retarding force on the little ball than on the heavier one.[21] Now he's doing physics. He was after Nature's "real rule" which was hidden behind the apparent motion, affected by the air. He was exploring what's the same about falling objects, not what was different about them.

[21] Test this yourself. Take two identical pieces of paper and crumple one of them into a tight ball and the other only slightly. Drop them side by side and what happens? The air resistance matters a lot.

Air Resistance

Little g is a large acceleration, but raindrops don't usually bruise us and hail stones don't usually kill. Were there not for air resistance, nice spring showers would be dangerous as rain drops fall a long way! It turns out that things dropped from high up reach a speed in which the viscous drag of air friction pushes back with the same force that the Earth pulls, causing a falling object to reach equilibrium. The result is that the speed of falling becomes constant to what's now called the Terminal Velocity, which depends on an object's size.

For example, a regulation major league baseball will reach terminal velocity in about two and a half seconds after falling about 100 ft. At that point, its speed will be just about 60 mph and will stay at that value. For reference, the "Green Monster" left field wall in Fenway Park is about 40 feet (11 m) high, so you can see that high fly balls would be high enough to reach terminal velocity on their way down.[22]

[22] There have been stunts performed with baseballs dropped from the Washington Monument, skyscrapers, and balloons for players to catch. The speed that a baseball would achieve is not much more than that experienced by a catcher on a high school baseball team. However, famously a professional catcher knocked himself out missing a ball dropped from a blimp.

The saga that he dropped balls from the Leaning Tower of Pisa while a faculty member there, surrounded by all of the members of the student and faculty body was told by one of his ardent followers after Galileo's death. There is no evidence that this public event ever happened, and Galileo himself never described doing such an experiment. This same disciple also told a story that as a child he sat in the Cathedral of Pisa and while bored during mass timed the candelabra overhead and from that experience reasoned his pendulum rule later. The tale had young Galileo making these measurements with a candelabra that were not installed in the cathedral until many years after the event was said to have taken place. So there's much to be wary of in Galileo-lore as told by his young admirers.

While Galileo was on the faculty at Padua, he did other experiments with motion which lay dormant until in 1638 he wrote the second of his great books, *Discourses and Mathematical Demonstrations Relating to Two New Sciences.* He finished this work while in house arrest outside of Florence as an old, sick man. How he got into trouble is a story that we'll leave for our discussion of early astronomy.

The Equations of Constant Acceleration

Acceleration at g is called... well, "gees." And "pulling gees" is a measure of acceleration that fighter pilots must contend with: in order to not black out, humans can tolerate accelerations up to around 6 g's, or $6 \times 9.8\ \mathrm{m/s^2}$ = about $60\ \mathrm{m/s^2}$. That's moving right along as you will see from the Porsche-experience below.

The plots for motion had their origins with Galileo, but their algebraic form waited for the 17th and 18th Centuries to become standard. Our formulas so far can be summarized here: [23]

[23] I've added one for constant acceleration that is easily derived from the others, but I've not done so here. Trust me. I've also generalized the equations that I did use to include the possibility that the initial conditions might not be when $x_0 = 0$ or $v_0 = 0$.

$$\text{(page 82)} \quad x_{\text{ave}} = \frac{\Delta x}{\Delta t} = \bar{x} \tag{3.7}$$

$$\text{(page 91)} \quad v = v_0 + at \tag{3.8}$$

$$\text{(page 94)} \quad v_{\text{ave}} = \frac{1}{2}(v + v_0) = \bar{v} \tag{3.9}$$

$$\text{(page 92)} \quad x = x_0 + v_0 t + \frac{1}{2}at^2 \tag{3.10}$$

$$v^2 = v_0^2 + 2ax \tag{3.11}$$

Obviously, if the motion starts from rest, then the initial velocity is zero ($v_0 = 0$) and we can often easily define our coordinate system so that the initial starting point is set to zero ($x_0 = 0$).

Let's work out what the acceleration would be for cars that you may or may not drive. How long does it take your car to go from 0 to 60 mph? If you own, say, a Mitsubishi Mirage ES, it will take you around 12 seconds to get to sixty. If you own a Porsche 911 Carrera S, it will take you closer to 4 seconds.

Example 3.1

The Slow Ride

Question : What's the acceleration of the Mitsubishi in m/s^2?

Solution:
From the above equations, we can see that if we start from rest, then $v_0 = 0$ and we can calculate:

$$v_M = a_M t_M$$
$$a_M = \frac{v_M}{t_M}$$

But we need metric units, so we can refer to Fig. 2.4 and see that 60 mph is just about 26 m/s. So for 12 seconds, we get

$$a_M = \frac{v_M}{t_M}$$
$$a_M = \frac{26}{12} \approx 2 \text{ m/s}^2$$

Not exactly like falling off a log...or falling off of anything for that matter, since it's almost 5 times less acceleration than 1 g! If you black out while speeding up with your Mitsubishi, it's not because of acceleration.

You Do It 3.3. Mitsubishi

or copy the solution

What is the acceleration of the Mitsubiishi and the Porsche respectively in m/s^2. And what fraction is that acceleration relative to that of gravity, $g = 9.8\text{m/s}^2$?

These two states of motion—constant velocity and non-constant velocity—are very different. We'll see in our discussion of Relativity just how different, but let me let the cat out of the bag right here. Have you ever been in a train or a car where you're relatively enclosed and can only see a car or train next to you and not the earth? If the adjacent vehicle starts to move you can be suddenly confused: who moved? You might actually not know! I find that unsettling, and my brain quickly tries to figure it out. This is an inherent feature of constant velocity motion: there is no "right" answer as the same unsettled feeling would be felt by a passenger on the other train. Both are equivalent situations relative to one another. There. I've said the word "relative" and you now have the hint of only a part of what's interesting about Einstein's theory of *relativity*.

3.6.4 Going In the Right Direction!

Now we need to add one more piece to the modern story. As we saw in the Tools chapter, distances—displacements—are vector quantities. Because distance is in the definition of velocity, and since distance is a vector, so is velocity. Since velocity is a vector and acceleration is defined in terms of it? Yes, acceleration is a vector too.

Velocity Is a Vector

Here's our first physics-vector. I've been loose in the use of the words speed and velocity. In fact, there's a difference: velocity is a vector, $\mathbf{v}$, and so it includes a magnitude and a direction, and the magnitude of velocity is the speed. Now the definition should really be:

$$\mathbf{v} = \frac{\Delta \mathbf{x}}{\Delta t}$$

Notice, that I've implicitly included distance ($\mathbf{x}$) as a vector quantity, but not time. North and east are different directions and $\mathbf{x}$ makes that clear. Like mass, temperature, and many other quantities, time is not a vector. At least, not yet!

So 60 mph east is not the same velocity as 60 mph north. The speeds are the same, but the velocities are different. And, certainly if you're trying to go north, you don't want to deploy a *velocity* that points east. So both the magnitude (speed, here) and the direction are required to specify a velocity. We simply draw an arrow, the direction of which points in the direction of travel and the length of which is defined by some scale of speed magnitudes.

Now when you're walking I want for you to invent your own speed scale—how many inches equals 1 mph of speed— and then imagine this arrow sticking out of your chest. As you speed up, the arrow gets longer and as you slow down, the arrow shrinks. If you turn left, the arrow turns with you. Everywhere you go, you imagine your very own velocity arrow preceding you. You should also imagine arrows coming out of everyone you see, people on bicycles would have longer arrows and cars would have even longer ones. When they come to a stop sign? The arrow disappears. Keep this in mind as we move on to those length-changing arrows.

Wait. *That's silly. But now I'm seeing arrows everywhere.*

Glad you asked. :)

Acceleration Is A Vector

Since velocity is a vector and acceleration is defined in terms of velocity, it too is a vector:

$$\mathbf{a} = \frac{\Delta \mathbf{v}}{\Delta t}.$$

While the little arrow that represents your personal velocity is always point out in front of you, its direction seems rather arbitrary. It goes where you go, but there's nothing special about the direction. But that's not really the case since our movement is inside of a coordinate system of streets, sidewalks, hallways, etc. and they all have relative directions.

So let's for a minute remember the obvious...if we take our coordinate system to be (arbitrarily) that to go east is to go in a positive direction and west, a negative direction then you could describe your vector that way. If you're walking east, then you could say your velocity is 2 mph, east, or you could say that your velocity is +2 mph. If you're walking to the west, your velocity would be -2 mph. The direction and the sign are mingled. Because we will work in one dimension most of the time, this dual-role for a sign will matter but hopefully be pretty easy.[24]

[24] The alternative is to use a full-on vector notation which is more complication than we will need in QS&BB.

This is important as we consider the direction of an acceleration. Suppose your speed is 60 mph and after a certain time—let's say 2 hours—your speed has increased to 70 mph. Remembering our ordering in the Δ notation of (now - before), then then the magnitude of your acceleration would be:

$$a = \frac{\Delta v}{\Delta t} = \frac{70-60}{2} = 5 \text{ miles per hour}^2$$

Now what if your original speed is reduced because you applied the brakes from 70 mph to 50 mph. Then we'd have:

$$a = \frac{\Delta v}{\Delta t} = \frac{50-70}{2} = -10 \text{ miles per hour}^2$$

What are we to make of that negative sign? Of course it means we're slowing down, and we learned that the term for that is deceleration. But that pesky sign change between accelerate and decelerate is again related to our coordinate system. If we continue to take east as positive, and have both of our circumstances—the acceleration to 70 mph and the deceleration from 70 mph—be easterly motion, then that algebraic negative sign that came from the calculation plays a geometrical role.

Now you have two vectors point out of your car's hood. First we have the velocity vector, which is always positive as long as we're traveling east, albeit getting longer in the one case and shorter in the other. Then we have also an acceleration vector that points to the east when we accelerate (corresponding to the change of the speed being positive) but points west when we decelerate. That's what the negative sign does, it changes the direction of the acceleration vector. Let's see how this works.

What Goes Up, Must Come Down

Armed with Galileo's discovery and our notion of vectors, the acceleration felt by all falling objects near the Earth is constant, we can look at some examples. Let's drop something from the Leaning Tower, in honor of fictional accounts of famous scientists.

Mr. Google tells me that the height of the Leaning Tower is 183 ft. By now, you know how to convert that to meters and it's 57 m. The first thing to realize is that a baseball dropped from the Leaning Tower at this height would likely reach terminal velocity. But let's ignore that for a moment—we will pretend that the atmosphere doesn't exist—and look at Fig. 3.2 for the arrangement.

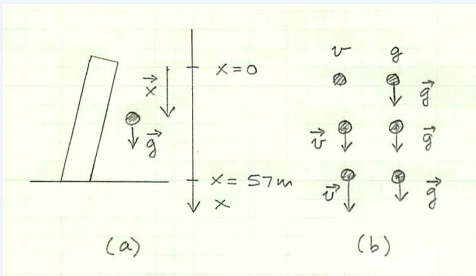
Wait. *Why do you keep doing that? You take a perfectly normal situation and then replace it with a pretend situation and explain that instead!*

Glad you asked. *This is what Galileo taught us: We try to understand the basic rules by which Nature operates and must work very hard to understand why our measurements might be corrupted by effects that mask the details. Sometimes what we can calculate are only the simple situations and we have to add in the complications, like friction or air resistance as small effects. Sometimes we can't calculate these effects perfectly so we have to build...yes, models...to approximate them. But the overall goal is those hidden, underlying regularities that dominate.*

Example 3.2

That Leaning Tower

Question:



Drop an apple from the Leaning Tower of Pisa, all 57 meters.

- Draw a picture that shows the rock on its way down twice...near the top and near the ground: draw a representation of the distance vector, the velocity vector, and the acceleration vector on the rock.
- How fast is a rock moving just before it hits the ground?
- How long does it take to hit the ground?

Solution:

Always draw a picture. We'll take the positive x axis to be pointing down, in the direction of $\vec{g}$ and the direction of v, so both are positive and increasing in magnitude. Figure 3.2 shows the situation and the various vectors. The distance axis is on the left side. On the right side of the figure are the kinematical vectors. The velocity vector gets longer as the apple falls and the acceleration vector stays the same. Notice that there is an acceleration at the top.

We can use various of the equations from the set in Eqs. 3.7. To find the time, for example, we can use $x = x_0 + v_0 t + \frac{1}{2} a t^2$. We defined our coordinate system so that $x_0 = 0$ at the top of the tower. And since it's dropped from rest without being thrown down, $v_0 = 0$. And we can then calculate the time when $x = 57$ m:

$$x = \frac{1}{2} g t^2$$

$$2x/g = t^2 \text{ so } t = \sqrt{2x/g} = \sqrt{\frac{(2)(57)}{9.8}} = 3.4 \text{ s} \tag{3.12}$$

In order to calculate the speed at the bottom, we could use a couple of equations. We could use:

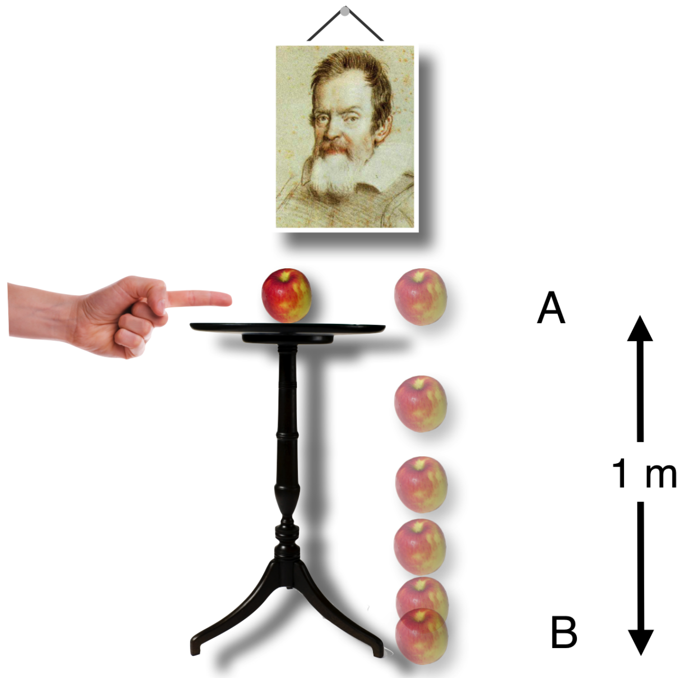
$$v^2 = v_0^2 + 2ax = 0 + 2(9.8)(57) = 1117$$
$$v = \sqrt{1117} = 33.4 \text{ m/s}$$

Now, you do it. Closer to home.

You Do It 3.4. Apple Motion

or copy the solution

Instead of dropping something from the Leaning Tower, drop an apple from a table to the floor, 1 meter down. What's its speed just before it hits? Calculate it in m/s and pretend that $g = 10$ m/s^2...it's close!



In fact, while the above illustrations are elementary, they lead us to practical and interesting questions. For example, suppose I throw a ball straight up. We could analyze the velocity vector on the way up and the way down and the acceleration vector during the same trip. Galileo's assertion that the acceleration is constant for all objects, we interpret as the acceleration vector of constant length pointing *down* throughout the path, up and down. What about at the very top? Is it accelerating? Yes. Is it moving? No.

One of the biggest indignities that Aristotle's motion suffered was trying to explain projectile motion. Galileo has that covered too.

3.7 Projectiles

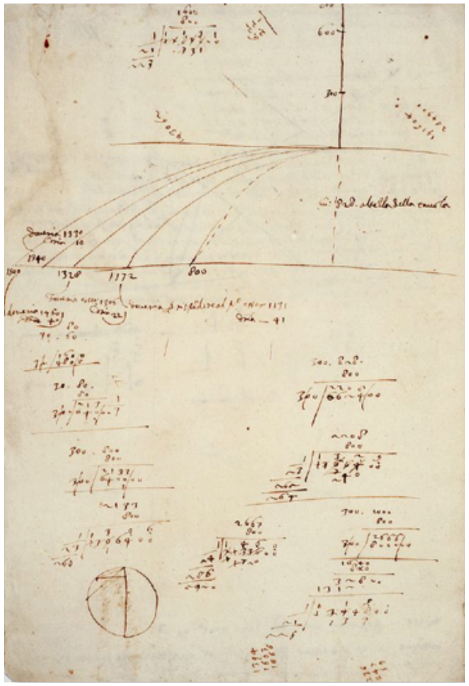


Figure 3.14: Galileo's notes showing his table-floor measurements.

Although we've been dealing with 1-dimensional motion, Galileo wasn't done when he figured out that everything falls at the same, constant acceleration of gravity. He also tackled the other problem that Aristotle messed up: throwing something. With an understanding that objects would happily move at a constant speed without being pushed (his two inclined planes) and that objects fall towards Earth with a common acceleration, he had the genius idea to embed these two different motions together into the motion of a single, thrown object. Let's follow his experiment by looking at his notes in Fig. 3.14.

What you see are various measurements of rolling a ball off the surface of a table and marking where it lands according to how fast it was going. How did he repeat measurements so that each attempt would be the same speed upon lift-off? He rolled them from an inclined plane where he could dial-up the speed he wanted by how far up the plane he let go of the ball as shown in Fig.~3.15. Clever, no?

So let's think like he did. If the speed of the ball at B is v_x, then what's the speed of the ball just at the edge of the table, C? v_x. Now the ball acquires two separate motions that are combined: the ball *continues* to have the horizontal speed, v_x. But as it falls, it starts to acquire a *new vertical speed*, accelerated down by g. The two motions are separate, but together! So throughout the trajectory, the ball's *horizontal speed remains* v_x. Nothing has happened to change that. The ball's *vertical speed increases* as the ball "falls" along this curved path.

Now here's the neat thing. Suppose at the same point where the ball leaves the table, another ball is allowed to just drop from that point, say at point C in Fig. 3.15 so that it falls (under the influence of gravity!) to point E. This one has only one kind of motion, vertical and accelerated. In fact, it's accelerated by the same amount as the first one and from the same vertical height, at the same time. So which one reaches the floor first?[25] They both reach the floor at the same time.

This bundling up of two separate motions into one object was pure genius. Nobody had ever conceived of that sort of thing. By analyzing the landing point and some sophisticated solid geometry, he was able to extrapolate to perfect conditions and assert that the trajectory that the ball followed was that of a parabola.

Now let's take this to the act of actually throwing something. Refer to Fig.~3.16.

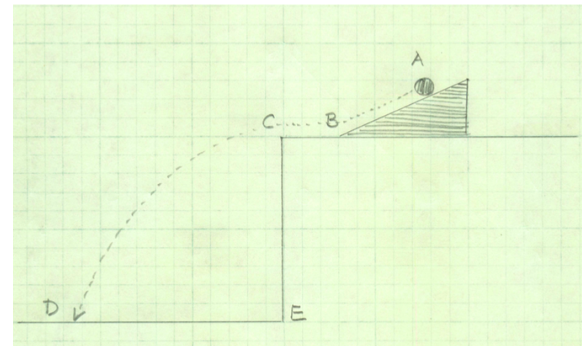


Figure 3.15: This is the arrangement for Galileo's projectile experiment. He released a ball at point A on an inclined plane so for each trial, every ball at B would have the same speed, which would be the speed upon "launch" off the edge of the table at C. The ball then flies through the air and lands at point D.

[25] This is a "Who's buried in Grant's Tomb question.

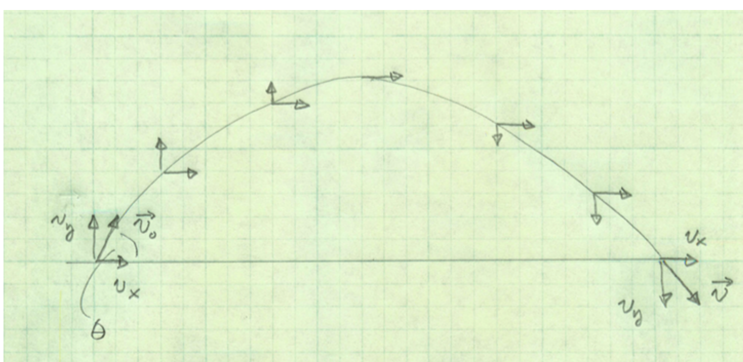


Figure 3.16: This shows the path of a projectile—like throwing a ball—launched at the left and landing on the right. The velocity is shown twice at the beginning and at the end. $\mathbf{v_0}$ is the total velocity when the ball is first thrown, a little out and a little more up. This velocity is also drawn on top in terms of its x and y components. This is drawn again for the landing point. At each point along the way, the components alone are shown.

Here I've put on the diagram the separate components of the ball's velocity, horizontal and vertical. Throughout the path the horizontal velocity that the thrower originally provided (because she didn't throw the ball straight up, but at an angle relative to the horizontal) is unchanged.

But the vertical component of the velocity points up until the top. Then, down on the way, well, down. If you've ever watched a towering home run, you know that the baseball does not follow a perfect parabola. Rather it falls to Earth relatively more quickly than it took off and so the trajectory is shorter down than up. The forces that an actual ball feels are two: there's the force of the Earth's gravity, always down. But there's also the force of resistance that the air presents to the ball and its direction is precisely the opposite of the ball's actual direction. The faster the ball goes, in most instances in a fluid, the harder that force of resistance becomes. So as the ball gains speed from gravity, it loses some speed due to deceleration from resistance force. . . which increases as the ball falls and hence changes the trajectory from parabolic.

Projectiles on the Earth follow essentially parabolic trajectories. *Key Observation 3*

In Chapter **??** we'll begin to develop a graphical way to approach things that move. We're going to learn to draw Feynman Diagrams in Spacetime. Not your father's physics book, this one.

3.8 The Beginning of Physics

When Galileo reached the conclusion that all objects fall to Earth with the same acceleration, he was going against what he—and Aristotle, and everyone else—actually observed. An iron cannonball *would* be observed to reach the ground faster than a wooden one and as you saw in your crumpled paper experiment from page 102, the same object would be observed to fall at different rates depending on its shape. So how in the world could he insist that there is a uniform rule, in spite of these undeniably different outcomes?

Today you'd say "wind resistance" as an explanation of any differences observed of falling objects and you would be right. But for a 17th century thinker, this was a stretch. Galileo said that it's not sufficient to just observe and describe what happens because effects like friction are hiding what Nature *really wants* to do! And uncovering those hidden rules is his science...and ours.

3.8.1 The Red Pill, or the Blue Pill?

This is very un-Aristotle. First of all, Aristotle would never have countenanced doing experiments. You can look at nature, but don't touch. But Galileo was all about constructing experiments and making quantitative measurements of what he observed. This was new and recognizably "modern."

But there's more. To him Nature's rules are hidden to us. We can get close to them by reducing marginal effects like friction but then we have to extrapolate from what we see in our rough and ready laboratory to the hidden rules of a more perfect world. The pendulum bob gets close when it swings back, but the actual rule driving the motion would instruct the pendulum bob to come all the way back to the original height. That's what's real.[26]

[26] It's not for us to get too deeply involved into the question of "what's real." Is what happens to us what's real? Or is the underlying, mathematical regularity what's actually Real. Exactly the premise of *The Matrix* and other science fiction stories.

I cannot over-emphasize how important this is. This is very much Plato's view of nature, not Aristotle's. For Plato, The Real was perfect and with our poor, corrupt visual tools we can only perceive inferior copies of the real things—Platonic Ideals. While there is much that's wrong with Plato's philosophy, the uncovering of nature's hidden order—free of imperfection—is the goal of modern science.[27]

[27] Ask Mr Google about Plato's Cave.

The Father of Physics

Nobody had ever done what Galileo did in all of history. Let's summarize his strategies:

First, Galileo chose not to explain nature (motion in his case) on the basis of logical argument from within pre-conditioned philosophy (neither Aristotle's nor the Church's): Galileo confronted nature without precondition: he assumed that Authority did not dictate how Nature is.

Second, rather than sit passively and observe, Galileo created artificial circumstances designed to explore particular questions: he assumed that Nature can be characterized by doing experiments.

Third, rather than report results as a narrative, Galileo made quantitative measurements under the assumption that arithmetical and geometrical constructions were descriptive of nature's behavior: he assumed that Nature is mathematical.

Nature appears to behave according to mathematics. *Key Concept 6*

Why is Nature inherently mathematical? *Key Question 7*

Those three strategies alone would make Galileo the first, great scientist. But he went further, and his Physics Paternity comes from his Platonism.

Fourth, Galileo chose to interpret nature as consisting of rules that can only be discovered by going beyond the rough regularities in our observations. Nature is best assumed to be simple and mathematical, but unfortunately all that we can observe is complicated by extraneous effects and the true nature...um, of nature is hidden, just out of grasp. Strip away complications like friction as best you can and through mathematical modeling and extrapolation we can uncover Nature's hidden rules.[28]

[28] By now we can go further and add into our models effects of friction and air resistance, for example. This is one of the important features of models as I described in Chapter 2. But it's not perfect, just better.

Without this **fourth** strategy, physics would be impossible.

The rules of Nature are often hidden from experiment and must be inferred by a combination of theoretical modeling and experimental confirmation. *Key Concept 7*

Taken together, these four strategies form the modern-looking nature of physics.