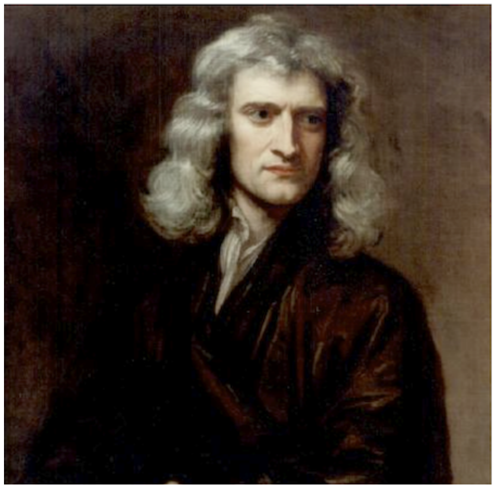
Chapter 4

Momentum and Force

The Big Mo.



Isaac Newton when at his political prime at his scientific prime. Sir Godfrey Kneller, 1689

Isaac Newton, 1642-1727

““I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.” ”*Correspondence.*

Imagine a three year old boy, fatherless and abandoned by his mother to a grandmother he feared and an unwelcoming grandfather. Imagine that this boy is solitary by nature and surrounded by half-sibling girls until late in his teenage years. Now try to project that child into a well-rounded and comfortable future. Hard to do, right? That’s the beginning of the most extraordinary mind ever.

4.1 Goals of this chapter:

- Understand:
  - How to calculate a force required to produce a particular acceleration in one dimension.
  - How to calculate the average change in momentum induced by the application of a force through a given time.
  - How to calculate the centripetal force and acceleration for an object moving at a constant, circular speed.
- Appreciate:
  - Forces create accelerations.
  - That if a momentum vector is changing in direction or magnitude, a force had to be involved.
  - That an object moving in a curved path must have had a force applied to it perpendicular to the trajectory.
- Be familiar with:
  - Newton's third law
  - Newton's Life

4.2 Introduction

We're all about modern particle physics and the collisions we create in order to see into the deepest depths of reality. But in order to get there we need forces to accelerate our beams to high momenta and a language to describe their energies. In addition, we learned long ago that the most intriguing feature of the particles we produce is: mass. All of these concepts have their roots in Newton's notebooks: Force, acceleration, momentum, and mass.

But even though our subject is very contemporary, we're stuck with the definitions and terminology from the 1600's when our heroes emerged: Galileo, Kepler, Descartes, Huygens, Leibnitz, and Newton. While some of those fellows are important for their brilliant intuition (Galileo, Kepler, Descartes), only Isaac Newton, Gottfried Leibnitz and Christiaan Huygens wrote down mathematical relations which form the language and, yes the Metaphors of our modern models of how things move. Typically, we divide mechanics into two parts:

- *Kinematics* is the description of the motion of objects without regard to what caused them to move. If a ball is accelerating by some amount, the rules of kinematics will tell you how far it will go in a

The study of motion without regard to cause.

given time, how fast it's going after a given distance: time, distance, speed, and acceleration are the parameters of Kinematics. When you estimate how long it will take you to arrive at a destination, you're doing kinematics. We tend to attribute the important ideas of Kinematics to Galileo, but he didn't have algebra and the actual mathematics of the kinematical rules came later with Newton and others. What we did in Chapter 3 is sufficient for our purposes.

- *Dynamics* is the study of forces—their causes and their consequences. According to Newton, forces create accelerations in objects by pushing or pulling. That's how the ball in the previous paragraph got its acceleration—something pushed on it. This is a big subject, but we will only consider motions in one dimension, except for circular motion. Here's the rule of thumb for dynamics:

Definition: Dynamics.
The study of forces which cause accelerations.

A force applied to a body will cause it to accelerate. *Key Concept 8*

4.2.1 A Little Bit of Newton

The biggest scientific life of all, is Isaac Newton's. His childhood was a mixture of pain and some accidental fortunate associations. His father was a farmer in Woolsthorpe,[1] not far from Nottingham and a little more than 100 miles north of London. Isaac senior died before tiny, premature Isaac was born. When he was three years old, his mother, Hannah—a semi-literate woman for whom the farm and manor was a big job—married the 63 year old Rector of North Witham who wanted nothing to do with a frail toddler who was then left in the care of his maternal grandparents and female cousins.[2] Seven years later, Hannah, a widow yet again, returned with two young children in tow. While she had been away, young Isaac had been sent to school, a privilege that might not have happened, had his father been alive.

It's often said that Newton was born the same year that Galileo died, but that's not quite correct. Britiain waited until 1752 to convert to the Gregorian Calendar which required an 11 day adjustment. Since Newton was born on Christmas day in Britain, these famous events are not quite the same year.

[1] Of course, an historic treasure in Britain today: `http://www.nationaltrust.org.uk/woolsthorpe-manor/`

[2] When Isaac was 19 years old, he wrote a list of his sins—he always kept voluminous private notebooks—among which he listed, "Threatening my father and mother Smith to burn them and the house over them."

When he was 12 years old, he was sent to a free grammar school in Grantham where the emphasis was Latin ("grammar," after all), which was the language of intellectuals and in which he wrote his great works. He lived with the apothecary, another lucky break as that family indulged his precocious abilities with tools and crafts. In what must have been one of the most frivolous activities of not just his adolescent, but entire odd life, one night he constructed dozens of kites with firecrackers, which he flew over the town at night "...wonderfully affrighting all of the neighboring inhabitants for some time, and causing not a little discourse on market days..."

He was a loner then, and for most of his adult life. "His school fellows generally were not very affectionate toward him. He was commonly too cunning for them in everything. He who has most understanding is least regarded." When he was 17 Hannah brought him home and tried to turn him into a farmer, but

Figure 4.1: Isaac Newton's childhood home where he first conceived of gravity, optics, and calculus. (Copyright GP Williams and licensed for reuse under this Creative Commons Licence)

he was a disaster. She gave up and sent him back to Grantham to prepare him for university...as the only outlet for his increasingly apparent unusual mind.

He was sent to Trinity College at Cambridge University where he was enrolled as a "Sizar" which was essentially the role of servant to an upperclassman. He was 18 years old, considerably older than most of the students and as a studious person, different from the mostly carefree student body. He made a single friend, a most unlikely event for the difficult Newton. He bumped into John Wickins while he was alone on a bench and they became roommates for 20 years, until Wickins married and left. Wickins was Newton's assistant as Newton began his life of changing the world.

The curriculum at Cambridge was terribly old-fashioned: Aristotle, from top to bottom. But Rene Descartes from the Continent was all the rage and read and discussed secretly by the students. To Descartes, the world was mathematical in a manner beyond what Galileo might have imagined. It was he who blended geometry together with the brand new algebra so that equations might be presented as curves and curves, as equations. Of course we call our axes, "Cartesian" after their inventor. Descartes was also a mechanist: meaning the motions of all things were caused by mechanical interactions of matter. Nothing spiritual, nothing occult. Motion was inserted into the cosmos by God originally, who then apparently abandoned His creation in order to pursue other interests? In any case, Descartes believed that this original motion persisted[3] as the universe formed and that God did not drop in and adjust things. The world was predictable according to Laws (yes, capital L for Descartes) and he proposed to start with the Laws and draw observable conclusions from them. Very top-down, was Descartes. He was all about Why. Newton weaned us from Why to How. Even the planets moved by being carried by vortices of invisible balls. Mechanical modeling was his goal, and analytical mathematics was his tool.

Figure 4.2: Newton's faculty rooms at Trinity College, Cambridge. (http://heirloomheritagetours.com/blog-2/isaac-newtons-cambridge-trinity-college/)

[3] This is very close to one of the so-called conservation laws that are so important. We'll see that next chapter.

The young, inquisitive Newton ate Descartes up. Here was an escape from Aristotle, but a systematic and mathematical way out, and that was right up Newton's alley. But Descartes' conclusions were problematic for Newton and much of what he wrote later was in reaction to his disagreement with Descartes.

But the Frenchman's[4] mathematics stuck and the neoPlatonists at Cambridge carried Descartes' mathematics program forward and the leader of that movement was Isaac Barrow, who was the first Lucasian Professor of Mathematics. Isaac Newton was the second.[5] He became a student of Barrow's and eventually, his benefactor. But disaster struck in London—in 1664 the Bubonic Plague arrived and before it was over, 20% of the population was dead. By 1665 it hit Cambridge forcing the university to close and the 23 year old Newton went back to Woolsthorpe where he remained to himself for more than a year. There, he consumed all mathematics known at the time and went beyond. While at the farm, he basically invented calculus, had his first ideas about gravity (the famous apple was to have fallen in his presence during this period), and reinvented optics.

[4] French, indeed. But Descartes saw what happened to Galileo and relocated to the Netherlands to avoid possible prosecution by the French Catholic Church. While we think of Descartes as the father of analytical mathematics, academia thinks of him as the Father of Western Philosophy. He of "I think, therefore I am" fame.

[5] Other famous Lucasian Professors: George Biddell Airy, Charles Babbage, Paul Dirac (of whom we will fawn over later), and Stephen Hawking

> In the beginning of the year 1665 I found the Method of approximating series & the Rule for reducing any dignity of any Binomial into such a series. The same year in May I found the method of Tangents..., & in November had the direct method of fluxions & the next year in January had the Theory of Colors & in May following I had entrance into the inverse method of fluxions. And the same year I began to think of gravity extending to the orb of the Moon & (having found out how to estimate the force with which [a] globe revolving with in a sphere presses the surface of the sphere) from Kepler's rule...I deducted that the forces which keep the Planets in their Orbs must [be] reciprocally as the squares of their distances from the centers about which they revolve: & thereby compared the force requisite to keep the Moon in her Orb with the force of gravity at the surface of the earth, & found them answer pretty nearly. All this was in the two plague years of 1665 & 1666. For in those days, I was in the prime of my age for invention & minded Mathematicks & Philosophy more than at any time since.

When he returned to Cambridge he rose quickly, his mathematical skills having surpassed those of all around him. In 1667 he was elected as a Fellow (like an assistant professor) with a salary. In 1669, Barrow resigned the Lucasian Chair insisting on Newton appointment to the highest post in the college. ("Mr Newton, a fellow of our College, and very young ... but of an extraordinary genius and proficiency in these things.") One of the only duties of the Lucasian Chair was to offer a single course a year, which he dutifully fulfilled, but usually lecturing to a literally empty classroom as nobody could understand him.

Among his discoveries during the plague years was that sunlight was composed of all colors, which was in conflict with the standard idea that white light was a color of its own and that the colors we see are mixtures of white and dark. He passed light through a prism and found that it spread out into a continuous spectrum of colors. This led him to experimenting with light passing through glass and eventually to grinding his own lenses for telescopes. By this time telescopes were beginning to be unwieldy in their length and the effects of light's spreading of color in the lenses ("chromatic aberration") was limiting precise viewing. He eventually changed the design completely. By using mirrors rather than lenses, he could create images of higher quality and with higher magnification in a compact form. His original 6 inch

Figure 4.3: Newton's second model telescope. From the Royal Society.

"Newtonian reflecting telescope" would magnify by 40, which would have required a less precise conventional telescope six feet long to achieve.

The telescope and his explanation was his first entry into membership in the Royal Academy of Sciences. His explanation of how it worked led to the explication of his theory of light, which the Curator of Experiments, Robert Hooke, thought was stolen from his (incorrect) ideas. They became bitter enemies for life, one of a number of such vicious rivalries that Newton suffered through his whole life. So galling was this dispute—and the criticism that his theory of colors attracted from all over Europe—that Newton went into nearly complete isolation, vowing to keep the products of his research secret, rather than ever again suffer such public antagonism. He communicated almost exclusive through voluminous correspondence, much of which still exists. Newton did not suffer fools well, and even legitimate dispute would send him into a towering rage, or stony silence. His response to Hooke was to write a book on optics...which he inadvertently destroyed when a fire from an alchemy experiment that went out of control.[6]

[6] Newton was an accomplished alchemist, probably damaging himself with the noxious chemicals that he inhaled and tasted. He believed that God had hidden first-knowledge of the physical universe to the Ancients, and that among those ideas were alchemical. His lodging was near to his laboratory on the Trinity campus and a huge furnace fire was never put out as he single-mindedly attacked chemistry. He also was an unusual religious fanatic. He learned Hebrew and translated the Bible from original texts and became convinced that fourth-century changes made the false claim of the divinity of Jesus. Rather, he should have been treated as a prophet on par with others. He was not a Christian, but a believer in a God of Nature. He wrote way more about alchemy and his fanatic religious histories than about science. These works are still being studied today.

What we all know as Newton's enduring scientific work came as the result of a wager and we'll pick up that story when we study his law of Gravitation and the first-ever attempt at a scientific cosmology in Chapter **??**.

Box 4.1 Newton and the Book

Newton was in his 40's when he basically sequestered himself in his rooms working on his alchemy and intensely weird religious researches. In 1684 three of his colleagues (the famous London architect Christopher Wren, the scientist Edmund Halley[7], and his arch nemesis Robert Hooke) were trying hard to figure out the shape of an orbit if the force of gravity varied like the inverse square of the distance from the Sun. Hooke claimed in his obnoxious way that he knew the answer, but he would not produce a calculation—because he couldn't since he had no mathematical training. Hooke's instincts often guided his brilliance as an experimenter. But, a little of Robert Hooke went a long way and Wren and Halley got tired of listening to him so they deputized Halley to go ask Newton. So he did. He showed up unannounced at Newton's messy room and asked him. Immediately came the recluse's famous response: "An ellipse." "Why?" asked Halley. "Because I have calculated it." But, typical of the paranoid Newton, he'd not told anyone.

[7] Halley was quite a guy and is worth reading about. http://en.wikipedia.org/wiki/Edmond_Halley

He'd worked out the mathematical rules for the motion of the planets. . . and kept it a secret! Well, the small problem was that while Halley waited, Newton could not find his calculation. A little while later, Halley received 9 pages from Newton that showed: if the force on a planet varies like the inverse square of the distance from the center, then the orbit's shape must be a conic (a parabola, ellipse, circle, or hyperbola). And, he showed that if the orbit is an ellipse, that the force of attraction must be an inverse-square. This pamphlet became known as , *De Motu Corporum in Gyrum* (*On the Motion of Revolving Bodie*s). *De Motu*, as it's known, was a summary of the first book of his magnificent work. Figure 4.4 shows a page of De Motu in Newton's hand that he later prepared for his correspondent-friend, John Locke. This electrified Cambridge and London and set Hooke's teeth on edge as he'd guessed some of the same conclusions and again insisted that Newton had stolen his ideas, this time on gravitation.

Halley realized what Newton had done and implored him repeatedly to write it all. Newton finally agreed and went into one of the historically most intense periods of concentration ever embarked on by anyone. For two years he worked night and day, forgetting to eat, wandering around Cambridge without regard to his surroundings. Thousands of pages of manuscript littering his quarters along with days' worth of uneaten food. Two years! Eventually he emerged with the first book of what was to be three volumes of *Philosophiæ Naturalis Principia Mathematica*, or the *Mathematical Principles of Natural Philosophy* affectionately known ever after as "The Principia." It was all there in Latin. His laws of motion and gravitation, but also of fluids and the strengths of materials. He'd pestered scientists and astronomers from around Britain for data on the planets and the tides. He'd made measurements of motion in his own lab. He let his alchemy furnace go out forever as he worked solely on his system of the world. The arguments were mathematical and constituted the first workable system of nature. He continued to hide his calculus, preferring to speak in terms of limits and extrapolations using geometrical constructions, surely backed up by his own private calculus based calculations. Principia went through three editions after the original 1686 start, often with him revising his last chapter, which was more philosophical, but also with successive furious deletions of the names of rivals.

Hooke had persuaded the Royal Society to act as the publisher of Principia and Newton dedicated it so. But the coffers of the Society were dry when it came time to print as they had used up their entire accounts in a lavishly illustrated two volume History of Fishes. So Halley took a deep breath and paid for the initial publication himself. This of course led to his active interest in encouraging Newton to push the book off at booksellers and libraries himself. Never was there a more generous gift to science than Halley's unselfish gesture. And for a book that only a few people in the world could read, but a book that quite possibly initiated the Enlightenment and people's relationship to our universe.

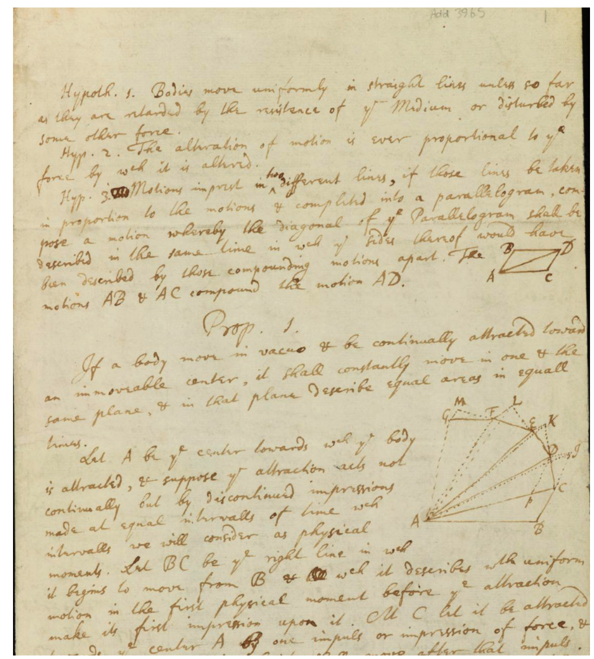


Figure 4.4: A portion of De Motu in Newton's hand in 1684 written for John Locke who was an early reader of Principia.

4.3 Getting Going

We don't need to deeply study Newton's laws for our purposes in Particle Physics like you might in a general physics course, but his whole system is built on two ideas that do matter to us: they are the idea of **mass** and the concept of **momentum**. Let's move.

In Chapter 3 we discussed the rules that govern any object's motion, whether it moves at a constant speed or accelerates. What we conveniently avoided was how that motion is caused—on that Galileo had little to say. Of course Aristotle had something to say about everything and he insisted that motion[8] is not for free, that one always needs to apply a force to keep something moving, or to start it moving from rest.

[8] Remember, unnatural motion requires a push, natural motion just happens. For him.

One of the many ways that Isaac Newton got into the textbooks was to say to Aristotle: "no." Constant motion *is* free. It's only accelerated motion that requires payment in the form of a force. Further, while Aristotle simply declared what his rules were, Newton built the first-ever mathematical model describing all motion. Remarkably, his model has functioned for four centuries and still forms the basis of mechanical and civil engineering projects.

Definition: state.
The circumstance of an object's position and it's momentum define its "state."

Here's what he said: in order to change the "state" of motion of any object requires the application of a force. To start something moving from rest? Apply a force. To speed up or slow down something already moving? Apply a force! To cause something to deviate from a straight line? Yes. Another force. To keep something moving at a constant speed? No force required, thank you. So whenever there's a change of velocity,[9] a force is at work, so forces are responsible for acceleration.

[9] Hold that thought. The difference between speed and velocity is crucial here. More in a bit.

4.3.1 Impulse

To get something up to speed, you must whack it or shove it—either a sharp collision or a steady push increases the speed of an object. Push harder? More speed. Push longer? Again, more speed. And as you know from any sport involving a collision, something that's moving fast can in turn exert a bigger force than something that's moving slowly.

So let's codify that everyday notion into a formula. Let's imagine a force, F that pushes during some time interval, Δt. A whack means that Δt is small (like a golf club hitting a ball) while a steady shove (like a rugby scrum) means that the force is slowly applied so Δt is larger. Here's a trial formula to reflect the fact that either (or both) circumstance changes the state of motion of an object:

Definition: $\propto$.
"proportional to" or sometimes "goes like" as in "this function goes like the square"

$$F\Delta t \propto \Delta v. \tag{4.1}$$

The application of a force for a time interval means that the speed changes in proportion. Increase F, Δt, or both and the speed goes up. The quantity on the left side is called the Impulse in physics. It's the sports-quantity. Any game involving a ball involves impulse. The quantity on the right implies that the speed changed and of course if the speed changed, then the object accelerated.

But now let me ask you: Suppose I apply a force of 100 pounds for 60 seconds to a Volkswagen and and you apply a force of 100 pounds for 60 seconds to a little red wagon. Will the resulting Δv be the same for both vehicles? Of course not. The little red wagon will gain more speed than the Volkswagen (regardless of what color it is). So Eq. 4.1 is not the whole story. What's missing is the reluctance that any object has to being accelerated, which has a name: inertia.

Definition: Interia.
Inertia is the reluctance that an object has to being accelerated.

4.3.2 Newton's Mass

Mass is a toughy. Here's how he defined it in the *Principia*:

> Mass is the ...“quantity of matter...arising from its density and bulk conjointly.”

There you go. Useful? No? What he seems to be saying is that mass is the product of density and volume. But, he doesn't tell what density is which is why it's not helpful. When people invent whole sciences, they also need invent a language! That his words don't quite work shows that his concepts and his mathematics are a bit ahead of him. In any case, you do perfectly well—at least in solving homework problems or building bridges—to accept the idea that mass is the amount of the “stuff” in an object and that it's also the quantitative measure of an object's reluctance to be coaxed into changing its motion.

Here begins our love-hate relationship with mass.

Figure 4.5: In 1971 Alan Shepard smacked a golf ball on the moon with a makeshift 6 iron. He'd need to apply the same force up there that he would on Earth in order to achieve the same acceleration—but on Earth there would be air resistance, so it would not have gone as far.

Mass is an object's resistance to being accelerated. *Key Concept 9*

What is the nature of Mass? *Key Question 8*

At the deep level of elementary particles, mass confuses us, perhaps in a different way from how it confuses college freshmen. We think that mass may actually not be an actual property of object, but rather a result of an object's interaction with a spooky field that sprang into existence just after the birth of the Universe. Now, in the 21st century, we've got a whole new set of neuroses about this subject, as understanding it occupies almost the entire Particle Physics community. So, Mass has been a problematic subject since its beginning in Newton's hands.

4.4 The "Quantity of Motion"

We just developed a sense that our hand-built, car-pushing formula, Eq. 4.1 has to depend on speed and mass and so we'll just add it in on the right-hand side to get:

Pencil 4.1. ✎

$$F\Delta t = m\Delta v \tag{4.2}$$

Definition: **impulse .**
is the quantity which is the force applied to an object throughout at time span, Δt. It's equal to the change of the mass times the velocity experienced by that object.

Being more explicit, impulse which when spelled out is

$$F\Delta t = mv - mv_0 = \Delta(mv). \tag{4.3}$$

So since the mass of the Volkswagen is much bigger than the little red wagon, the same force applied through the same time results in a smaller speed change for the former, rather than the latter. Think of it this way: the numerical product of $F\Delta t$ (which was the same for your push as it was for my push) is shared by m and Δv so more m leaves smaller Δv and of course, a smaller m means more is left for Δv. This collecting m and v together proves to be useful.

Everyone knew that Aristotle's ideas about motion were silly—what's pushing on a projectile? He danced around this and said: the air rushing around from the front to push the object from the back. What if it's an arrow that slices through the air going forward? If you shoot it tail first are we to believe that now the air pushes on the arrowhead that before sliced through it? Seriously? That's all Aristotle's got.

4.4.1 Momentum

Newton's second good idea is the concept of "momentum" which he called the "quantity of motion"—a nice description, I think. The idea that a moving object possessed *something* —some quality—was pretty hard to ignore. But, nobody could figure out how to describe it for 2000 years before him. Aristotle just denied it: "No," a moving object doesn't possess any quality. Galileo vaguely said "yes," there is something "in" a moving body that he called *impetio*. Kepler seemed to say "yes." Descartes definitely said "yes." Newton agreed with his 17th century predecessors but made the idea useful.

What he concluded was that the "quantity of motion" is **momentum**. Keeping with tradition by using the symbol "p" as its nickname, momentum is:

$$p = mv. \tag{4.4}$$

Definition: **Momentum.**
$\mathbf{p} = m\mathbf{v}$
Momentum is proportional to speed and mass. This is specifically "linear momentum." It is a vector because $\mathbf{v}$ is a vector.

We can continue the manipulation of Eq. 4.3 and restate it one more time in terms of momentum:

$$F\Delta t = \Delta p. \tag{4.5}$$

We're going to find that momentum is most important in our particle physics story—much more so than Force, velocity, or acceleration. We'll use it over and over in different guises.

Equation: **Momentum.**
$p = mv$

Box 4.2 The relationship for all sports using a ball.

Equation 4.5 works in three ways: either you know the force and you use the formula to calculate the change in momentum in a given time. Or, you know the change in momentum and you use it to calculate the total force in a given time, or you adjust the time for a given force.

Now you've gotten the formula that governs all sports involving whacking one thing with another. . . like baseball or tennis—or football. Think about what you almost always want to do: you want to make the ball go faster after you hit it. That means, you want the change in the momentum to be the highest possible. So, Equation 4.5 tells you how: you hit the ball as hard as you can (that's a large F) and you get "good contact" (which means you hit the part of the bat or racquet or club where you can touch the ball as long as possible. . . which is the largest Δt).

This also explains how airbags and bumper-crumple zones in automobiles work. There, you know what the change in momentum is. . . it's

$$m \times (v_{\text{after}} - v_{\text{before}})$$

That's the *change* where the final velocity is zero (the car stops). The initial velocity is fixed and so v_{before} has to be divided up between the force and the time in $F\Delta t$ where the force is applied, say to your bumper. High force is not good for the occupants inside the car, so this leads to the design goal of spreading out the time—large Δt so that the force will be smaller. This is the same reason that you bend your legs when you jump off a table and hit the floor.

Momentum Is a Vector

Because velocity is a vector, momentum is also. In the next chapter when we consider collisions the direction-part of the momentum vector will play a crucial role. For that matter, since momentum and force are vectors the actual general statement about Impulse is:

$$\mathbf{F}\Delta t = \Delta \mathbf{p} \tag{4.6}$$

4.5 Newton's Famous Three laws

Newton's Momentum and Mass are at the heart of his three laws of motion. Let's go through them in words, and then one of them in more detail in symbols.

Newton's 1st law of Motion says that anything that's moving at a constant speed (which could be zero) will continue in that way unless a force acts on it. That's a statement about inertia—resistance to acceleration. (He inherited this from Galileo, but gave it a quantitative meaning.)
Newton's 2nd law of Motion says that momentum is changed when a force acts on an object for a duration of time. Or, you might have learned it as a defining statement about "force" namely:

Force is equal to the rate of change of momentum. *Key Concept 10*

Newton's 3rd law of Motion is subtle. It says that if you push on something—anything and with any amount of force—that object will push back with exactly the same force. We'll think harder about the 3rd law when we talk about collisions.

Your immediate reaction might naturally be to wonder how anything moves at all, since there appears to be a balance from this rule. What matters for an object to feel an imbalanced force *on it*. That it exerts a force *on something else* doesn't affect its own motion. So, if a donkey pulls on a wagon, the wagon pulls back...but the force *on the wagon* is itself imbalanced and so it moves.

4.5.1 Newton's Second law

Most of those problems you might have worked in a physics class are related to the 2nd law, which is all about momentum and how to change it from from the vector version of Eq. 4.3.1. A simple arrangement of that impulse equation, yields the real mathematical definition of Newton's 2nd law:

Pencil 4.2. ✎

$$\mathbf{F} = \frac{\Delta \mathbf{p}}{\Delta t} \tag{4.7}$$

Maybe you've maybe seen this equation (on a tee shirt?) but written in a different way. Inserting back the definition of momentum, $p = mv$ (in one dimension, so we'll drop the vector notation):

$$F = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t}. \tag{4.8}$$

Let's assume that the mass of the object doesn't change through the push, so we pull it outside of the Δ and get

$$F = \frac{m\Delta v}{\Delta t} \tag{4.9}$$

and, what's $\Delta v/\Delta t$? That's nothing more than acceleration, so

$$F = ma \tag{4.10}$$

which is the tee shirt version of the famous Newton's Second law of motion.[10]

Notice that if you isolate the acceleration, you get the force divided by the mass. Therein lies the "inertia" nature of mass since

$$a = \frac{F}{m} \tag{4.11}$$

explicitly showing the inverse relationship: the larger the mass (so that $1/m$ is a small number) the harder it is to accelerate for a given force (the Volkswagen). Conversely, if that force is now applied to a lighter object (so that $1/m$ is a larger number) then the acceleration will be more (that's the wagon).

[10] The more complete statement of Newton's Second law is Eq. 4.8.

Equation: Newton's Second law.
$F = ma$ (for constant mass)

If you think about it, there are three ways for **p** to change:

1. As we've just seen, if the speed changes, the velocity vector changes, so then the momentum vector changes. That's easy.
2. If the *mass* changes, then the momentum changes. That's easy mathematically, but huh?
3. If the velocity vector changes, but the speed stays the same? Ah. That's interesting and we'll look at that carefully in Section 4.11.

#1 is obvious. If you change the speed, you will change the velocity. #2 is less obvious but think about pinching the opening of a balloon that you've blown up. The balloon's mass consists of the stretchy material of the balloon plus the mass of the air inside. Held out in front of you there is no net force and it's at rest in your hand. If you open the pinch, then the air rushes out *fast* and so the mass of the air inside the balloon rapidly decreases. The consequence of that decrease in mass, a Δm in a time Δt by itself results in a change in the momentum of the balloon! That's how rockets and jet turbines work.

Item #2 at the left requires some thought. Remember what "change" means: something at the end minus something at the beginning. So, the mass in a balloon is decreasing, so $\Delta m = m - m_0$ which is a negative quantity. in fact, the mass escapes out the nozzle and the momentum change is in the opposite direction (the negative sign of Δm becomes a directional sign of the opposite of the direction that the mass goes.

Weight

We tend to mix up the units of weight and mass at the grocery store, for example. A gram is a unit of mass, while an ounce is a unit of force. A kilogram is just a 1000 grams and still a measure of mass and a pound is 16 oz and still a measure of weight. But we get away with using both systems since we tend to buy things and compare them on Earth. If we had a Mars colony, well then there would be trouble. That 5 Earth-pound bag of Gold Bond flour at Kroger would be 13 Mars-pounds. But in each location, it would still be a 2.27 kg bag of flour. How can that be?

Table 4.1: Units for quantities used in weight calculations.

	English	Metric	Conversion
acceleration	ft/s^2 $g = 32.2$ ft/s^2	m/s^2 $g = 9.8$ m/s^2	1 ft/s^2 = 0.305 m/s^2
mass on Earth:	slugs a mass of 1 slug → weight of 32.2 pounds	kilograms (kg) a mass of 1 kg → weight of 9.8 N	1 slug = 14.59 kg
force	pounds (lb)	Newtons (N)	1 lb = 4.45 N

Weight is the force that the Earth exerts on objects on its surface while as we've seen, mass is the amount of inertia that an object possesses. The inertia is determined by measuring the force that it would take to accelerate the object to a given amount. What Galileo showed was that the acceleration due to gravity on the surface of the Earth is a particular value—he presumed was a constant. If the ground suddenly disappeared, then everything with mass would start to fall with that acceleration, g. But happily the ground pushes back and things are stable on the surface. So from Newton's Second law, when we have a mass and we have an acceleration, we can calculate a force and we define that particular force of attraction by the Earth the weight. Let's call it w and we can write it out:[11]

[11] Don't be confused. This is just Newton's Second law, but when the acceleration is the particular value of g, the force is the particular kind of force we call "weight."

$$w = mg \tag{4.12}$$

We can measure the weight by making use of the fact that the Earth pushes back with the same value as the weight... when you think of weight, probably a spring is doing the pushing-back. That's your bathroom scale which is calibrated in the U.S. to read that push in pounds. We'll see how the Earth does this in a bit when we get to Newton's other law, that of Gravitation.

Unfortunately in the English system, the unit of mass is " slugs." So we can collect our units appropriate to Newton's Second law in Table 4.1. In the last column, the standard abbreviations are shown as well.

Example 4.1

Weight

Question : If I weigh 200 pounds on Earth, what is my mass in slugs? In kilograms?

Solution:

$$w = mg$$
$$m = \frac{w}{g}$$
$$= \frac{200}{32.2}$$
$$m = 6.2 \text{ slugs}$$

To convert to kilograms, we can do conversions within the correct quantities using Table 4.1. So let's do $w(\text{English}) \rightarrow w(\text{metric})$:

$$w(\text{metric, in N}) = w(\text{English, in lb})\frac{4.45 \text{ N}}{1 \text{ lb}}$$
$$w(\text{metric}) = 200 * 4.45 = 889.6 \text{ N}$$

Now calculate the mass in kilograms like before:

$$w = mg$$
$$m = \frac{w}{g}$$
$$= \frac{889.6 \text{ N}}{9.8 \text{ m/s}^2}$$
$$m = 90.8 \text{ kg}$$

We can check with Fig. 2.4 and see that we got the right answer.

Example 4.2

Jumping from a Step

Question : Suppose I'm 200 pounds, or 90.7 kg and I jump from a step which is 1 meter high, about a yard. Now, with my artificial knees, I probably shouldn't do that, but were any of us to do so, we'd automatically flex our knees on landing and here's why.

Solution:

Equation 4.3.1 tells the story. On the right hand side is the change in momentum. Now, just before my feet hit the floor, I'm traveling downward with the greatest velocity possible. The *change* in momentum is

$$\Delta p = p - p_0 \tag{4.13}$$

where p is the final momentum, and p_0 is the initial momentum. Now, the final momentum is easy. It's m times the final velocity, which is "stopped" or 0. So,

$$\Delta p = 0 - mv_0$$

where v_0 is the initial velocity. Although we didn't talk about it, if I start from rest, then the velocity of an object falling under gravity is $v = \sqrt{2gx} = \sqrt{2 \cdot 9.8 \cdot 1} = 4.4$ m/s.

The final momentum is then $p_0 = mv = 90.7 \cdot 4.4 = 402$ kg m/s. Now, all of this momentum has to be taken up by the combination of F (which is applied through my legs to my poor knees) and Δt which is the amount of time that that force is applied. If I land stiff-legged, say my impact happens in 0.1 seconds, then the force applied to my knees would be

$$\begin{aligned} F\Delta t &= \Delta p \\ F &= \frac{402}{0.1} = 4020 \text{ Newtons} \end{aligned} \tag{4.14}$$

The metric system of force is "Newtons" and here 4000 N is about 900 lbs.

If, I can spread out my shock-absorption by bending my knees...to maybe as much as a second, then I would relieve the force transmitted by a factor of 10!

Example 4.3

Biking at a constant acceleration

Question : In Chapter 3 we described the bicyclist's constant acceleration of 2 m/s^2 and the subsequent increase in speed in time and consequent quadratic increase in distance covered. If I weigh 200 pounds, we found in Ex. 4.1 that my mass is 90.7 kg, how much force do I have to apply to the ground through the pedals and the tires in order to keep up that constant acceleration? What fraction of my weight is this force?

Solution:

This is a simple application of the popular form of Newton's Second law, Eq. 4.10.

$$\begin{aligned} F &= ma \\ &= (90.7)(2) \\ F &= 181.4 \text{ N} \end{aligned}$$

To find my weight, we again can use the same formula with an important difference (we'll call my weight W) and I'll approximate the acceleration due to gravity, which is $g = 9.8$ m/s^2, as $g \approx 10$ m/s^2

$$\begin{aligned} W &= ga \\ &= (90.7)(10) \\ W &= 907 \text{ N} \end{aligned}$$

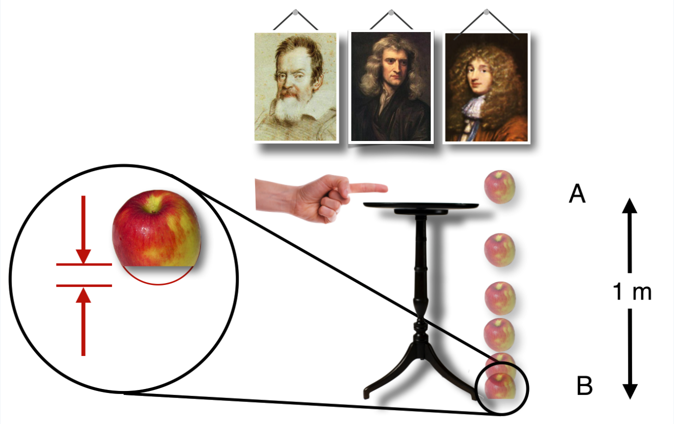
So the force that my legs would have to continuously apply to the pedals, and in turn to the ground through the friction between the tires and the road is about 20% ($\approx 180/900$) of my weight.

How much sustained force is this? Well, suppose we have a stationary bike hooked up to a pulley and a bag with five bowling balls. The force required to keep that bag 'O balls aloft—forever—is the amount of force that I'd have to sustain—forever—to maintain that acceleration. Now is that sensible? Figure 3.8 shows that after 10 seconds of this acceleration I'd be traveling at 20 m/s, which is about 45 mph. So obviously, that's too fast to imagine pedaling a bicycle for 10 seconds. Rather, if it were possible for me to exert 181 N, after about a couple of seconds, I'd be moving around 10 mph and surely at that point I'd stop trying to accelerate and apply just enough force to maintain that speed.

Example 4.1

Apples falling again

Question:



In Example **??** you calculated that the speed that an apple would attain if it was dropped 1 meter would be 4.4 m/s. You did, right? Look at the figure at the left for our new situation. The apple at A is dropped onto the carpet and bruises flat at B, slowing it down to a stop. The carpet applies a force to it which would be pointing up. (You can see the damage in the inset.) If it takes 0.090 s for the apple to stop, what is the average force that the carpet applies to bring it to rest? The mass of an apple here is 0.1 kg.

Solution: This is another application of Newton's Second law where we have:

$$F = \frac{\Delta p}{\Delta t} = \frac{mv}{\Delta t}$$

The change in velocity is of course the velocity that the apple has just before it hits since it's dropped from rest. Putting the numbers:

$$F = \frac{(0.1)(4.4)}{0.09}$$

$$F = 4.9 \text{ N}$$

which is about half of the force of gravity. The apple would probably not bruise.

Now you try it. Drop it on the carpet.

You Do It 4.1. Apple Force

or copy the solution

If the apple in the above example is dropped on a hard floor, then it will stop more abruptly. Llet's pretend that it takes only 0.005 seconds to come to rest. What is the average force that the carpet applies to the apple to make bring it to rest? Would it bruise more or less than on the carpet?

4.6 Circular Motion

Item #3 back on page 127 of how a change in momentum can occur is subtle, but you use it and experience it every day. Suppose you're a passenger in a car going around a curve. When you enter the curve, you're moving north. When you emerge from the curve, you're pointing west. Watch the speedometer and make sure that your driver stays at the same speed through the whole path. So did your speed change?[12] No!

[12] a "Who's buried in Grant's Tomb" question

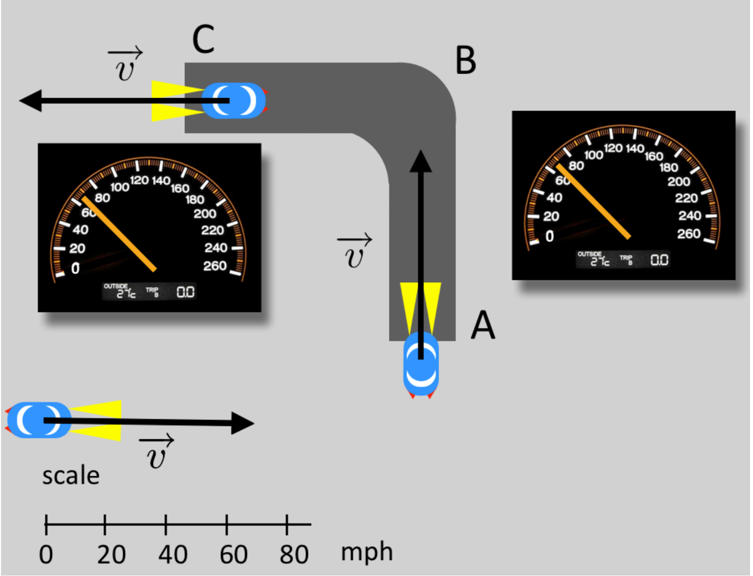


Figure 4.6: At the beginning the car is headed north at point A with a velocity vector shown and the speedometer pictured at that point. The car curves to the left at B and at point C the car's speedometer still looks identical to point A, but the velocity vector is pointing to the west. The speed is constant, but the velocity is different. The scale at the bottom left shows that the length of the velocity vector corresponds to just about 65 mph.

Did your velocity change? Yes! Because, the *direction* of your speed changed. This is shown in Fig. 4.6. What do you *feel* during the curve?

Let's recite the series of events that follow from going around this curve: You stay in the car, so you are being "forced" to deviate from straight-line motion. A variety of mechanisms cause that to happen: your seatbelt, the friction of your pants and the seat, the door pushing on your shoulder. All of these apply a force in a direction to the *inside* of the circle that your car is moving along.

- Your speed didn't change.
- But, your direction changed, so your velocity changed.
- If your velocity changed, and your mass remained the same your momentum changed (#1)
- If your momentum changed, there was a force applied on you (Second law).

These various forces on you all cause you to go in the same circle as the car (Second law) while you are trying desperately to continue to go straight (First law!). This force that causes motion to deviate from a straight line is called "centripetal force" and this is another genius idea of Newton's.

Definition: centripetal force .
is a force that causes an object to go in a curved path. It points to the center of the curve.

The essence of circular motion can be visualized in Fig. 4.7 where a figure is twirling a ball attached to a rope in a circle. Let's ignore gravity for a second and concentrate on the motion in the plane of the rope and ball...and his fist. You know by now what would happen if he let go of the rope. Without the

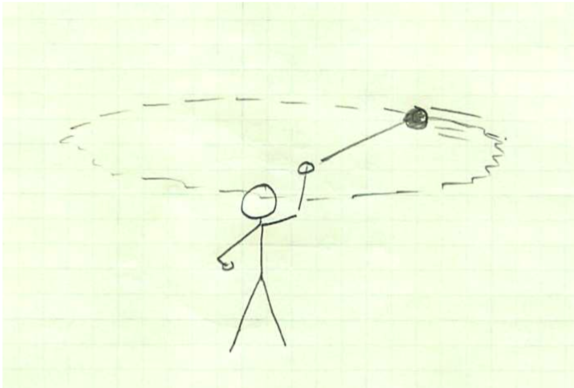


Figure 4.7: Twirling a ball in a circle with a tether.

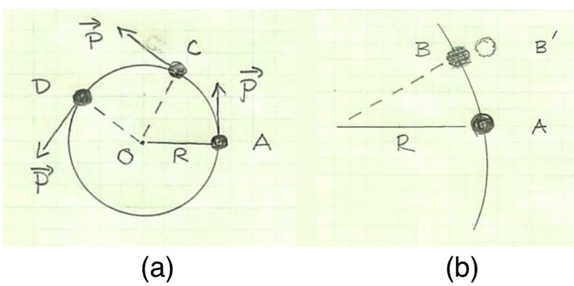


Figure 4.8: Looking down from the top of a ball's path.

rope pulling in towards the center, there is no horizontal force and according to Newton's First law, the ball would go straight. So the rope is causing the ball to deviate from a straight line.

Figure 4.8a shows the view of the ball over the head of the figure. The rope, R, is shown and the location of the ball is represented at three spots around the circular trajectory, A, C, and D. This is meant to be uniform motion, which here means that the speed is a constant...like the speedometer in the opening discussion. But of course the velocity is changing, by virtue of the changing direction. But what's represented in this figure is now the momentum, **p**, or here, $\vec{p}$, which is the mass m times the velocity, **v**. Since the speed is constant, the momentum vector has a constant length, but because the motion's direction is around a circle ("not straight"!), the momentum vector is tangent to the circle at all points around the path. Newton's brilliance was to explain this using his three laws.

Figure 4.8b shows a segment of the circle as the ball passes point A in Fig. 4.8a. Newton reasoned that the ball would "like" to go straight, to point B' but that the rope tugs it back to point B. So the ball goes a little, gets tugged back, goes a little further, gets tugged back, and so on. These little tugs were in his mind acting all around the circle, which in the limit of being infinitesimally spaced create a continuous, circular trajectory. This notion of "infinitesimal" was kin to the habit of mind he was developing in the invention of calculus.

Pencil 4.3. ✎

But let's carry this further. Since the momentum at, say A is different from the momentum at, say C (because the direction is different), even though the magnitudes are the same, there is still a changing momentum, a non-zero $\Delta\vec{p}$. If there's a change in momentum of any kind, there's a force:

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t} \tag{4.15}$$

Let's see what he found. Figure 4.10 shows two strategically placed points on the circle "here" and "there" and the corresponding momenta of the ball associated with each point, $\vec{p}_{\text{here}}$ and $\vec{p}_{\text{there}}$. So,

$$\vec{F} = \frac{\vec{p}_{\text{there}} - \vec{p}_{\text{here}}}{\Delta t} \tag{4.16}$$

Let's get a feel for what this means by actually manipulating the momentum vectors and look at what the numerator gives us in Eq. 4.16. We'll remove them from the diagram and take their difference according to the rules of vector subtraction...by adding. Figure 4.9 shows the process.

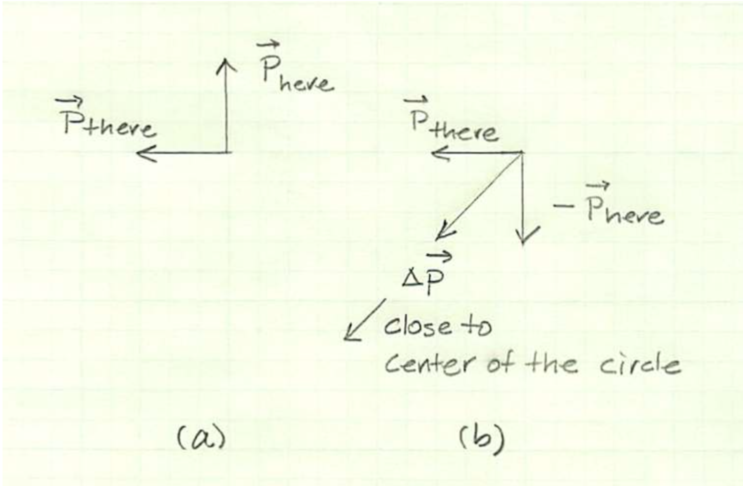


Figure 4.9: (a) shows the vectors brought tail-to-tail with their original orientation unchanged. (b) shows them with $\vec{p}_{\text{here}}$'s direction reversed.

In the left side of the figure, the momentum vectors have just been redrawn. We need to form the difference as in Eq. 4.16 and we remember that the difference of two vectors can be:

$$\vec{F} = \vec{p}_{\text{there}} - \vec{p}_{\text{here}} = \vec{p}_{\text{there}} + (-\vec{p}_{\text{here}}). \tag{4.17}$$

So by reversing the direction of $\vec{p}_{\text{here}}$, we can just add it to $\vec{p}_{\text{there}}$ and get the required difference from Eq. 4.16. This is shown in the right side of Fig. 4.9. That difference is labeled $\Delta\vec{p}$ and its direction is very close to the center of the circle! This was his brilliance! If the "here" and "there" points were closer and closer to one another, then the difference would point closer and closer to the center.

So like we knew all along, the rope is what causes the ball to go in a circle, the combination of Newton's First law with his Second law, and the crucial recognition that momentum is a vector, leads to the demonstration of the force towards the center is responsible for the change of momentum.

This force is that special "centripetal force" that we encountered going around the curve in the car above. All non-straight motions are caused by an centripetal force. If the trajectory is circular, it's easy to see that it points to the center of the observed circle. If the trajectory is uniformly curvy, at each point there can be an instantaneous "circle" and the force would point towards it.

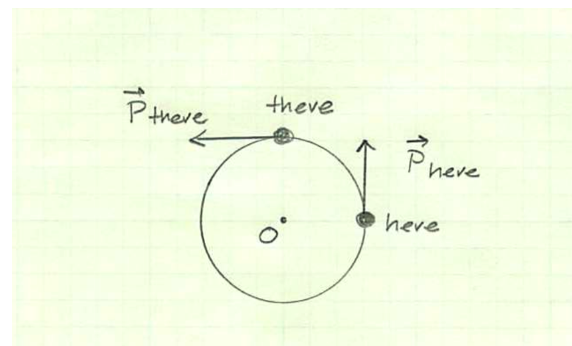


Figure 4.10: Two momenta shown for two stratetically placed points around the circular path.

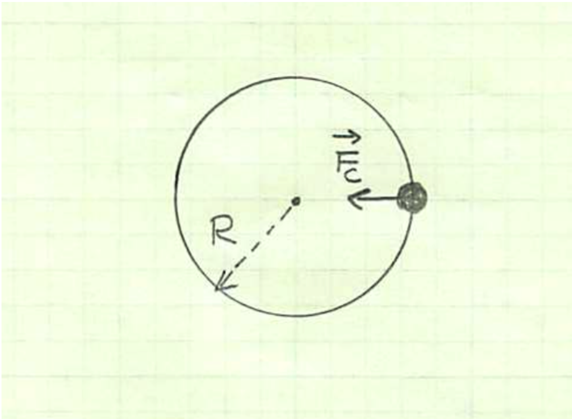


Figure 4.11: The centripetal force shown for a circular path of radius R.

Equation: Centripetal force..
$F_C = m\frac{v^2}{R}$ (mass times centripetal acceleration)

Figure 4.12: Cynthia Watt who won the 2015 Big Ten Outdoor Track & Field Championships in East Lansing, Mich. (Photo by Matt Mitchell)

4.6.1 Centripetal Force and Centripetal Acceleration

If this understanding of circular motion weren't enough, he went a step further in his paranoid sort of way. He actually found a relationship for what the centripetal force would be and did it both using his new calculus and in a strictly geometrical fashion. The latter he published in *Principia*, and like other such derivations, kept the calculus version to himself. Why? He feared being scooped. Calculus was his private tool for a long time.

Without going into those details, I'll just report the results. The centripetal force is special, and I'll call it F_C:

$$\vec{F}_C = m\vec{a}_C$$
$$\vec{F}_C = m\frac{v^2}{R} \tag{4.18}$$

Here, a "centripetal acceleration" is also assigned and is related to the distance from the object to the center as shown in Fig. 4.11.

$$a_C = \frac{v^2}{R} \tag{4.19}$$

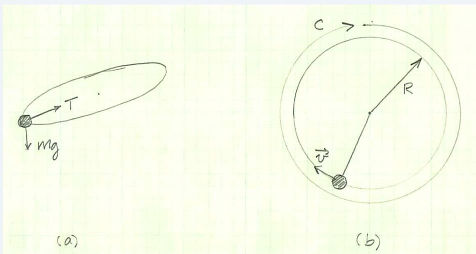
An object traveling on curve requires a force directed to the center. *Key Concept 11*

There are two ways to use this concept: If you want something to move in a curved path at a particular speed, you can calculate and apply a precise (centripetal) force—tug it—to make that happen. If you see that an object is *not moving in a straight line*, then you must be able to identify a centripetal force being applied to it! Sometimes identification of such a force is tricky. For example, for our car, what actually causes the car itself to go around a circular curve? Obviously that force is the force of friction between the road surface and the tires of the car. Reduce the stickiness of the road (ice, snow, rain?) and that force of friction is reduced and the force that's possible is reduced, sometimes considerably. You instinctively know this, so you drive slower (reducing v in the numerator of Eq. 4.18 to match the F_C that *can* be produced given the conditions.

Example 4.1

Hammer Throw.

Question:



The Hammer Throw is an old track and field event. For men, a 16 lb ball (7.3 kg) is attached to a chain that's approximately 4 ft long (1.22 m) and whirled around a circle and let go. Olympic-class hammer throwers spin their bodies incredibly fast—in their last "wind" before release they are spinning less than a second per revolution. Let's call it 0.3 seconds. Figure 4.12 shows a collegiate hammer champion at work.

1. Calculate how fast the ball is moving at that rotational rate.
2. Using that speed, what is the force that their arms must exert in order to keep the weight moving in a circle?
3. What fraction of the weight of the hammer is that force?

Solution:

1. In order to calculate how fast the hammer is traveling around its arc, we have what we need to know: We know how long it takes to make a complete revolution and we know how far it goes in one revolution is the circumference, C, of that circular path. Figure 4.1 shows the forces and the distances for our situation.

   $$C = 2\pi R = (2)(\pi)(1.22) = 7.67 \text{ m}$$

   So the speed is:

   $$v = \frac{C}{t} = 7.670.3 = 25.6 \text{ m/s}$$

   (This is about what the measured "escape velocity" is for world-class throwers, who can toss the hammer more than 80 m. Mr Google will quickly tell you that this is about 60 mph.)

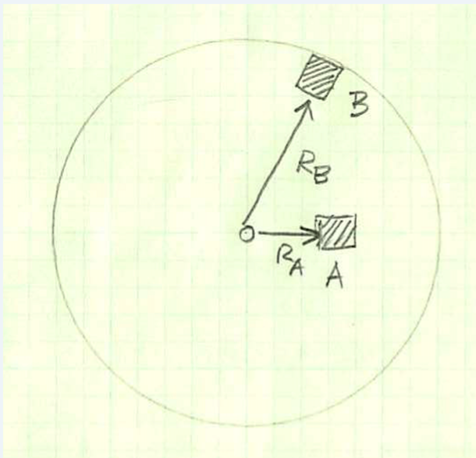
2. The force on the thrower's arms would be the centripetal force of the hammer, $F_C = m\frac{v^2}{R} = (7.3)\frac{25.6^2}{1.22} = 3900$ N which is about 880 pounds! The womens' hammer is 4 kg, so the force that they would experience for the same speeds would be about 460 pounds.

3. The weight of the men's hammer is $W = mg = (7.67)(9.8) = 75$ N which is 1/50th of the centripetal force: $\frac{3900}{75} = 52$.

Example 4.2

Playground merry-go-round.

Question:



The figure shows an upper view of a merry-go-round with two children at two different distances from the center. What is the force of friction required to hold child A on board? Is the force of friction required less or more to hold that same child on at B? $R_A = 3$ m and $R_B = 5$ m. The merry-go-round makes one complete revolution in 10 seconds and the child weighs 50 pounds, so 22.7 kg.

Solution: In order to know the force of friction required, we need to know the speed, which we get just like we did in the hammer throw example.

$$v_A = \frac{C_A}{t} = \frac{2\pi R_A}{t}\frac{(2)(\pi)(3)}{10} = 1.9 \text{ m/s}$$

The force is then $F_A = m\frac{v_A^2}{R_A} = (22.7)\frac{1.9^2}{3} = 27.3$ N which is about 6 pounds of force. Maybe sticky tennis shoes?
If the child stands all the way at the rim, is it harder or easier to stay on? A different, but related question: is that child moving faster or slower than the child who's closer to the center? You've all done it and you know that it can be very hard sometimes at the rim of such a playground device. So the faster it goes, the higher is the force required. If the force is just a constant (like friction), then we can derive a relationship that will tell us the force as a function of radius. Stay with me here. Your pencil is out, right?

$$F_C = m\frac{v^2}{R} = m\frac{(2\pi R)^2}{R} = m(4\pi^2)\frac{R^2}{R} = 4\pi^2 mR$$

...the further out you venture towards the edge, the higher is the force you need to apply to stay moving in a circle, linearly with your distance from the center. By the way, have you seen the multitude of YouTube videos of idiots speeding up a merry-go-round with a motorcycle laying on its side with the tire powering the rim? This is Darwin at work.

Example 4.3

Racing.

Question:

The first turn at the Indianapolis 500 raceway has a radius of curvature of about 800 feet (about 244 meters) as you can see in the picture. Racing tires are "flats" and have maximum rubber on the road surface for the most friction.

1. If the force at which sliding would start to happen is 1500 lb (6700 N), what is the maximum speed that a driver can achieve? An representative weight of an indy car is about 1600 pounds, or about 725 kg of mass and 7100 N.
2. How many factors of g does this force represent?

Here the force is fixed—it's determined by the road surface, tires, and weight of the car—and we need to know the speed. A little bit of speed increases the force quickly, since it's proportional to v^2. A race is all about speed, so a lot of engineering goes into keeping the car firmly on the track.

Solution: From the centripetal force equation, we can solve for the velocity:

$$F_C = m\frac{v^2}{R} \rightarrow v^2 = \frac{RF_C}{m} = \frac{(244)(6700)}{725} = 2250 \text{ So the speed is } v = \sqrt{2250} = 47 \text{ m/s ...which is about 103 mph.}$$

The number of "g's" that the driver would feel is the centripetal acceleration divided by g, g's $= \frac{1}{g}\frac{v^2}{R} = \frac{2250}{(9.8)(244)} = 0.94$. In fact, an Indy car can take that curve above 200 mph. Why is this different? First, the tracks are banked at about 9°; next there are front and rear air-foils that create a down-force, so hundreds of pounds of more "stick" between tires and road (the driver can adjust this); and finally, drivers "aim" at the turn more strategically than just going in a circle.

Drivers pull as much as 3-4 g in each turn! They must fight this force in order keep his or her neck straight four times around the track, for 500 miles: 800 times that the driver has to exert this strenuous resistance. Now you know why this race is referred to as "grueling."

The big use of centripetal force will re-emerge in the great triumph of Newton's, namely his law of Gravitation which we'll encounter in Chapter **??**. This really made his bones. But next, let's bang things together.