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Quarks, Spacetime, and the Big Bang

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"Nature Loves to Hide"

Heraclitus

Preface:

Quarks, Spacetime, and the Big Bang

QS&BB is a book designed to accompany a general education course of the same name that I've taught at Michigan State University for a number of years. Why? Well, there's a story there.

The North American approach to university education is nearly unique in the world. Citizen-students come to college in order to become proficient in a focused few areas of study (your "major") but are also broadly educated in many other areas ("general education"). So an English major would dive deeply into literature but also take courses in maybe physics, astronomy, chemistry, biology, geology, history, anthropology, psychology, etc. Likewise a physics major would study physics and mathematics, but also biology, literature, psychology, and so on. Every U.S. campus manages this deep-plus-broad approach to higher education its own way.[1]

Creating courses for non-specialists in the sciences is especially challenging, but important since many of society's big problems are scientific at their roots.[2] An informed citizen needs to understand some scientific facts, but also appreciate the scientific method as all too often, controversy swirls as much around what is or isn't "science" as it does to the details. How best to do this in physics?

There are many physics courses for non-science college students. The traditional course is often called "Physics for Poets," which is a conceptual (less mathematics) version of the otherwise full-physics curriculum taught to science and engineering students. But there are other paths which teach physics by shining a light on particularly interesting topics in accessible presentations.[3]

[1] This approach to higher education is credited to the Harvard University president Abbott Lawrence Lowell who began transforming undergraduate education in 1909. Under him, fields of concentration (majors) were established along with required sampling of courses outside of majors, the distribution requirement. "A well-educated man must know a little bit of everything and one thing well." affected college education across America to this day.

[2] Climate change. Energy production. Evolution and big bang in schools. Nuclear power. Nuclear proliferation. NASA. NIH. Vaccination. Pandemics. Weather. Health effects (or not) of common radiation sources. Peer review. Basic versus applied research. And so on.

[3] Many physics departments will offer astronomy courses (or of course, astronomy departments will when they exist), physics of music, physics of energy issues, physics of light, and so on. Our department is no different in that respect. By the way, 50,000 students take college-credit astronomy every year in the United States!

What QS&BB Isn't

This book is not a comprehensive survey of all of physics. A student will not be expected to solve many of the standard "physics class" problems—QS&BB is intentionally, mostly conceptual. Many topics which would be in a conventional course are not covered here, or touched on lightly. For example, there is no chapter on thermodynamics nor on energy production or climate. Motion and forces are only presented for one-dimensional situations and only sufficiently to appreciate relativity. Electricity and magnetism are covered in a descriptive way, with only a few quantitative examples. "How things work" is sometimes covered, but less so than from the usual survey course.

We cut a strategic path through "classical" areas of physics in order to accumulate the concepts, quantities, and vocabulary that would apply to a conceptual appreciation of relativity and quantum mechanics, both of which are the jumping-off points to our two main topics.

The level of scientific literacy among college-educated young adults in the United States always ranks among the top two or three among all nations of the world. This research has been done over decades by Professor Jon Miller of originally, Northwestern University and Michigan State University, and now the University of Michigan. In an article for the Association of American Colleges & Universities ("What Colleges and Universities Need to Do to Advance Civic Scientific Literacy and Preserve American Democracy" `https://www.aacu.org/node/2139`) he explains why U.S. results are so positive: "The answer is college science courses." He goes on to note that "The United States is the only country that requires all college students to take one or more science courses as a part of a general education requirement. In a series of statistical analyses using structural equation analyses of both cross-sectional and longitudinal data, I have shown that exposure to college science courses is a strong predictor of civic scientific literacy in young adults and in adults of all ages (Miller 2010a, 2010c)."

What QS&BB Is

My aim is to help you appreciate two of the more exciting "fundamental" topics in physics, particle physics and cosmology. You'll come to appreciate our current picture of how our universe began and what open questions continue to motivate thousands of us around the world. In order to get there we need a working knowledge of some of the classical subjects and these are presented in the early chapters in a gentle way. We start with a conventional, but abbreviated approach to the classical subjects of mechanics and electricity and magnetism with some simple algebra-based descriptions and examples in the early chapters. After about a third of the book, this light-mathematical approach evolves into a more conceptual narrative where we tackle modern-day topics. The Chapter 1 describes how the book—and the Michigan State course—are organized in more detail.

I emphasize biography. We'll meet intellectual giants whom everyone has heard of, but also our professional scientific heroes whose images are *not* on tee-shirts. The history of physics and astronomy is full of unusual people—and a lot of just plain folks—and I'm eager for you to think of us without white coats and strange manners. We're regular people who chose career paths that are a little outside of the mainstream. But we're not so special except that we are privileged to be supported by the public in order to do our work.

I'm an experimental particle physicist and I've been teaching physics to physics majors and especially non-science students for more than three decades—I hope well. I know I have fun doing it. I'm lucky enough to be continuously supported by the you[4] for my research in Particle Physics for three decades

[4] The National Science Foundation, specifically.

and I'm grateful. In some ways, this book and course are in partial repayment for that support.

I like knowing how the universe works and I've never met anyone who didn't share my curiosity. Even after a lifetime daily immersed in these matters, I'm constantly in awe at how beautiful it all is and how lucky we are to know as much as we do. I enjoy talking about it and teaching some of the details.

Figure 1: You can find more about me at `http://www.pa.msu.edu/~brock/`. You'll get to know me as I tell you stories in the pages that follow. Unfortunately, I'll not be able to meet you!

I'm not stuffy. I've tried to write here like I teach, which is informally and hopefully without pretense. I'm deadly serious about the science and passionate about the subject-matter. But I also like to have fun and hopefully I'll make you smile every once in a while and help you to grasp complex ideas. Stay with me, and you'll be able to explain Special Relativity at parties just like I can![5]

[5] Wait. That's not necessarily a selling point.

Chapter 1

Introduction

Studying the Smallest and the Largest

The Large Hadron Collider, looking south across Lake Geneva and the Swiss Alps

"In the beginning, the universe was created.
This has made a lot of people very angry and been widely regarded as a bad move." *Douglas Adams*

We're about to follow a Big Story —the "just so" story of the beginning of the universe. Yes, that one: Everything. The plot of this story seems to have all sorts of twists and turns that we're still unraveling. Surprises await.

Of course, the details are where the devil resides and they are fiercely complex. So much so that two entirely different scientific communities are deployed to battle with nature: those of us who work on the "outside" and those who explore the "inside." The outside crew are astronomers and astrophysicists. They measure and characterize the constituents and nature of the cosmos. They look *out.* The inside teams mimic the earliest picoseconds of the universe by recreating its incredibly hot, adolescent conditions in laboratories here on the Earth. These are the particle physicists and they look *in.* This is the story of both.

What's the smallest real thing that you can know about? For people of my *grandparents' generation*, the sophisticated answer would be "what you can see." I was born in the year 1950, and so my grandparents would have been children about 1900 which is when physics got interesting. They would have been taught that to claim existence for an object that the naked eye could not see would be to utter an absurdity. Chemists spoke of atoms, but were disdainful of anyone who thought they were real. They were just a shorthand picture for how to visualize elements. Physicists were even less flexible.

For people of my *parents' generation*, the answer would have been "protons, neutrons, and electrons." The atom had been thrust into believability around the turn of the century, and then refined during the next two decades. But the neat planet-like picture of the atom was where it all stopped for many.

In *our generation*, the answer to the "smallest" question has been "quarks and leptons" ...but with the full expectation that they may not be the end of the "smallest" story. We approach this question differently now. We're hard at work, as "we speak" using brand new tools to explore further than ever before.

"Quarks"? "Leptons"? Lots of jargon and I'll keep it all straight for you as we go along. For now, quarks are itsy-bitsy pieces of the proton and leptons include the electron and others.

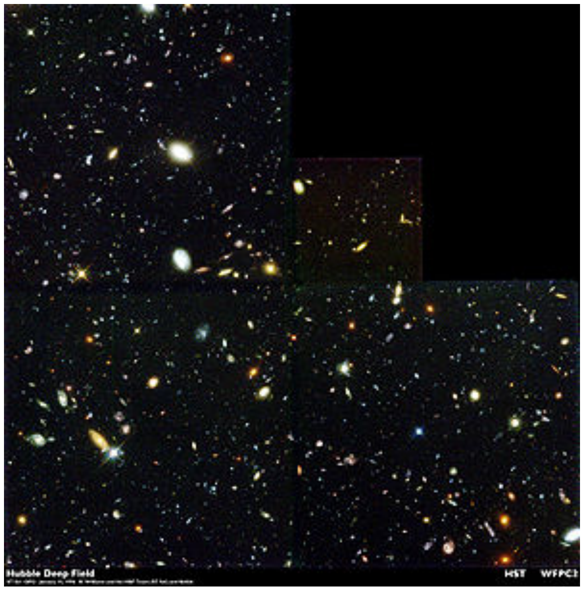


Figure 1.1: The so-called "Hubble Deep Field" view of a tiny spot in the sky, filled with 3,000 galaxies.

In *your generation*? The sky's the limit! We've hints at solving some old puzzles and we'll undoubtedly find new ones. We're developing and deploying amazing new instruments and theoretical ideas now rub shoulders with not just nature, but philosophy and the deepest questions asked by humans. Your generation is going to see amazing things.

Through decades of intense experimentation and imaginative theorizing, the tiniest bits of reality are turning out to be a fascinating collection of objects. In the 1950s and 1960s, we just stood back and tried to catch the hundreds of particles that our experiments spit out at us. New particles every year! Names that nobody could remember. Hundreds of them, which was ludicrous! Didn't nature have some plan?

Well we uncovered a plan that we think is a very good picture of how much of the fundamental particles of the universe work together and we've been exploring it since the 1970s. We've knitted that earlier mess together into a coherent picture of the entities themselves as well as the rules that govern how that stuff behaves.

But we're unhappy. Our grand synthesis of the Tiniest Bits Story—called the Standard Model — now looks a little shakey. While its been the gold-standard of the successful scientific theory, we expect that *new* tiny bits are lurking in our experiments and we will be astonished if nothing shows up as we dig deeper.[1] This new anticipation would have been met with blank stares only a couple of decades ago.

So much for inside effort.[2]

[1] These new states of matter might be: "additional quarks, the Higgs Boson, Supersymmetric Particles, Weakly Interacting Massive Particles (WIMPs), Dark Matter particles. . . ," all famous candidates for future discovery. Of course whenever we get too cocky, nature plots to surprise us with something completely unexpected—more often than we'd like to admit! So, we're instinctively wary of being too sure of what's coming.

[2] See the frontispiece of this chapter!

Okay. So what's the biggest real thing you can know about? For people of my *grandparents' generation* the learned answer to this question would be "the size of the Milky Way," which they would have been taught constituted the whole universe. Everything visible in the night sky was thought to be a part of one big, but still cozy cluster of stars which we see to be densest around the southern sky (from North America). Not only was my grandparents' universe compact, it was supposed to be permanent—static and unchanging— built of three kinds of objects: planets, stars, and clusters of stars. Stars twinkled, planets were steadfastly bright, and clusters of stars were fuzzy, indicative of their presumed distances from us. Sure, they all moved with regularity during each night and shifted slightly in a year, but the large scale structure of my grandparents' universe was simple: a nice, intimate, dependable universe.

For people of my *parents' generation*, the universe suddenly become huge. Those fuzzy clusters were found to be other galaxies outside of the Milky Way which are surprisingly far from us—we're not alone in our comfy galaxy. They were taught about thousands—we now know, billions—of others, of which the Milky Way is a relatively modest and ordinary example. But, the real shocker was the overthrow of the static universe of my grandparents. My parents' universe was found to be flying apart—expanding—at a breakneck speed. No longer a tight-knit, stable thing...the universe is now huge and reckless.

The really unsettling piece of news for *my generation* is that the Big Questions of antiquity are now legitimate scientific research programs: Was there a beginning to the universe?[3] Are we alone? Will the universe end? Are there other universes? Was there anything *before* The Beginning? What drives the expansion of our universe to accelerate? The outside crowd thinks big thoughts now and this is a development of only the last couple of decades.

[3] There was a battle royal between two competing models of the universe in the 1950s. The first was dubbed by a proponent of the second, the "big bang"—not as a compliment. The second model was called the "Steady State" model. We'll talk more about these later. This battle raged until I was in high school.

When I was in graduate school, a professor told me that Cosmology was "physics knitting." Not any more! Cosmology in my and especially *your generation* is going to be flat-out amazing!

1.1 An Auspicious Beginning

Yes. The observable universe had a beginning and quite a beginning it must have been: a roiling mess of radiation and elementary particles at temperatures never to be seen again. Everything that is would have been confined into a size smaller than the smallest particle we know of.[4] Unthinkably dense and with growth that was stunningly rapid, our early universe defies imagination. It's so outrageous that comprehending it seems a job for fiction and not science, yet my generation has also found ways to explore it: we probe it through direct telescope observations and we remake it in particle collisions. This is the blending of the outside with the inside pictures that motivates me.

[4] Maybe. Maybe not.

Wait. *I don't believe in the big bang. You appear to, but Isn't what you think just another "belief"? Aren't we each entitled to our own beliefs?*

Glad you asked. *"Believe" is a tricky word that we all use, although in our context, we should be clear. When I say "I believe in X," treat that as shorthand for the sentence: "X is highly confirmed by experiments and X likely to survive any future experimental test." If I'm an expert in the field of X, then I have the obligation to describe those experimental tests. If I'm not an expert in X, I should expect that an expert could also enumerate its experimental successes in detail. There are do's and don't's about this in science. About scientific belief, I can't do three things: 1) I can't say that I believe in X because I want to, 2) I can't say that I believe in X because my gut or a "feeling" tells me to, and 3) I can't say that I believe in X because a non-expert or an ancient text tells me to. Likewise, I* can't *say that I don't believe in X for any of those same three reasons. Stay with me. What I'll show you are amazing things and a record of success that's hard to ignore. Science is a process as well as a collection of theories!*

Quarks, Spacetime, and the Big Bang (which I'll affectionately refer to as "QS&BB") tells the interleaved stories of the two sciences of Particle Physics and Cosmology and how they have come to be blended together into a believable picture of how we all came to be. We're deep into the narrative—the plot is well understood, the characters are developed, and a "can't put it down" fever has set in. We're eager to see how it comes out and we're doing experiments all around the globe—and in orbit *above* the globe and in deep underground laboratories *inside* the globe— to push ourselves to the story's climax.

Definition: Particle Physics.
The study of the smallest bits of energy, matter, and the rules that govern their interactions.

1.2 The Inside Game: Particles and Forces

Sure, we've learned a lot in the last four decades about the Particle side of this story—my whole professional life. But, what's particularly interesting about this coming decade in EPP is that we've reached an impasse. We have bushel baskets full of theories about what should come next, but we're starved for new data which will direct us on how to sort out the various theories. You and I are going to explore that situation because new data are coming in right now at extraordinary international laboratories. The coming decade is going to be interesting.

Definition: Cosmology.
The study of history and the future of the whole universe.

The inside story is that of "Elementary Particle Physics," (aka "EPP") or as it's often called, just "Particle Physics," while the outside story is that of "Cosmology." We'll travel these narratives sequentially from their common beginnings.

The Particle Physics side is a well-established field practiced by about 10,000 of us in nearly every country and with major labs on four continents: North America, Europe, Asia, and Antarctica.[5] We build accelerators to provide beams of electrons or protons and crash them together. We then collect the debris from those collisions in gigantic "detectors" that allow us to unravel the products of those collisions.[6] Or we build detectors that are exposed to cosmic particles. EPP is one of many sub-disciplines in physics, but it's a little different. The questions of most of science have evolved in time as people became smarter and new problems became interesting. New disciplines sprang up as things got more complicated and challenging.[7] In contrast, while Particle Physics has become specialized and sophisticated, its goals have always been intensely focused on two questions:

[5] Isn't Antarctica a continent? Yup. Lots of experiments at the South Pole.

[6] Chapters **??**, **??**, and **??**.

[7] For example, Nuclear Physics and Particle Physics were practiced by the same people until the 1950s when they naturally split into two different subfields of physics. One group pursued the intricacies of more and more complex nuclei and the other pursued the complexities of the simplest objects. Each approach requires specialized devices and each separate theoretical tools.

What are the most elementary particles in nature? *Key Question 1*

What fundamental forces act among those elementary particles? *Key Question 2*

We think that getting closer and closer to answering these two questions will lead us to a deep understanding of the early universe. Paradoxically: understanding the tiniest things in nature will help to understand our "origins" which have been debated and argued for 2500 years.

Box 1.1 A little philosophy

By the way, do you see how these two key questions are different? The first one asks about the existence of "things." An inventory. The second question asks about physical laws *among* the things. We're realists, which is to say, we think that things are real and that our theories are about real processes. While these two ideas are debated in philosophy, scientific realists would refer to these two questions as "entity realism" and "theory realism." The former is more easily defended than the latter. But, we're not philosophers. We're scientists and we believe that the discovered laws of nature are factual statements about how things work. Enough of this.

These first two questions were stated carefully, so let's take them apart: "elementary," "particles," and "forces" are all three specific ideas in my world that have different meanings from normal peoples' worlds! How about parts?

1.2.1 What's "Elementary"?

The most basic qualification for some entity of nature to be "elementary" is: no parts. Most things have parts: stars, trees, molecules. Even an atom has parts—the nucleus, which can be multiple protons and neutrons, and the atomic electrons all constitute "parts." So, an atom is not elementary and not a subject of our investigations in particle physics. Every nucleus has parts and those parts—the proton and neutron—have parts![8] The electron? No parts. It's elementary.[9]

[8] Chapter 23

[9] So far.

An elementary particle is a bit of matter and energy that has no constituent parts. *Key Concept 1*

"Elementary ≡ no parts" is a simple idea.[10] But, as you'll see if there was ever a pattern in 20th century physics, it's that thinking hard about simple ideas quickly leads to the weird. A part of the theme of this book is to emphasize how simple ideas about nature can become wonderfully complex.

[10] The symbol ≡ means "defined as" or "equivalent to."

"Here's your moment of zen": "Simple means complex"!

1.2.2 Why "Particles"?

Here's one story: It appears that nature "clumps" energy in particular ways. If somehow we could prepare a bundle of energy—say, really-really hot, radiant energy—it will appear to condense quickly into very specific objects which go by various names: the fancy designation would be "quanta." But, what is left over after such an energy-bundle settles down are the "particles" we know and love: electrons, protons, and neutrons...and the other sets of particles that are relatively new: quarks, leptons, hadrons, bosons, and presumably those that we've not yet found. Nature makes only these particular states, some of which are produced readily, some rarely. Why? What governs how this happens?[11] Further, we get only whole electrons, not half electrons. Why?[12]

[11] We don't know!

[12] We don't know!

> Before Maxwell, Physical Reality ...was thought of as consisting in material particles.... Since Maxwell's time, Physical Reality has been thought of as represented by continuous fields, ...and not capable of any mechanical interpretation. This change in the conception of Reality is the most profound and the most fruitful that physics has experienced since the time of Newton. Albert Einstein, in "Maxwell's influence on the development of the conception of physical reality," in *James Clerk Maxwell: A Commemorative Volume* 1831-1931, (The Macmillan Company, New York, 1931), pp. 66-73.

But, things are not quite as simple as that first story of everything as particles. There's another story, and it's actually closer to the truth. As we'll get a taste of later, when we combine the theory of quantum mechanics with the theory of relativity, we find that the basic "stuff" that eventually arranges itself into atoms, people, and stars is actually a set of continuous *fields*.[13] Now that's disturbing since a field is everywhere, but a particle is "there." So there's the appearance of a conceptual contradiction and physicists have been working it out for more than 80 years. Notice, I didn't say that there's a logical contradiction in quantum mechanics. It's the most accurate description of nature ever devised! No, the problem is ours: this "conceptual contradiction" is one that exists between our ears as we try to translate our mathematics into pictures that we can keep in our heads and yes, write in words.[14]

[13] Chapters 13 and ??

[14] Chapters ??, ??, and ??

I have to admit to the mental crutch of *particles*. Even though the mathematics seems to require that all states of matter have both wave-like and particle-like properties for EPP it's easiest to mostly use the mathematical language of particles and that language came to us from Richard Feynman.[15]

[15] Chapters 6.5, 19, 21, 22, and 25

So, one side of my brain is full of the sophisticated symbols and manipulations of the relativistic quantum field theory that precisely describes this stuff. But the other side of my head is full of images of billiard balls bouncing off of one another: colliding particles. It's not an entirely satisfying picture since in order for this analogy to be precise, my mental quantum billiard balls should also randomly decay into other billiard balls—or into baseballs or bananas,— should pass right through other billiard balls, and even spontaneously leave my pool table and appear on someone else's! But, we have to cling to some picture in our heads and that's mine.

1.2.3 What "Fundamental Forces"?

"Force" is one of those words that has many colloquial meanings.[16] But in physics a force is a precise concept—a noun and not just a verb. Here's the simplest notion of a force: if you've changed the motion of an object, you had to exert a force in order to do so.

Definition: Force.
Anything that alters the state of motion of an object is a force.

[16] "May the Force be with you."
"You can't force me to eat that!"

Everyday Forces

You and I deal with three kinds of forces every day. Let's talk a little about all three...and then how the forces in Particle Physics are different from these.

First take regular pushing on something—whether it's a push that's through muscles against something, or the push of a tire (or your shoe) against the road—seems to be a separate force of its own. You have this mental picture that when you touch an object and push that this mechanical thing-to-thing contact is the cause of the object's change of motion. You might be satisfied with the phrase "mechanical force" as all you need to say and you'd be consistent with its modern usage in engineering. File that away for a moment as a particular kind of force.

But what about a magnet? Surely at one point in your young life, you've played with a pair of magnets and marveled at the fact that they seem to "communicate" with one another. Without touching, and without any obvious connection between them, a force is transmitted through thin air. Here's another one that doesn't need direct contact: your hair's state of motion is affected on a cold, dry day by a comb—your 'do rearranges itself as if by magic without touching the apparent cause of the hair motion—a statically charged comb.

One of the neat stories we'll uncover is that the relationship between your hair's unruliness in January and the dog's picture sticking to your refrigerator is an intimate one: they are both examples of a single force, the "electromagnetic force" and understanding that will take us into Albert Einstein's young life.[17]

Shh. Now, a well-kept secret: the mechanical thing-on-thing pushes and pulls of everyday life are actually electromagnetic: the reason your hand doesn't go right through the box you're pushing is because the electrons in your hand are repelled by the electrons in the surface atoms of the box and so you... push it.[18]

So the idea of a force in EPP is not what you obviously experience in everyday life. Nature's fundamental forces are very precise and very selective pushes and pulls that exist among particles. "Precise" because there is no wiggle-room. The force between two electrons some distance apart is precisely the same as the force between any two electrons with that separation. "Selective" because that same force would exist between two protons of that separation, but be zero if the electron or proton is replaced by a neutron. The electric force only acts on particles with the attribute of electric charge. And finally, the forces are all of different strengths. Your dog's picture stays on the refrigerator and doesn't fall to the floor because the force of gravity is very much weaker than the force of electromagnetism.

[17] Chapters 13, ??, and 15

The electromagnetic force is blind to anything but the amount of electric charge. If we place a charge of $+Q$ on two ants, they will be repelled by that electrostatic force between them. If we place a charge of $+Q$ on two elephants they too will be repelled...by exactly the same electrostatic force. Electricity "sees" only electric charge.

[18] The reason you don't pass right through the floor is due to the same electrostatic force.

Definition: Electromagnetic.
The theories of electricity and magnetism were shown in the 19th century to be actually a part of a more fundamental theory which has come to be called "electromagnetism." The forces, both electric and magnetic, are called the Electromagnetic Force.

1.2.4 Particles, Forces, and Theories

Just enumerating particles is like physics-stamp-collecting. But if there was a confirmed theory that linked the forces and the particles into *a single, deductive story*? Well, that would suggest that we'd learned something important about nature.[19] One of the amazing accomplishments of the last three decades in particle physics is that we do have a particular theory, called colloquially "the standard model" that explained how the forces originated and predicted the existence of particles which were eventually discovered.

[19] Chapter 30

How Many Forces Are There?

How many forces nature knows about (we just talked about two fundamental forces: electromagnetism and gravity) and how they act on different constituents is for us to figure out. Unlike the everyday mechanical force, where I can push on a wooden box as easily as I can push on a lawnmower, fundamental forces are related to rather specific qualities of particles. The electrostatic force only "sees" electric charge, the gravitational force only recognizes mass: anything that has a mass feels an attractive force to all other massive things. It doesn't matter what color it is. It doesn't matter what material the two objects are made of, it also doesn't matter what their electric charge is, if they have mass, then the gravitational force is going to act... and do so by attracting them together.[20]

[20] No gravitational repulsion here...or is there? Stay tuned.

Now, the story is a little more complicated than this introduction. For the record, just to be complete, it's not only mass, but also energy that is affected by the gravitational force. And, we've some reason to think that traditional gravity may even have a repulsive component to it on cosmological scales! Finally, the gravitational force is the only one that resists an explanation using quantum theory, and so it holds some really well-kept secrets that we would very much like to uncover.

Nature is pretty economical. If there are 12 kinds of particles in the universe, you might guess that maybe there's one force, or 6 or 12. But, it turns out that there appear to be only 4. We've encountered two of them: gravity and electromagnetism. But, by experimenting for nearly 100 years, we've found that there are two more forces. And, like electricity and gravity, these forces also pick out particles with particular attributes and ignore the others. Some particles respond to just one force. Some of them respond to two or three. We want to understand this. Badly.

Besides electromagnetism and gravity, the other two forces are called, get ready: the Weak Force and the Strong Force. They are, as you might guess, weaker and stronger than some others.[21] We'll talk a lot more about these later, but from weakest to strongest, the forces order themselves:

[21] We have a lot of fun naming things in particle physics.

1. the Gravitational Force,
2. the Weak Force,
3. the Electromagnetic Force, and
4. the Strong Force.

So, in reality, the only two forces that we experience in everyday life are electromagnetism and gravity. The others act behind the scenes.

1.2.5 Particle Confusions

Our standard model now has no missing pieces. It's a complete description of just about everything that we've ever manipulated on Earth! This result was sealed in 2012 with the announcement that we had found a strange particle called the "Higgs Boson" in our experiments at the Large Hadron Collider at CERN. But we're not happy. And we're not happy for two reasons. First, there are experimental reasons: something's going on in the universe that causes galaxies to move oddly (see below) and something's going on with nothing. That is, with the vacuum, which we tend to think of as related to that theory of fields that I described above. The second reason that we're not happy is that the standard model has some formal features about it that don't quite sit right with us. We need to do an odd thing in the mathematics to get it to work and we're pretty sure that this "odd thing" should have a formal basis and not be quite as *ad-hoc* as it seems.[22] Let's go large.

[22] Want to know what that odd thing is? We take an equation, and we change the sign of one term from negative to positive. No particular reason. . . except that it works. Stay tuned, you'll see.

1.3 The Outside Game: The Big Bang

As I've indicated the big news of the 20th century is that our cosmos appears to have had a beginning. Astrophysicists have made huge strides in the last three decades with amazing instruments on Earth and in orbit. Once the big bang was hinted at in the late 1960's, satellite observatories have sealed the deal. Our universe had a beginning.

Definition: Astrophysics.
The study of the dynamics and the origins of astronomical objects.

Stand back and think about the implications: this is the most remarkable scientific discovery in history. Of all of the ways people have thought about their place in the world, over thousands of years there was only speculation and myth about a possible Beginning. After decades of patient research, we know: there was a time—before which there was nothing. Suddenly, in the blink of an instant space, time, and the energy of matter and radiation were born and the subsequent cooling eventually caused our universe to evolve. Into us.

Cosmology is an old, old metaphysical or religious subject (habit?), but it only became a *science* in the last century. Traditionally, Cosmology is the story of the whole of the universe. From the creation stories to the "just-so fables," humankind used mythology and belief to orient themselves with the universe they could see. There was the strong sense that the whole of the universe was bigger than what humans could imagine.

Some would call this later version, Physical Cosmology in order to distinguish it from the precursor story-telling. (I'm looking at you, Wikipedia.) But we'll just call it plain, old Cosmology.

Well, we don't "imagine" any more. We measure. Cosmology is a new science and it became one in the hands of Albert Einstein in the early twentieth century. Things didn't quite go as he'd planned, as we'll see. But he laid the groundwork for a human-based study of the universe using mathematical rules rather than mythology or belief. Today it's among the most exciting branches of all of physics.

The two basic questions that modern cosmology tries to understand the answers to are these:

What are the past and future histories of the universe? *Key Question 3*

What are the ingredients of the universe? *Key Question 4*

These questions area alsoo carefully stated. So let's unpack "universe," "history," and "ingredients."

1.3.1 Histories of the Universe?

You know the meaning of "Universe," right? It's...well...it's everything. At least that's what it used to mean. We'll consider a growing suspicion is that a *universe* might be a relatively local object and that there might be room for an interpretation of the whole cosmos that could incorporate other *universes*.[23]

[23] Now, did you ever think that there could be a plural of that word?

Perhaps you've read that there is consideration of a "multiverse" in which there are an infinite number of universes which are born and die spontaneously and for eternity. All of them would have different physical laws and so different particles and varying potential for life. To some, still unconfirmed mathematical models push to this conclusion. To others, this is speculation that's beyond wild.[24] We'll talk about why the multiverse is a topic for science seminars and not just comic books. On this, we'll be agnostic. Just the facts, ma'am.

[24] For some, even reckless and unscientific.

But in order to be specific, we should try to define what our universe would entail. Our universe is

1. the one in which we (or our original elements) reside,
2. the one where the same physical laws work throughout, and
3. the one that had the big bang that our evidence points to..[25]

[25] It ain't much, but it's home.

Certainly, the past history of the universe is the hot[26] topic in all of cosmology.

[26] No pun intended.

Past History

Our inference to the need for a beginning—a big bang—comes from a) the fact that the universe is expanding, b) that we therefore infer that it was smaller in the past, and importantly, c) that we have a plausible, predictive model that describes this situation. Both the fact of the big bang and the stories that led us to this conclusion are fascinating and we'll spend quite a bit of time unraveling them.[27] But just how this happened is a matter of urgent research.

[27] Chapters ??, 27, and 29

We can play the universe-movie-camera backwards in our models and know where "the beginning" should be in time. The original "$t = 0$," nominally called the big bang. In the conventional model of cosmology we can reliably predict[28] the times at which atoms were formed, then when nuclei would have formed, and then even when protons and neutrons would have been formed. At that point the universe would have been unbelievably hot and dense and only consisted of the most elementary of particles. This birthday of matter is about a picosecond after the big bang: when the universe was about

[28] post-dict?

0.000000000001 seconds old.

Let's call this the "Electroweak time," which we'll study later.[29]

But as tiny as the Electroweak time is, it's still not zero. If we keep pushing our models earlier, we reach a point where our mathematics fails us with infinities. This is at called the "Planck time" and is at 10^{-43} seconds, or...dare I write it?...it's at

0.0000000000000000000000000000000000000000001 seconds.

This is a defining point. Before the Planck time, the very concept of "time" would not exist.

Anyway, between the Electroweak time and the Planck time a lot of physics had to have happened. We've models of what events to insert in that interval and tests we can perform to lend credence to them. [30] So the good news is that from the Electroweak time forward until now, we can explain how just about everything evolved. The bad news is that before that point, we come up against the 800 pound gorilla-question: **what "banged"?** The idea of an explosion of the loud-noise sort is not correct. So the past history of the universe is an active area of research, world-wide. Theory and experiment, in astrophysics and particle physics all work together on this.

Now put on your seatbelt. Could there have been a "before the big bang"? The general consensus is "yes" and the front-runner model—what some have called "an amendment to the big bang"—is called **Inflation**. This 30 year old idea predicts that at about 10^{-35} seconds the universe went through a phase transition, not unlike when water boils. Before that point, there was only the vacuum...a bubble of nothing. After the inflationary event, radiation and particles were created and our universe evolved until today.

This going from *nothing* to *something*—dubbed the "ultimate free lunch" by inflation's inventor—is heady stuff. But it's testable stuff. And it's bizarre stuff since inflation is part of the inspiration for the far-out notion that ours is only one of a "multiverse." This hypothesized infinite collection of other vacuum bubbles would be eternal (time wouldn't exist) and would be spawning other universes for all eternity. Some might become full-fledged universes with particle and laws amenable to making stars, galaxies, and carbon-based life. Some might not.

Future History

So with that out of the way, what is a future history? Well, I'm playing a word-game with you since we'll see that in physics the direction of time becomes a different sort of thing than what we're used to. But the eventual fate of the universe has been a matter of mathematical modeling since the 1960s. The universe could logically expand forever; stop, shrink, and collapse; or slow down and become static. Nobody was prepared for the surprise of 1999.

[29] Chapter 30

[30] To complicate matters, space and time are intimately involved in this event in ways that we can't yet rigorously pin down.

The results of independent measurements of a particular type of supernovae and their speeds and distances led to the conclusion that not only is the universe expanding, but that expansion appears to be *accelerating*. Something seems to be pushing space to stretch faster and faster and we're not sure what it is (but Inflation can accommodate it). Taken at face-value, the future seems grim for this universe. At some point the expansion will be so fast that light would not be quick enough to be able to travel from one galaxy or star to another. Every celestial object will become isolated. Anyone left alive on any planet in this universe would see only... **black**. It will be a lonely place.

Another future history comes from a model from physicists at Princeton in which after the universe's novel birth and then big bang-ish evolution it would actually experience a contraction of space, all the way to an eventual collapse ("Big Crunch"). And then the whole process would start over: the universe would be cyclic. An endless repetition of groundhog day cosmic repeats. In this scenario there is no unique beginning, but rather an endless series of beginnings.[31]

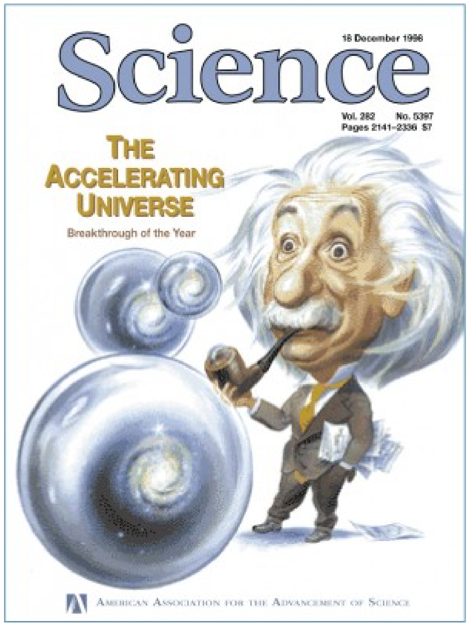


Figure 1.2: sciencemag

[31] This model is also consistent with the accelerating universe, but ascribes the cause differently from inflation.

So you can see that while the knowing the past and future of the universe are age-old quests their unraveling might be puzzles that humans can actually solve. Our two most compelling models are physically different and even *philosophically different*! Inflation assumes that time had a beginning, while in the cyclic picture time is perpetual—never starts and never ends. Appreciating the details of these and other advances are a part of the QS&BB mission.

Time to lie down for a bit and let this sink in.

1.3.2 Ingredients?

In order to inventory the ingredients of your world, you just look around you. Houses, clouds, Earth, the Moon, the Sun, stars, and so on. But the ingredients that I'm speaking of are courser-grained. First, the universe is incredibly big—and we'll get a sense of that—but the average amount of actual stuff is actually quite small, not much more than about 3 protons per cubic meter overall. So the overall density of the universe is minuscule, pretty smooth, and pretty much dominated by hydrogen atoms. So ingredient number one? The simplest element of all. All of interstellar and intergalactic hydrogen was born out of the big bang. All of the other elements[32] are made in stars.

[32] except for tiny traces of helium and lithium

An inventory of the other ingredients depends on the epoch in which we make the list. During our current era, we'll care about galaxies, a few spectacularly destructive stars (supernovae), and some stellar and galactic black holes. The atomic hydrogen and these shining objects are what we can apparently study directly since they all emit radiation. Thirteen billion years ago, we couldn't have included galaxies and thirteen and a half billion years ago, there would have only been particles and radiation. So understanding the evolution of the ingredients of the universe is a major undertaking, backed up with very sophisticated computer modeling and very precise satellite observatories.

In addition to the regular stuff of which stars are made, there are other ingredients which are more exotic. There is the radiant energy all around us left over from about 300,000 years after the big bang, and this Cosmic Microwave Background is now the object of many precise space missions.

Even more strange is whatever it is that dominates the motions of galaxies. They don't rotate the way we expect, given the otherwise reliable laws of gravitation. No, their motions suggest that they're (we're!) surrounded by unseen (not shining) stuff that gravitates but doesn't radiate: Dark Matter is our intriguing name for this stuff. There's room for Dark Energy in both the Inflation and cyclic cosmologies.

Finally, the most fascinating ingredient of the universe seems to be nothing. That is, the unseen force that seems to be pushing everything into that newly discovered accelerated expansion, might be a feature of the vacuum.[33] When we don't know what something is, we name it! "Dark Energy" is the placeholder name for the mysterious "something" that also is a target of frantic experiments and theoretical work.

Figure 1.3: vacuum

[33] We'll talk a lot about the vacuum, which until this discovery was the province of particle physics. Now both cosmology and particle physics intellectually own nothing!

1.3.3 Cosmological Confusions

In Cosmology we face some flat-out observational or *experimental* embarrassments. For example, when we add up all of the mass-energy of all of the objects that we can see using all of our observational tools (optical telescopes, infrared telescopes, microwave satellite telescopes, radio telescopes, etc.), 95% of the mass of the universe is missing. No kidding.

A part of the missing stuff appears to be that Dark Matter (about 30%) and the rest seems to be made up of Dark Energy. When you take the paltry 5% of shining stuff and add in these two "Dark" ingredients, it actually works out to 100%! This is a major victory for the "standard model of cosmology" or the "hot big bang model" (two names) and getting there is a part of the QS&BB story.

But we're confused about what Dark Matter and Dark Energy actually are. Embarrassed even. So there are major programs all over the Earth to study them.

Want stranger? Where are the antimatter galaxies? We don't see any evidence of relic antimatter in the universe. Only matter—the stuff we're made of. So either the universe began with an artificially enhanced matter dominance—an "initial condition" that is not scientifically acceptable—or at some point the originally *symmetric* matter-antimatter soup became our *antisymmetric*, matter-dominant outcome.

It gets still stranger. If you look at the sky to the West and do a careful analysis of the distribution of matter and temperature and then do the same thing to a part of the sky in the East, you will find that they are identical to a tiny fraction of a percent. The problem with that is that in the evolution of the universe, there is no way that the two opposite sides of the cosmos could have been in communication with one another.[34] By that, I mean that in order for these two patches of sky to be so precisely identical, they must have been in contact with one another in the past. The hot big bang model doesn't allow that.[35] They would have always been so far apart that even mixing propagated at the speed of light, the conditions of one part could not reach the other. Yet something connected them, but what? Let's play together.

[34] This is called the Horizon Problem among aficionados. Namely: *you* by the time we're done.

[35] while Inflation encourages that!

1.4 Particle Physics and Cosmology, Together

After 50 years of successes and surprises in both fields, one thing is clear: the reality of a big bang means that there was a period when the universe consisted of only particles and forces. No protons, atoms, stars, galaxies, or Starbucks. Just elementary particles and the forces among them.

That epoch was less than 0.000001 seconds long, but critical since the particles and forces were created just before it and what happened after was determined *by* the ingredients and rules of that period. What's more, we suspect that the set of forces *then* was different from those we know of *now* and that the set of primordial elementary particles might have included whole species that we've not yet found in terrestrial experiments.[36]

[36] As a tantalizing tease each of the cosmological problems above has candidate particle physics solutions!

These eras are not connected by a single story thread—yet. But they must be! So we have a lofty goal: we're working toward a model of *everything* about the universe from the big bang through to today. Theories abound, but experiment will decide. We can explore the earliest moments of the universe with the most powerful telescopes, but in order to investigate the times earlier than about 3 minutes after that Beginning, we need to do experiments in laboratories on our Earth. It's a bold extrapolation: by colliding protons head-on at very high energies, we're reproducing that early hot cosmic cauldron.

Wait. *How do you know that this is the right connection to make? Maybe the conditions in the big bang were totally different than those in proton collisions?*

Glad you asked. *It's a plausible story, and, frankly a nice one. But as pleasing as it is, we have to test it and what's neat about the state of affairs right now is that particle physicists are joining astrophysics collaborations and astrophysical measurements are directly testable in our labs on Earth. It could be wrong! But we have to pursue it with a vengeance since the stakes are so high.*

We're currently mounting experiments in both EPP and Cosmology that are going to hit these issues squarely in the next couple of decades. Their results will completely change the way we think. Textbooks will be rewritten. If the first 40 years of the twentieth century were wacky, the first couple of decades of the twenty first are likely to be amazing.

In my professional lifetime, these two fields have become kin. Theoretical and experimental advances (or surprises) in one field directly affect the other and visa versa.

The stakes are so high, that we can add a third focus for EPP:

How did elementary particles and their forces affect the evolution of the universe? *Key Question 5*

Like the ancient Ouroboros, the snake eating its own tail. Cosmology—the science of the biggest— is dependent on the science of the smallest, particle physics, and *visa versa*. That's our story: Elementary Particle Physics and Cosmology are now united in a single path of discovery and this book will show you how.

QS&BB is not old "dead white guy physics"! It's all new and the details are still being worked out so we're going to be talking about matters of very current interest. If you make it through with me, you'll be in a good position to appreciate the surprises when they start to occur at the Large Hadron Collider, Fermilab's LBNF and DUNE, Mu2e, g-2, numerous underground laboratories, as well as the Planck Explorer, James Webb Telescope, the Fermi Gamma-ray Space Telescope, and other space-based laboratories. They'll be in the newspaper (if we still have newspapers). You wait.

Figure 1.4: Ouroboros

1.5 How QS&BB Will Work

Here's how QS&BB is going to work. As you read through the book you'll see a number o f repeating features: Goals, Biography, Sides, Flags, Notebooks, Diagrammatica, and the Crank. Let's see what these each are.

1.5.1 Goals

The first section of every chapter will itemize three categories of goals that I hope you'll achieve. After completing each chapter, I hope you will:

- **Understand**. This will often mean some facility with a set of calculations and/or graphics interpretation. It means that you've followed a simple mathematical argument interactively (see Notebooks, below). For example to **Understand** a recipe means that you've prepared a meal using it. It doesn't mean that you created it.
- **Appreciate**. This is less quantitative than **Understanding**. To **Appreciate** a recipe you would realize that to sweeten it you'd add sugar, but not actually do it or even predict exactly how much.
- **Familiarize**. This is a fly-by of some story or feature of a bit of our physics story. To be **Familiar** means that you know to go to Mr Google for information, because you can't remember the details before that step. Continuing with the food analogy, you might be **Familiar** with the idea that recipes for chocolate cookies exist, but you'd need the web or a cookbook in order to **Appreciate** or **Understand** one.

1.5.2 Biography

I'll bet you might think of physics as strange symbols and dry prose memorialized between the covers of big books and journals. But at its most basic, what is physics? It's people. Scientists carry on daily tasks, most of which are routine and not very risky. But every once in a while exceptional people accomplish exceptional things—they see some phenomenon or interpret some idea differently from everyone else.[37] This is a stressful place to be! Our heroes—the ones in textbooks—pursue their visions sometimes at personal cost.

[37] Everyone I work with is smart. But there have been some scary-smart people in the history of science and I'd like for you to meet many of them.

I've found that sometimes the content of the physics stays in students memories because they associate it with the people, so rather than stick a little scientific biography in a sidebar like many books, I highlight the people. The second section of each chapter includes a story: "A Little Bit of Einstein" (or someone) will introduce you to someone you've heard of ("A Little Bit of Einstein,"A Little Bit of Newton," and so on) or someone maybe you've not ("A Little Bit of Huygens," "A Little Bit of Kepler," "A Little Bit of Dirac," and so on).

Now while many of these folks are pretty special—and indeed some were a little odd—most were just everyday people with skills. That's most of us.[38] I'd muck up the preparation of a legal opinion and you wouldn't want me to treat you for an illness. Those are skills practiced by others. My colleagues and I have different skills, no fancier than those required of many other jobs. We're moms and dads, mow the yard, and fix dinner just like everyone else. But we have these heroes to whom we're professionally connected[39] our chapters will highlight them. I hope you enjoy this part of QS&BB.

[38] Perhaps you're not surprised at my impatience with the "mad scientist" image. Marty McFly's friend, Doc Brown, is my least favorite example of a scientist.

[39] A fun exercise that all of us have played at some part in our lives is to trace our Ph.D. degree supervisor, to his or hers, and so on back in history. For example, mine was Lincoln Wolfenstein. His was Edward Teller, who came from Werner Heisenberg, who in turn came from Arnold Sommerfeld, who came from Ferdinand von Lindemann, who came from Felix Klein, who came from Julius Plücker, who came from Christian Ludwig Gerling, who came from Carl Friedrich Gauss who came from Johann Friedrich Pfaff who came from Johann Elert Bode who came from Johann Georg Büsch who came from Johann Andreas Segner who came from Georg Erhard Hamberger who came from Johann Adolph Wedel who came from Georg Wolfgang Wedel who came from... well, you get the idea.

1.5.3 Sides

Pay attention to what appears in the side margins. To your left are examples of the items that will appear regularly.. Footnotes[40] will be there, for easy reference. Side comments—sometimes even serious ones—will be placed in margin notes. Little, tiny essays. And there will be three kinds of named sidenotes: definitions, equations, and constants.

[40] Here's a footnote.

Just a regular margin note here.

Definition: Some word.
Followed by the definition of that word.

Equation: Tee shirt equation.
$E = mc^2$

Constant of nature: A constant of nature..
Gallon = 4.0 quarts.

[41] And hopefully, sometimes hysterical.

There's a lot of jargon in this business and so I'll call out words or phrases that you'll need to keep in mind for later use. Those will get the name definitions, just like the dictionary.

There are also a handful of equations that will be useful and so when one of them appears in a margin, take it seriously. You'll need it. In fact, as you'll see below, I'm serious about taking notes and frankly copying the definitions and equations in a notebook, which you'll add to with each chapter, would be a good reference for you and an extremely important part of mentally processing what you write. So: write them for exercise and for safe-keeping.

1.5.4 Flags

While our coverage is largely historical[41] we'll come across ideas and concepts that will play various important roles as we move through the decades. These I call "flags" and they appear in the text, and then will be recalled at the back of each chapter so they will all be in one place. There are four kinds of flags:

A concept is just what it sounds like: an important idea worth highlighting. *Key Concept 2*

An observation is an experimental fact of profound consequence. *Key Observation 1*

A question is just that: something that we need to understand. *Key Question 6*

Then there is a particle-flag. We'll be accumulating a number of particles as we go along and I will provide this table each time. For example, the electron was discovered in 1895 and the particle-flag for it will read:

Particle 1

Electron

symbol:	e				
charge:	$-1e$	mass:	$m_e = 0.511\ \text{MeV/c}^2$	spin:	1/2
category	fermion			category	elementary

1.5.5 Notebooks

Actually, there is much of this account that I *don't* want you to "read" in the normal way. I want you to walk through the book—like the phone book—with your fingers doing the walking.[42] One thing I've learned over a few decades of teaching smart students who study subjects that are not mathematical (you?) is that if you come to the university as a freshman to major in, say Political Science or English or Psychology... that initial semester of college might be the first time in 13 previous school years in which you aren't taking a math course. At that point, after about a year away, you might find that your math muscle has atrophied. Trust me, I'm a doctor. You do have a math muscle and it needs periodic exercise to keep it fit.

I'm convinced that your brain is directly wired to your fingers.[43] Unless you've spent many years at this, you really can't *read* mathematics like you might read a history textbook: you have to interact with it. There is an enormous cognitive benefit from tactile reading: forming the symbols and numbers along with the text and allowing the logic to happen in your brain *by writing it out.* So this book will urge you to participate in the mathematical story-telling and I've got two ways for you to do it.

The Pencil.

The first way is by following along with your fingers: Buy a spiral-bound notebook into which you'll record your reading notes.[44] Then, when you're reading, you're using a pencil.

[42] There used to be this book. It had phone numbers, names, and addresses in it. The Phone Company's slogan was "let your fingers do the walking." This seems a century ago.

Just like I can't do 100 pushups any more—and I'd be pretty anxious if I were asked to do that in front of a class—I know that you might not be able to do some mathematics that you once were able to do! That's the famous "math anxiety."

[43] Or is it only my brain?

[44] Or your instructor might wish for you to use the template at the end of this chapter for your work. Notice that the "Pencil" has a number and that would be transferred to your paper.

When I get to a point in the text where need you to use that direct connection from your fingers to your brain, I'll indicate it with:

Pencil 1.1. ✎

What will follow the pencil will be short sections of content where you need to drill down a little deeper than what just passively reading will do for you. To me that means, start recording detailed notes. In fact I'm happy if you even *just copy* the numbers and formulas and that will be good enough. It will still penetrate your brain...in a good way.

When it's done, I'll congratulate you with a thumbs-up and you can go back to just reading.

I guarantee you that if you don't do this and simply kick back and read without pencil in hand, what comes after will mean less. Further, I can guarantee you that if you *do do this*, the logic of the mathematics and the inevitability of the narrative will be escorted to your brain and be there when you need it later.

The You Do It

The second way is more active and requires you to actually fill in blank spaces in the book. For example, I will sometimes come across an algebraic equation that needs to be manipulated a little or evaluated by plugging in numbers in order to keep going with the narrative. Or I'll have a graph that we need to look at for a specific number or an ordering exercise that will inform the narrative. When this happens, you'll see a QR code, some short instructions, and some white space. That tapping sound you'll hear is me waiting for you to fill the space.

You Do It 1.1. Example QR

This is an example of the kind of thing that you'll see: Newton's Gravitational Law is $F = G\frac{mM}{R^2}$.Solve for G.

or copy the solution

The You Do It will include enough blank space for you to actually do the manipulation. The QR code (or the link underneath) will provide you with a smartphone screen-sized derivation or some written coaching as to what you should write in that blank space. The idea is that first *you try it on your own*, then you use your phone to see how I did it.[45] Even if you simply copy what your phone shows you symbol by symbol, there's still a huge benefit to your understanding the physics. It will be in your brain, through your fingers. I *want* you to copy my work!

[45] If you can't make the QR code work, or don't want to, the phrase underneath it "or copy the solution" is a hot link in the pdf version of this document and when clicked will take you to the QR code's destination.

Wait. *I know how to read. Do I really have to do this?*

Glad you asked. *No, of course not. But if you can absorb what's coming without your pencil connecting to your brain then you're a lot smarter than I am. Take a chance. Write in your book. I won't tell.*

1.5.6 Digrammatica

I will need many diagrams. Sometimes these will be graphs of characteristic physical quantities (like distance versus time). Sometimes, these will be diagrams of phenomena (like an electric field). Sometimes these will be iconic items that go together in useful ways, like Feynman Diagrams. Rather than interrupt the flow in the narrative, I'll follow that chapter of interest with a special kind of chapter which I'll call *Diagrammatica*.[46] The contents of Diagrammatica chapters will be little more than a descriptive inventory of the diagrams of interest. Don't expect much lyrical prose in the Diagrammatica chapters. They're all business.

[46] The name is actually borrowed from a venerated little book on Feynman Diagrams by Nobel Laureate and University of Michigan Physics Professor, Martinus Veltman ("Tini"), *Diagrammatica: The Path to Feynman Diagrams (Cambridge Lecture Notes in Physics).*

1.5.7 The Crank

Finally, in a course like this the emphasis is not on the details of calculation but on the conceptual ideas. But calculations do happen and I think we should be able to identify what goes into a particular calculation and what comes out. In Chapter 2 I'll talk a little bit about models and the scientific process. Every prediction includes the following components:

- A Hard Core of unquestioned assumptions, models, data, and so on. A modern publication in aerodynamics doesn't need to go back and justify the use of Newton's laws of motion. It's assumed to be correct. So there is always a Core.
- Sometimes a prediction requires mixing data with mathematics. So an input might include Data.
- Every prediction is a prediction of a model, sometimes as a test of the model and sometimes as a test of an experiment. So the primary input are the ingredients of a Model.
- Most calculations involve a strategy of how to proceed using the Core and the Model.
- Then, there is a result! A prediction can be purely mathematical (we'd say "theoretical") and so the calculation really is a test of the logical consistency of the Model (does it "hang together"). Usually though, we expect the outcome to predict the results of some measurement.

I know that you've all used the phrase "turn the crank." The assumption is that somewhere someone simply followed through with the rules of a mathematical calculation. Well, a crank is so 19th century! I'll repeatedly use a graphic of a nonsense circuit that uses a little fictitious microprocessor[47] which is doing the crank-turning. Figure 1.5 is my silly image which will emphasize the inputs, what's being tested, and the conclusion. We'll take it for granted that someone with the right expertise can turn that crank, just like a computer might. You'll see how this works in the next chapter and then in many to come.

[47] I've plopped on top of my nonsense circuit an FPGA (Field Programmable Gate Array) from Xilinx Corporation. This is their newest model, the UltraScale+™.

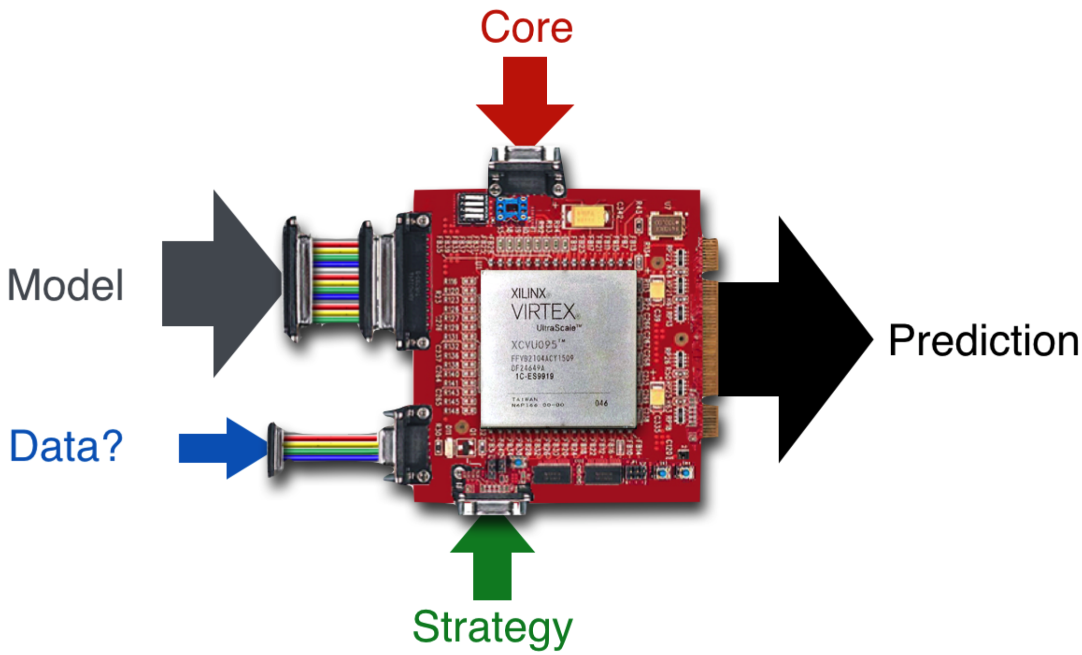


Figure 1.5: Our QS&BBcrank. The inputs are the Core, the Model, and sometimes Data. The outputs is some prediction. The Xilinx FPGA is essentially a little computer-on-a-chip used in many industrial and research applications, including those designed at MSU for our CERN ATLAS experiment.

Here's an example. In the early 18th century Newton's ideas about momentum and mechanics were being tried out on various phenomena. Daniel Bernoulli, a part of the most dysfunctional scientific family in the history of physics[48] had the idea that maybe the pressure that gases exert on a container were a function of collisions that hypothetical gas molecules exert on the walls of the container. This idea was expanded on later and actually resulted in a new understanding that temperature is nothing more than the average kinetic energy of a gas. This explained Boyle's Law, which maybe you remember from high school. It says that $PV =$ constant. Figure 1.6 is how I would short-circuit the calculation that one would go through to reach this conclusion. Get it?

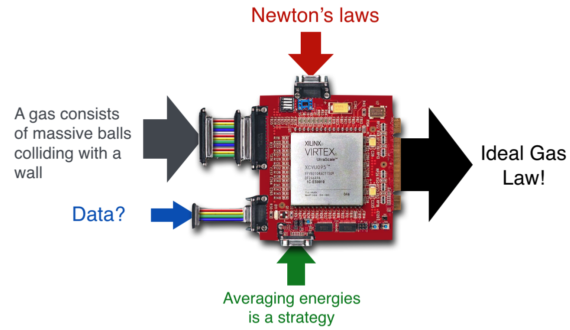


Figure 1.6: Newton's laws were not questioned, and so the Core. The Model was that a gas is a collection of tiny, massive balls that collided with the walls, and the strategy was to not treat each of them individually, but to average over their motions.

[48] Look them up! `http://www.daviddarling.info/encyclopedia/B/Bernoulli.html`

1.5.8 QS&BB Organization

I've organized QS&BBinto four Parts.

0. **Tools** We'll use minimal mathematics and the next chapter stands alone as a refresher (and hopefully, a calming influence) for all that we'll need to follow QS&BB.[49]
1. **Physics and Cosmology of my Grandparent's Generation.** Before the turn of the 20th century, known physics included the well-confirmed physics of Newton's mechanics, optics, and the relatively new electromagnetism. These subjects form the language for all of the 20th and 21st centuries and are the individual points of departure for the revolutions to come. We'll need to establish our foundations in these subjects.[50]
2. **Physics and Cosmology of My Parent's Generation.** From 1900 through the 1950's everyone was becoming comfortable (as much as possible!) with the quantum mechanical and relativity theories...and their merging in Relativistic Quantum Field Theory. These subjects are our theories, and our models all respect their rules.[51]
3. **Physics and Cosmology of My Generation.** Since the discovery of the fact that the universe is filled with microwaves left over from the big bang and that two of the most different-looking theories are actually a part of a single story, we've been hard at work on puzzles that these discoveries create. This is our work today.[52]
4. **Physics and Cosmology of Your Generation.** We are intensely pursuing a number of observational puzzles and inspired and compelling theoretical ideas. We will look to the future.[53]

[49] Chapter 2.

[50] Chapters 3 through ??

[51] Chapters 15 through ??

[52] Chapters 19 through 30

[53] Chapters 31 through 32

Okay. I lied. Five parts, but the first one doesn't really count as an actual part.

> What I need from you is an open mind and your pencil. Work the examples, do the Pencil-and-Thumb fill-ins, and enjoy our exploration of Outer and Inner Space.

Let's go to work!

Chapter 2

Everyone Needs Tools

A little math

René Descartes by Franz Hals, circa 1649

René Descartes (1596-1650)

"When I imagine a triangle, even though such a figure may exist nowhere in the world except in my thought, indeed may never have existed, there is nonetheless a certain nature or form, or particular essence, of this figure that is immutable and eternal, which I did not invent, and which in no way depends on my mind." *Meditations on First Philosophy* (1641)

It's always amazing to me , just how much we depend on the collaborative work of a handful of people from the 1600s. There must have been something in the water....in France, Italy, Britain, and Holland because this was a time of genius and courage. From people in this period—a number of whom we'll become familiar with—we received a way of thinking about, talking about, and poking at the world. René Descartes is one of my particular favorites. Let's learn a little bit about him.

2.1 Goals of this chapter:

- Understand:
  - Simple one-variable algebra.
  - Exponential notation.
  - Scientific notation.
  - Unit conversion.
  - Graphical vector addition and subtraction.
- Appreciate:
  - The approximation of complicated functions in an expansion.
- Be familiar with:
  - Descartes' life.
  - The importance of Descartes' merging of algebra and geometry.

2.2 A Little Bit of Descartes

The 17th century and just before saw a proliferation of "Fathers of –" figures: Galileo, the Father of Physics; Kepler, arguably the Father of Astrophysics, and Tycho Brahe, the Father of Astronomy. But the Grand-daddy...um...Father was René Descartes (1596-1650), generally considered to be the Father of Western Philosophy and a Father of Mathematics.[1] If you've ever plotted a point in a coordinate system, you've paid homage to Descartes. If you've ever plotted a function, you've paid homage to Descartes. If you've ever looked at a rainbow? Yes. Him again. If you ever felt that the mind and the body are perhaps two different things, then you're paying homage to Descartes and if you were taught to be skeptical of authority and to work things out for yourself? Descartes. But above all—for us—René Descartes was the Father of analytic geometry.

He was born in 1596 in a little French village now called, Descartes.[2] By this time Galileo was a professor in Padua inventing physics and Caravaggio was in Rome inventing the Baroque. Across the Channel Shakespeare was in London inventing theater and Elizabeth had cracked the Royal Glass Ceiling and was reinventing moderate rule in England. This was a time of discovery and dangerous opinion when intellectuals began to think for themselves. That is, this is the beginning of the end of Aristotle's suffocating domination as The Authority on everything.[3]

[1] Who's your daddy, indeed.

[2] Coincidence? What do you think.

[3] After all, by the time St. Thomas absorbed Aristotle into Catholic dogma, he was called The Philosopher.

Descartes' mother died soon after childbirth when he was only a year old and he was raised by relatives. His' father was an upper-middle class lawyer who spent little time with his children.[4] He was sent to a prominent Jesuit school at the age of 10 and only a decade later emerged from the University of Poitiers with the family-expected law degree. Apart from his success in school, the most remarkable learned skill was his lifelong manner of studying. He was sickly as a child and had been allowed to spend his mornings in bed, a habit he retained until the last year of his life.[5]

[4] When Descartes' father died, his brother failed to notify him (he found out through one of his correspondents) and he decided he was too busy to attend the funeral. Not exactly a close family. The similarities with Newton's childhood are striking.

[5] There's a story there...

One of the benefits of his schooling was a program to improve his physical conditioning, enough so that he became a proficient swordsman and soldier—he wore a sword throughout his life as befitting his status as a "gentleman."[6] And yes, he was essentially a soldier of fortune. During the decade following his graduation, he would alternate his time between combat assignments in various of the innumerable Thirty Year's War armies and raucous partying in Paris with friends.[7]

[6] He still worked in bed every morning until noon.

[7] He was a talented gambler, as befitting a mathematical mind.

Somewhere in that period Descartes became serious and decided that he had important things to say. He wrote a handful of unpublished books and maintained a steady correspondence with intellectuals in Europe, becoming well-known through these letters. Catholic France and of course Italy, were becoming intolerant of challenges to Church doctrine and he moved to the relatively casual Netherlands in 1628. Mostly a good move: he'd been inspired by Galileo's telescopic discoveries and became a committed Copernican and in 1633 was completely spooked by the Italian's troubles with the Inquisition.[8] However, he had trouble with some evangelical protestant leaders in Holland.

[8] That year, one of his major books, *The World*, was ready for publication, but he delayed it until after his death. In *World*, he expounded Copernicanism, but also provided for a reason why the planets circled the sun. A mechanism that Newton demolished with gusto.

Little did Descartes know that he was a mathematical genius. After study as a "mature" student at the University of Leiden, he found that he could solve problems in geometry that others could not. His devotion to mathematics and especially the rigor of the deductive method stayed with him and turned him into a new kind of philosopher. The logic of deduction and the certainty of mathematical demonstration were his philosophical touchstones.

Remember "deduction"? All squirrels are brown; that animal is a squirrel; therefore, that animal is brown kind of arguments? The important thing about this string of phrases is not that animal's color, but that the conclusion *cannot be doubted* if the two premises are true. Since Plato, "What can I know for sure?" was an essential question. For that particular Greek, things learned through your senses are untrustworthy. Only things you can trust are ideas which are eternal, outside of space and time. For other famous Greeks, you learn about the world through careful observation. Famously, Descartes convinced himself that he had discovered a method to truth: whatever cannot be logically doubted, is true.

2.2.1 Descartes' Philosophy

This is not the place to teach the huge subject of Descartes' philosophy. But there are two aspects of his work that directly influence the development of physics: what can we know and what is the nature of the natural world.

Descartes believed he'd found the formula for determining what's true: when an idea is clear and distinct, which means incapable of being doubted, then you can believe it. His method was to keep doubting *everything* until you reach a point in this thought-process that can't be doubted.[9] The point he reached was the recognition that *he* was doing the doubting. Since that can't be doubted, then what he's learned that's true is: thought exists. One more step to *I exist*, because it is I who is doing that thinking: "Cogito ergo sum"[10] was his bumper sticker for truth.

[9] In this way you reduce a complex problem to a more manageable one. . . one of his essential components to his "analytic philosophy."

[10] "I think, therefore I am." Words to live by.

The rest of his argument is a little shaky but this is the beginning of dispassionately and vigorously analyzing a philosophical problem, setting a high bar for argument. Of course, Medieval thinking was not friendly to the idea that everything can be doubted. The Bible and pretty much all that Aristotle wrote was off-limits. In fact, under the rules of thought not only could neither source be doubted, those sources were the only authority used to determine truth and falsity. Descartes pretty much changed that in philosophy.

He called his method "analytic" and it's essentially applying mathematical problem solving strategies to philosophical questions. Hence, history's assignment of paternity to him for Western Philosophy.

He said later that he made this discovery about doubt while still a soldier and holed up on a snowy night alone in a remote cabin. Sometimes his military escapades were real combat, but mostly it seems like he had a lot of leisure time.

For our purposes, what he decided were that true things about the world could be obtained through pure thought. This is the "Rationalist" philosophy of which he is the king. This is in the spirit of Plato, but unlike Descartes, he gave up on the sensible world as simply a bad copy of the Real World, which is one of Ideas..."out there" somewhere. By contrast, Descartes asserted that there are two substances in the universe. One is mind and the other is matter. Understanding the universe means gaining knowledge of both by blending thinking with observing.

We'll see that physics takes some inspiration through Descartes' approach. Theoretical physicists are often motivated by knowledge gained through thought—and always mathematics—and many work as if those thoughts are representing the world.

Definition: Rationalism.
The only test of and source of knowledge is reason.

Definition: Empiricism.
All knowledge originates in experience—through experiment and observation.

This two-part universe is now called Cartesian Dualism and was all the rage when Newton was a student. But the important thing to take away from this is that Descartes is the proud proponent of the notion that true knowledge can be obtained purely through thought. The counter to this Rationalist belief is Empiricist belief, that knowledge can only be obtained through observation (and in modern form, experiment).

The other aspect of Descartes' philosophy that matters[11] is his notion of Mechanism. The Renaissance was saturated with ideas of nature that we'd consider magic. Nature was infused with occult properties,

[11] no pun intended. . . sort of.

that it is almost alive with "active principles," even human-like in ways. Of course, astrology, alchemy, signs and numerology, Cabala, black magic and white natural magic, and so on were aspects of organized occultism. But it went deeper. People lived lives, tended the sick, and found explanations for natural phenomena based on the assumption that what we would call inert natural objects were alive and possessed magical powers. This continued a long-standing philosophical discussion about Qualities. Is the boiling pot hot because it possess the innate quality of "hotness"?

Magical thinking was a threat to the Church and Descartes also subscribed to the growing program of ridding nature of these features. Things in the world are not possessed of innate features like hot or cold, blue or red, and so on. These for Descartes are attributes not innate qualities. "Things" possess...place. Now we'll think a bit later about what constitutes space, but for Descartes and others, space is determined by the extent of objects. In fact the only aspects of matter that are "clear and distinct" (and hence true) are that matter has the properties of spatial extent (length, width, height) and motion.

He needed to have a mechanism to explain everything in the material world. He explained motion as the point-to-point pushing of material objects that we see (planets) by innumerable, small-sized, varied atoms which are indivisible. This "plenum" of stuff is moving, initiated by God, and they preserve that motion as they transmit it to all moving material objects.[12] It's communicated to the planets, through vortices, as in Fig. 2.1 from *The World*.

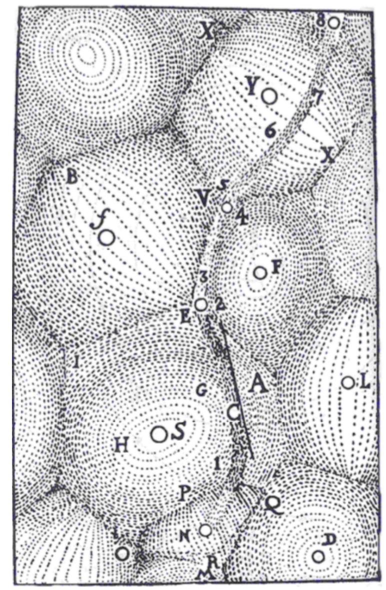


Figure 2.1: plenum

[12] Remember this when we get to momentum and energy!

Likewise magnetism. Boy, that's an occult-ish phenomenon if there ever was one. To Descartes magnetism was propagated by little, tiny left-handed screw-like object that find threaded holes in iron so as to attract or repel. Gravitation is another kind of material experience. First, Descartes hypothesized about a material cause for phenomena and then deduced the consequences.

Descartes paved the way for a reasoned approach to physics, that turns out to have been a part of the story. He motivated Newton and helped European thinkers to find their way to independent ideas, shedding the overbearing weight of Aristotelianism and Church dogma.

But this chapter is devoted to mathematics.

2.2.2 Descartes' Algebra-fication of Geometry

...or geometrification of algebra! Whatever. Descartes brought geometry and algebra together for the first time by reinterpreting the latter and inadvertently, rendering the former less important.[13]

[13] for a while.

Descartes pulled the very new, very unsophisticated new method of "algebra" to a role of supremacy over geometry. He did this by linking the solution of geometry problems—which would have been done with rule-obsessive construction of geometrical proofs—to solutions using symbols. He did this work in a small book called *Le Géométrie* (*The Geometry*), which he published in 1637, the same year he published his *Discourse on Method*.[14]

[14] *Geometry* can be considered an appendix to the *Discourse*.

He instituted a number of conventions which we use today. For example, he reserved the letters of the beginning of the alphabet $a, b, c, ...$ for things that are constants or which represent fixed lines. An important strategic approach was to assume that the solution of a mathematical problem may be unknown, but can still be found and he reserved the last letters of the alphabet $x, y, z...$ to stand for unknown quantities—variables. He further introduced the compact notation of exponents to describe how many times a constant or a variable is multiplied by itself.

Prior to Descartes, ab would be the product of a and b but explicitly refer to the area of a rectangle bounded by legs of lengths a and b. a^3 would be the volume of a cube. There would be no such thing as $abcd$ or a^4 because after all, nature has no more dimensions than 3. So the early algebra was confined to a strictly dimensional context. Descartes broke with that and explored equations of higher powers, even showing that equations of higher powers could be reduced to lower power equations and so on until a solution could be found. He did this algebraically and geometrically, side by side. In fact, *Le Géométrie* is just one example worked out after another: it's solutions-oriented. And it's abstract. There's no need to identify "things" to the variables, although one could do so if desired.

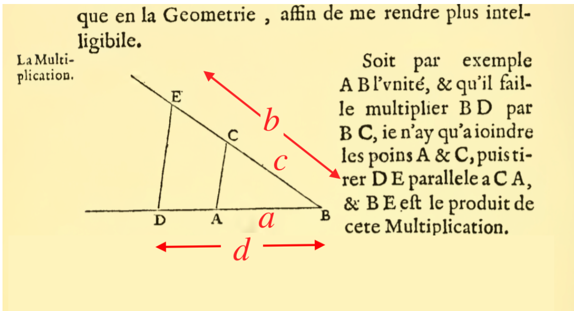


Figure 2.2: geometrymultiply

Just as arithmetic has addition, subtraction, multiplication, division, and square roots...so to he found geometrical interpretations of these operations. His geometrical description of multiplication—not referring to an area—is instructive of how he did things. FIgure 2.2 shows a figure from *Le Géométrie*. Using his notation, we immediately come upon a new "invention" of his: unity. A line of length "1" could be chosen arbitrarily, and then manipulated.

In Fig. 2.2 I've overlaid red letters in the fashion that Descartes would have, assigning a single letter to represent a line. The lines $\overline{DE}$ and $\overline{AC}$ are both parallel and so the triangles BED and BCA are similar. From elementary geometry, because of their similarity, we would have

$$\frac{b}{d} = \frac{c}{a}.$$

Now he does this clever thing with "1" and assigns the length $\overline{AB}$ to have length 1 so that we have

$$\frac{b}{d} = \frac{c}{1}.$$

and so the product of $cd = b$. No areas. A brand new use of the brand new algebra!

Here's another example from *Le Géométrie*. Supposed you want to find the square root of a quantity. Figure 2.3 is again from his book. His trick here is to assign the distance $\overline{GH}$ to be an arbitrary length x[15] and the distance $\overline{GI}$ to be y. His goal is to compute the $\sqrt{y}$ for this abstract situation. Again, he uses the

[15] See? Algebra with unknowns.

"1 trick" and makes $\overline{FG} = 1$. The end result is that $y = \sqrt{x}$ and the problem is solved in general terms and in a way that could be measured with a ruler. Like Euclid would have liked.

The early translators of algebra considered equations in two unknowns—some $f(x, y) = 0$—to be impossible. Descartes actually found a way by treating the locus of points on a line as indeterminant, some abstract x. GIven any particular location along x however, another corresponding to the other unknown variable could be identified. He called such a point y and then worked to find solutions to particular problems that might be different depending on what the value of x was...but he did it in a way that was general for any x. This is the first example of what we'd now refer to as an axis. He didn't actually use two axes, but he still solved problems for an unknown y in terms of a parameter x. He called one of these the abscissa and the other, the ordinate.

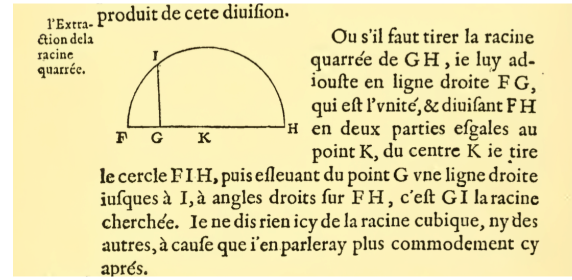
Mathematicians picked up on these ideas and extended them into the directions that we now love. One of those was John Wallis (1616-1703), a contemporary of Isaac Newton who learned from Wallis enough to construct the general Binomial Theorem.

The use of perpendicular axes, which we call x and y stems from Descartes' inspiration which is why they're called Cartesian Coordinates.

Descartes managed to get himself into a dispute with a Calvinist theologian, Gisbertus Voetius who wanted his university to officially condemn the teaching of "Cartesian Philosophy" as atheistic and bad for young people. Descartes responded by printing a reaction which was posted on public kiosks. This must have been quite a sight! In any case, Descartes began to imagine that his time in the Netherlands was coming to a close. An admirer, the Queen Christina of Sweden, was an intellectual of sorts and invited Descartes to Stockholm to work for her court and to teach her. She even sent a ship to Amsterdam to pick him up. He eventually accepted the position and this was the beginning of the end for him.

She required his presence at 4 AM for lessons. This, from the fellow who had spent every morning of his life in bed until noon! He caught a serious respiratory infection and died on February 11th, 1650 at the age of only 53.

We moderns owe an enormous debt to this soldier-philosopher-mathematician. Both for what he said that was useful and for what he said that was nonsense, but which stimulated productive reaction. In what follows from Section 2.5 there is a direct line from every word back to René Descartes.



l'Extraction dela racine quarrée.

produit de cete diuiſion.

Ou ſ'il faut tirer la racine quarrée de G H, ie luy adiouſte en ligne droite F G, qui eſt l'vnité, & diuiſant F H en deux parties eſgales au point K, du centre K ie tire le cercle F I H, puis eſleuant du point G vne ligne droite iuſques à I, à angles droits ſur F H, c'eſt G I la racine cherchée. Ie ne dis rien icy de la racine cubique, ny des autres, à cauſe que i'en parleray plus commodement cy aprés.

Figure 2.3: root

2.3 Introduction

In this chapter we'll do some old things and some new things. Some of the old things will be mathematical in nature, while some of the new things will include some terminology and some techniques. I promise that the math will not be hard and we'll get through it together. We'll develop just a few of these tools that we'll return to repeatedly: simple algebra, exponents, unit conversions, and powers of ten. It will come back to you.

But I want to start with some topics which are timely and confusing to non-specialists. What are we doing when we "do" science?

2.4 It's Theory, All the Way Down

Coming.

2.5 The M Word

The language of physics is mathematics, so uttered Galileo a long time ago (although he said that the language of the *universe* is mathematics). Well, he was right and we have no idea why that seems to reliably be the case! So the importance of that realization will become clear as we go, which is partly why I don't want to avoid mathematics altogether. But it will be relatively simple. You've seen everything I'll ask you to do in high school, at the very least. It will be fine. Let me show you.

> **Wait.** *I'm not a math person.*
>
> **Glad you asked.** *Actually, nobody is. Really mathematics is a habit of mind and strategy for how you read. Certainly for what we're going to do. I promise you. Read with your pencil out. Read every line with a mathematics symbol. You'll get it.*

2.5.1 Some Algebra

Our algebraic experience here will be some simple solutions to simple equations. I'll need the occasional square root and the occasional exponent, but no trigonometry or simultaneous equation solving and certainly no calculus. I'll refer to vectors, but you'll not need to do even two-dimensional vector combinations.

Algebra is pretty simple with basically one rule: Whatever you do to the left hand side of an equation, you must also do to the right side and visa versa. Period.

Let me make my point by going back to the Gravitation law and asking a simple scientific question of it. I mentioned that it's hard to measure G. Why is that? Let's try to answer this two ways. First, we'll ignore the mathematics and try to do it in words. Then, we'll use mathematics and algebra and see if insight happens.

First, in English as if the law is a sentence.

The words way:

"The force between two objects is the product of the two masses divided by the distance squared all multiplied by the gravitational constant."

Then we must analyze that sentence to figure out why the gravitational constant is hard to determine.

The mathematics way:

$$F = G\frac{mM}{R^2}$$

and then use the rules of algebra to ask about G and see what results. In fact, you do this:

for G in Newton's Gravitational law, $F = G\frac{mM}{R^2}$.1.5in

You Do It 2.1. Solve for G

or copy the solution

Solve for G in Newton's Gravitational law, $F = G\frac{mM}{R^2}$.

After a few lines, did you get: $G = \frac{FR^2}{mM}$?

Now you can see partly why G is so hard to measure. In order to do an experiment to measure G, you have to contend with the facts that:

- gravitational forces are tiny which means that since F is small, so that part of the numerator is a small number,
- in order to see an effect the masses used are often large, so m and M are big and since they are in the denominator they result in a small number, and
- you can't allow the masses to be too far apart from one another or *other* masses will affect the measurement so R is relatively small.

So everything on the right-hand side conspires to make G really tiny: multiplying by the tiny F, dividing by the enormous $m's$, and multiplying by the smallish-ish $R's$.

I suggest to you that none of this information would have been obvious if all you had only just looked at the original equation like a sentence in a book.

You needed to touch this equation and move the pieces around in order to gain any insight.

This is why I'll ask you to "touch" some of the material as we flow through an argument that might be mathematical. Even if it's a part of the text, you should copy it out while you read. Remember, these parts are marked by

Pencil 2.1. ✎

👍

Our appetite for algebraic complexity in QS&BB will be limited. For example, we'll not encounter formulas that are much more complicated than these:

$$y = a \times x = ax \quad \text{solve for } x \text{ to get} \quad x = y/a$$
$$y = x + z \quad \text{solve for } x \text{ to get} \quad x = y - z$$
$$y = a \times x + b = ax + b \quad \text{solve for } x \text{ to get} \quad x = \frac{y - b}{a}$$
$$y = \sqrt{a + x} \quad \text{solve for } x \text{ to get} \quad x = y^2 - a$$

You can do this, right? That's about all that you'll need to remember of algebra. Just remember the rule. Then...it's merely a game—a puzzle to solve.

There's an important reason I have chosen to include some mathematics in QS&BB: I'd hate for you to miss...dare I say...a spooky feature of the universe. It behaves as if mathematics is an essential part of how it works.[16]

We'll take it slow with the math, but even a little will add a lot to your understanding. So let's spend the rest of this chapter reminding yourself of things that you would have learned in high school.

[16] There has been this eyes-open discussion in physics for a century now. Is mathematics invented or is it discovered? The former would suggest that it's in some sense, man-made. The latter would suggest that it's a deeply embedded feature of nature... to be found out. In 1960 the famous mathematical physicist Eugene Wigner wrote a paper that's still read today called the *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*. Ask Mr Google about it. Almost 30,000 hits, almost all of them "reprints."

2.5.2 The Powers That Be

Once in a while, we'll need to multiply or divide terms that have exponents. There are simple rules for this, but let's figure them out by hand...so to speak. The first thing to remember about exponents is that in a term like x^n, a positive integer n tells you how many times you must multiply x by itself. So:

$$x^1 = x.$$

Here, there's just one x, so: $x^1 = x$.
The second thing to remember is that $x^0 = 1$. There aren't any $x's$ in the product and so all that could be there is 1. Armed with that, let's kick it up a notch.

Suppose I have

$$x \times x$$

You'd be pretty comfortable calling that "x-squared"[17] and from the above, the number of $x's$ there are in that product is two. So

$$x \times x = x^2.$$

If I add another product, then I'd have $x \times x \times x = x^3$. Get it? Notice that what we've also got in this equation is:

$$x \times x \times x = x^2 \times x^1 = x^3$$

and we've just developed our first rule on combining exponents:

$$x^n \times x^m = x^{n+m}.$$

Now you try it.

[17] From the discussion of Descartes, you can see why the word "squared" is used since this is a legacy of the early linking of algebra with geometry. Ditto for "cubed."

You Do It 2.2. Exponents

What is $x^2 x^1 x^4$?

or copy the solution

One more time, but different. Another rule:

$$x^{-n} = \frac{1}{x^n}.$$

If the same rule for adding exponents works—and it does—then we can multiply factors with powers by keeping track of the positive and negative signs of the exponents.

So here's an easy one, first by multiplying everything out:

$$\frac{x \times x \times x}{x \times x} = x$$

and now by using the powers and the rule:

$$\frac{x \times x \times x}{x \times x} = \frac{x^3}{x^2} = x^3 \times x^{-2} = x^{3-2} = x.$$

One more thing. The powers don't have to be integers.

Perhaps you'll remember that square roots can be written:

$$
\begin{aligned}
\sqrt{x} &= x^{0.5} = x^{1/2} \\
\text{so:} & \\
\sqrt{9} &= 3 \quad = 9^{0.5} \\
\text{or:} & \\
\sqrt{\frac{1}{9}} &= \left(\frac{1}{9}\right)^{0.5} = \frac{1}{\sqrt{9}} = \frac{1}{9^{0.5}} = 9^{-0.5} \\
&= \frac{1}{3}
\end{aligned}
$$

You Do It 2.3. Exponents Again

or copy the solution

What is $x^{-2}x^{1}x^{4}$?

That's it. Now we have everything we need to turn numbers into sizes of...stuff.

2.5.3 Units Conversions

Numbers are just numbers without some label that tells you what they refer to. Now not all number have to refer to something, pure number is a respectable object of mathematical research—prime numbers for example have been a topic of research for centuries. Irrational numbers–those that can't be expressed as a ratio of whole numbers, like π, –are likewise objects with no necessary relationship to..."stuff" in our world.

We're concerned with numbers that measure a parameter or count physical things and they come with some reference ("foot") unit that is a customary way to compare one thing with another.[18] Of course not everyone agrees on the units that should be used. Wait. There's *the world*, that agrees, and then there's the United States that marches to its own set of units.

[18] "Apples and Oranges" is a phrase that refers to units...you need to keep your fruit straight.

I'll not use Imperial units (feet, inches, pounds, etc.) very much, except to give you a feeling for something that you've got an instinct for... like the average height of a person. We'll use the metric system, in particular the MKS units[19] in which the fundamental length unit is the meter (about a yard).

[19] This stands for meter-kilogram-second, as the basic units of length, mass, and time. It's a dated designation as the real internationally regulated system is now the International System of Units (SI) which stands for *Le Système International d'Unités*. The French have always been at the forefront of this.

Just like an exchange rate in currency, so many euros per dollar, we'll need to be able to convert, among many different units. I've got a plan.

Let's get our bearings. What's a common sort of size in life? How about the height of an average male. Mr Google tells me that's about 5'10". How many inches tall is our average male? Here's the thought-process you'd use to calculate this.

Pencil 2.2. ✎

Three steps:

1. A foot is 12 inches.
2. So, 5 feet is $5 \times 12 = 60$ inches
3. and the combination is $60 + 10 = 70$ inches.

...which you could do in your head I'll bet. But this simple, almost intuitive calculation uses a more general conversion from one unit to another through the use of a conversion factor. All unit manipulations use a conversion factor, which we'll call $F(\text{to, from})$: it's a number,[20] which will be expressed as a ratio or fraction, of the conversion of one set of units ("from") to the new set ("to"). It will appear like this:

[20] . . . a number that's actually like 1 since it's really relating one thing in a set of units to the same thing in a different set of units. So if we use F in an equation, we're really just multiplying by 1, but in a fancy way.

$$\text{where you're going to} = \left(\frac{\text{to}}{\text{from}}\right) \times \text{where you're coming from}$$

$$\text{where you're going to} = F(\text{to, from}) \times \text{where you're coming from}$$

In this case, step 1 defines F, and step 2 uses it and in symbols, step 1 says:

$$F(\text{inches, feet}) = \frac{\text{number of inches in a foot}}{\text{a foot}} = \frac{12}{1}$$

So armed with this, we can do the conversion of feet to inches.

$$\begin{aligned}\text{five feet in inches} &= F(\text{inches, feet}) \times 5\text{ ft} \\ &= \frac{12\text{ inches}}{1\text{ ft}} \times 5\text{ ft} = \frac{60}{1}\text{inches} \\ &= 60\text{ inches.}\end{aligned}$$

Notice that we can cancel the units as if they were symbols: the "feet" (with the 5) cancels with "foot" (in the denominator of F)[21] That's the neat thing. If you set up the conversion factor right, the units will multiply and divide along with numbers so you can always see that you get what you want. While this is a particularly simple conversion, sometimes we'll need to do some which are either more complicated, or use units that maybe you're not very familiar with.

Now you do it. If a furlong is 201.2 meters, how furlongs are there in a mile?

How you do this might depend on where you like to start. What I always remember[22] is that a mile is 5,280 feet and that a foot is 12 inches and that an inch is 2.54 centimeters and that a meter is 100 cm. So I always start there. You might do it differently. So for me, that's 4 conversions, or 4 F's that I would use to do this conversion. If you were to do it my way, you'd need:

$$F(\text{meters, cm}) \times F(\text{cm, inches}) \times F(\text{inches, feet}) \times F(\text{feet, mile})$$

which I would chain together., and which you can chain together now in the next You Do It.

[21] I'm playing a little fast and loose with the singular and plural of the word that describes that part of your body that goes into a shoe.

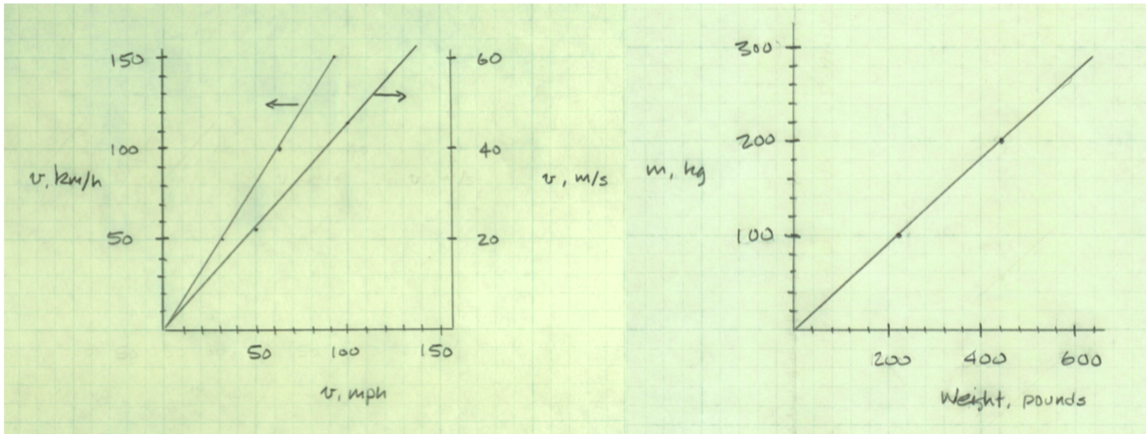
[22] ...for some reason

You Do It 2.4. Furlongs-mile

or copy the solution

How many furlongs in a mile?

Did you get that there are 8 furlongs in a mile? If not, get out your phone and look this up. **Understand conversions!** Conversions are a part of life and I've collected a number of the useful ones into graphs which you can use later.



(a) (b)

Figure 2.4: The right hand curve shows a constant speed of 4 m/s, holding steady for 10 s. The left hand curve shows the distance that an object will travel at that constant speed as a function of time.

2.5.4 The Big 10: "Powers Of," That Is

One of the more difficult things for us to get our heads around will be the sizes of things, the speeds of things, and the masses of things that fill the pages of QS&BB. Lots of zeros means lots of mistakes, but it also means a complete loss of perspective on relative magnitudes. Big and small numbers are really difficult to process for all of us.

As we think of things that are bigger and bigger and things that are smaller and smaller, where do you start to loose track and one is the same as another? Keep in mind our average-guy height of about a meter and half–for this purpose, thing... "about a couple of meters"–and here is a ranked list of big and small things with approximate sizes:

1. African elephant, 4 m
2. Height of a six story hotel, 30 m
3. Statue of Liberty, 90 m
4. Height of Great Pyramid of Giza, 140 m
5. Eiffel Tower, 300 m
6. Mount Rushmore 1700 m
7. District of Columbia, 16,000 m square
8. Texas, East to West, 1,244,000 m
9. Pluto, 2,300,000 m diameter

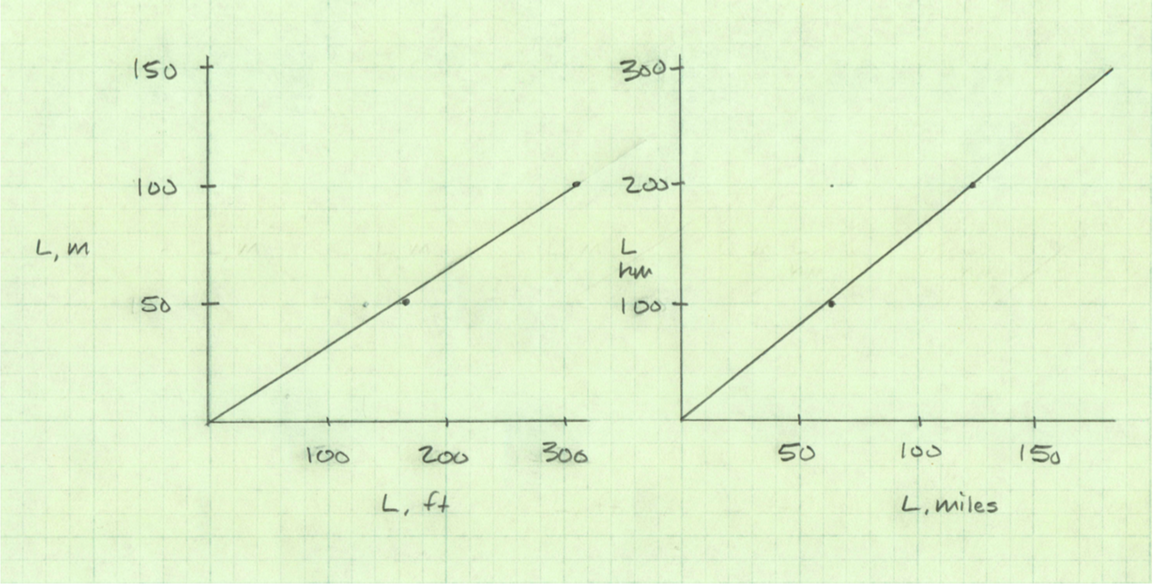


Figure 2.5: The right hand curve shows a constant speed of 4 m/s, holding steady for 10 s. The left hand curve shows the distance that an object will travel at that constant speed as a function of time.

10. Moon, 3,500,000 m diameter
11. Earth, 12,800,000 m diameter
12. Jupiter, 143,000,000 m diameter
13. Distance Earth to Moon, 384,000,000 m
14. Sun, 1,390,000,000 m diameter
15. distance, Sun to Pluto, 5,900,000,000 m
16. Distance to nearest star (Alpha Centuri), 41,300,000,000,000,000,000 m
17. diameter of the Milky Way Galaxy, 950,000,000,000,000,000,000 m
18. Distance to the Andromeda Galaxy, 24,000,000,000,000,000,000,000 m
19. Size of the Pisces–Cetus Supercluster Complex, our supercluster, 9,000,000,000,000,000,000,000,000 m
20. Distance to UDFj-39546284, the furthest object observed, 120,000,000,000,000,000,000,000,000 m

Do I need to go any further? Given what I know from my life, I have a pretty good idea of how big #1-8 are. Beyond that, I have no idea how much bigger the Milky Way Galaxy is than the size of Jupiter. It all blends together.

But there's a way: exponential notation. . . using our power rules and the number 10. It's easy.

A number expressed in exponential notation as:

$$\text{a number} \times 10^{\text{power}}$$

Let's think about this in two parts. First, the 10-power part.

The rules above work for 10 just like any number, so 10^n is shorthand for the number that you get when you multiply 10 by itself n times. This has benefits because of the features of 10-multiples, that we count in base-10, and how you can just count zeros. So for example:

$$10^3 = 10 \times 10 \times 10 = 1,000.$$

The power counts the zeros, or more specifically, the position to the right of the decimal point from 1. So if you have any number, you can multiply it by the 10-power part and have a compact way of representing big and small numbers. So, following through:

$$3 \times 10^3 = 3 \times 10 \times 10 \times 10 = 3 \times 1000 = 3000.$$

We can do the same thing with numbers less than 1, by using negative exponents for the 10-power part.

$$0.03 = \frac{3}{100} = \frac{3}{10^2} = 3 \times 10^{-2}.$$

So you just move the decimal place the power-number to the right to go from 3×10^{-2} to 0.03.

The second thing is the number in front that multiplies the power of 10. It's called the "mantissa" and that's all it is... a number.

Now that confusing list above can be written in a way that's more likely to allow your brain to compare one with the other, since now you'll immediately see that one thing is 10 or 1000 or so-on times another.

1. African elephant, 4 m
2. Height of a six story hotel, 30 m, 3.0×10^2 m
3. Statue of Liberty, 90 m, 9.0×10^2 m
4. Height of Great Pyramid of Giza, 140 m, 1.4×10^2 m
5. Eiffel Tower, 300 m, 3.0×10^2 m
6. Mount Rushmore 1700 m, 1.7×10^3 m
7. District of Columbia, 16,000 m square, 16.0×10^3 m, or 1.6×10^4 m
8. Texas, East to West, 1,244,000 m, 1.244×10^6 m
9. Pluto, 2,300,000 m diameter, 2.3×10^6 m
10. Moon, 3,500,000 m diameter, 3.5×10^6 m
11. Earth, 12,800,000 m diameter, 12.8×10^6 m, or 1.28×10^7 m
12. Jupiter, 143,000,000 m diameter, 143.0×10^6 m, or 1.43×10^8 m
13. Distance Earth to Moon, 384,000,000 m, 384.0×10^6 m, or 3.84×10^8 m
14. Sun, 1,390,000,000 m diameter, 1.39×10^9 m
15. Distance, Sun to Pluto, 5,900,000,000 m, 5.9×10^9 m
16. Distance to nearest star (Alpha Centuri), 41,300,000,000,000,000,000 m, 41.3×10^{18} m, or 4.13×10^{19} m
17. diameter of the Milky Way Galaxy, 950,000,000,000,000,000,000 m, 950×10^{18} m, or 9.5×10^{19} m
18. Distance to the Andromeda Galaxy, 24,000,000,000,000,000,000,000 m, 24.0×10^{21} m, or 2.4×10^{22} m
19. Size of the Pisces–Cetus Supercluster Complex, our supercluster, 9,000,000,000,000,000,000,000,000 m, 9.0×10^{24} m
20. Distance to UDFj-39546284, the furthest object observed, 120,000,000,000,000,000,000,000,000 m, 120×10^{24} m or 1.2×10^{26} m

So now you can compare and see that the distance from the Earth to the Moon is only a little more than three times the diameter of Jupiter. Now your "mind's eye" springs into action since you can sort of imagine three Jupiters between us and the Moon. With all of those zeros, I couldn't do that!

Powers of 10 have nicknames...Is "a google" really a power of ten?[23] Here's an official table of the names, size, and abbreviation for most of them:

[23] No. The word is Googol and it's $10^{1}00$. The rumor is that the Google founders misspelled it when they incorporated.

Let's work out an example. Something you can use at a party. I first worked this out for a class when I was in Geneva, Switzerland working at CERN. It was July 4, 2010, which was just another Sunday over there. The United States came into existence on July 4, 1776[24] which was 2010 – 1776 = 234 years ago.

[24] Actually, the Declaration of Independence wasn't fully signed until August 2, 1776—my birthday! The day, not the year.

So how many seconds had the United States been around if we start from midnight on July 4, 1776?

Pencil 2.3.

$$\begin{aligned}
\text{234 year per U.S.} &= 2.34\times 10^2\ \frac{\text{years}}{\text{U.S.}}\\
\text{86,400 seconds per year} &= 8.64\times 10^4\ \frac{\text{seconds}}{\text{year}}\\
\text{So:}&\\
\text{seconds per U.S.} &= 2.34\times 10^2\ \frac{\cancel{\text{year}}}{\text{U.S.}} * 8.64\times 10^4\ \frac{\text{seconds}}{\cancel{\text{year}}}\\
&= (2.34)*(8.64)\times 102 * 10^4 = (2.34)*(8.64)\times 10^6\\
\text{seconds per U.S.} &= 20.218\times 10^6\\
\text{seconds per U.S.} &= 2.0218\times 10^7
\end{aligned}$$

Wait. *You mean I treat the words of units as if they were algebraic variables?*

Glad you asked. *Yes. You can do that and even catch mistakes when the products and cancellations don't lead to what you expect. Had I gotten miles times hours, I'd know my actual formula was wrong even before doing it. No charge for this hint. Use it wisely.*

There are a few of things to notice here. First, that's a lot of seconds! Second (get it?), to multiply two numbers together, you separate the mantissas, and multiply them, and the exponents, and add them...separately.[25] Please understand these operations by doing them over by hand. The obvious thing happens when there are negative exponents involved. For example, convince yourself that 15% of the lifetime of the U.S. is 3,032,700 seconds, and do it by treating 15% as

$$15\% = 0.15 = 1.5\times 10^{-1}.$$

Finally, notice that I canceled the units of "year." You can always do that with units—set them up right, keep them in your equations, and you can quickly find mistakes. Here, the units on the right have to give you the units on the left, which we wanted: "seconds/U.S."

[25] Remember? The "mantissa" in $X\times 10^y$ is X and the exponent is the y.

2.5.5 Graphs and Geometry

One of the amazing mathematical discoveries of the 17th century was that geometry could be tied to algebra through the use of the growing notion of a function. This is almost entirely due to Rene Descartes and Leonhard Euler (1707-1783)[26]

[26] Euler was one of the most amazing mathematicians in history. He did so much that his work is still being analyzed and cataloged today. To him we owe the notion of a function. But he also worked in physical problems like hydrodynamics, optics, astronomy, and even musical theory. While Swiss, Euler lived and worked most of his life in St. Petersburg, Russia.

We will deal with some functions that would be very hard to evaluate on your calculator. But Descartes' gift is that I can show you the graph and evaluation can be done by eye, which is in effect solving the equation. We'll use some simple geometrical relations which I'll summarize here.

septillionth	yocto-	y	0.000000000000000000000001	10^{-24}
sextillionth	zepto-	z	0.000000000000000000001	10^{-21}
quintillionth	atto-	a	0.000000000000000001	10^{-18}
quadrillionth	femto-	f	0.000000000000001	10^{-15}
trillionth	pico-	p	0.000000000001	10^{-12}
billionth	nano-	n	0.000000001	10^{-9}
millionth	micro-	μ	0.000001	10^{-6}
thousandth	milli-	m	0.001	10^{-3}
hundredth	centi-	c	0.01	10^{-2}
tenth	deci-	d	0.1	10^{-1}
one			1	10^{0}
ten	deca-	da	10	10^{1}
hundred	hecto-	h	100	10^{2}
thousand	kilo-	k	1,000	10^{3}
million	mega-	M	1,000,000	10^{6}
billion	giga-	G	1,000,000,000	10^{9}
trillion	tera-	T	1,000,000,000,000	10^{12}
quadrillion	peta-	P	1,000,000,000,000,000	10^{15}
quintillion	exa-	E	1,000,000,000,000,000,000	10^{18}
sextillion	zetta-	Z	1,000,000,000,000,000,000,000	10^{21}
septillion	yotta-	Y	1,000,000,000,000,000,000,000,000	10^{24}

Table 2.1: More powers of ten than you ever wanted to know. Except that many of them we need to know.

Formulas From Your Past

I know that you've seen most of this somewhere in your past! So return with us now to those thrilling days of yesteryear.[27]

Equation of a Straight Line

A straight line with a slope of m and a y intercept of b is described by the equation:

$$y = mx + b. \tag{2.1}$$

[27] Google it!

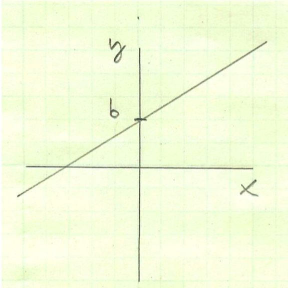


Figure 2.6: straight

Figure 2.6 shows such a straight line.

Equation of a Circle

A circle of radius R in the $x-y$ plane centered at a (a,b) is described by the equation:

$$R^2 = (x-a)^2 + (y-b)^2. \quad (2.2)$$

Of course if the circle is centered at the origin, then it looks more familiar as

$$R^2 = x^2 + y^2. \quad (2.3)$$

is described by the formula Figure 2.7 shows such a circle.

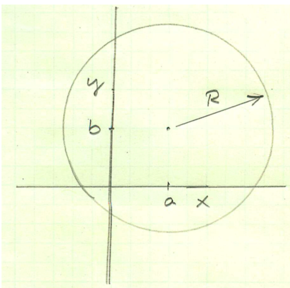


Figure 2.7: circle

Equation of a Parabola

A parabola in the $x-y$ plane with vertex at (a,b)

$$y = C(x-a)^2 + b \quad (2.4)$$

where C is a constant. Figure 2.8 shows a parabola.

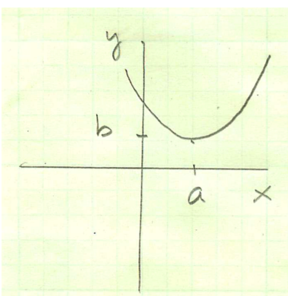


Figure 2.8: parabola

Area of a Rectangle

A rectangle with sides a and b has an area, A of

$$A = ab \quad (2.5)$$

Area of a Right Triangle

A right triangle (which means that one of the angles is 90 degrees) with base of a and height of b has an area, A of

$$A = 1/2ab. \quad (2.6)$$

For a right triangle, the base and height are equal to the two legs. But the formula works for any triangle. Figure 2.9 shows how that works.

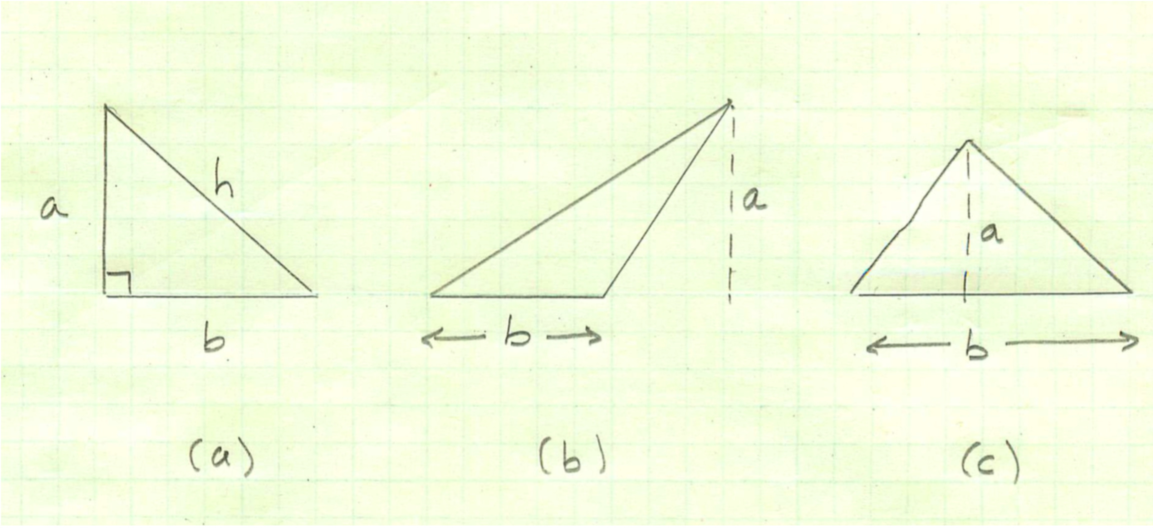


Figure 2.9: triangles

Area and Circumference of a Circle

For a circle of radius R, the area, A is

$$A = \pi R^2 \tag{2.7}$$

and the circumference, C is

$$C = 2\pi R. \tag{2.8}$$

Figure 2.10: You realize that two pizzas is a "circumference"? Because...wait for it...it's "2 pie are." You're welcome. (papajohns)

Pythagoras' Theorem

For a right triangle, the hypotenuse, h is related to the lengths of the two sides a and b by the Theorem of Pythagoras:

$$h^2 = a^2 + b^2. \tag{2.9}$$

2.6 Shapes of the Universe

One of the remarkable consequences of the mathematization of physics that began with Descartes is that we've come to expect that our descriptions of the universe will be in the language of mathematical *func-*

tions. Do you remember what a function is? The fancy definition of a function can be pretty involved, but you do know about function machines and I'll remind you how.

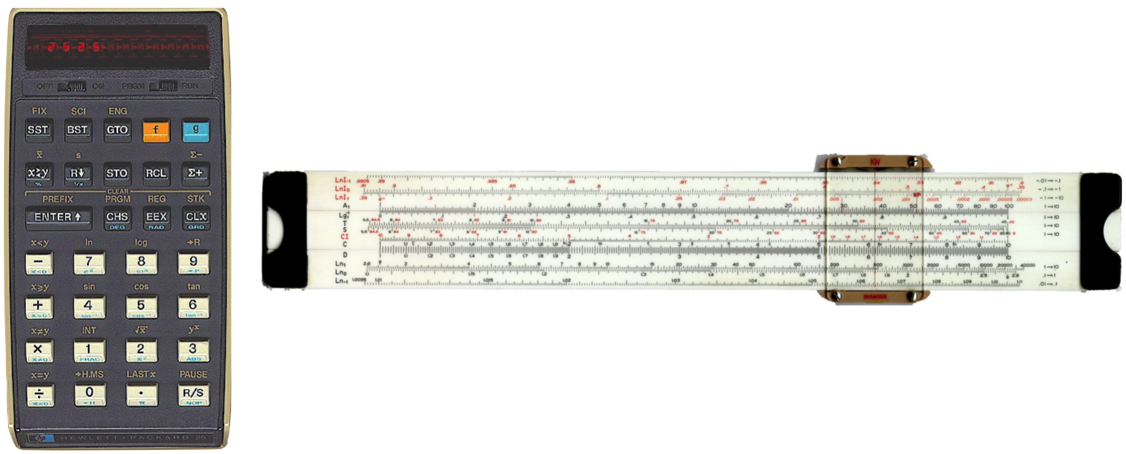


Figure 2.11: Left: the venerable HP-25 programable (!) scientific calculator. Right: a slide rule used for all calculations until the early 1970's. It was not programmable (although it was wireless).

When I was a senior in college, finishing my electrical engineering degree, our department had a visitor from the Hewlett Packard Company. It was either Bill Hewlett or Dave Packard, I can't remember which. But they promised to do away with the slide rule that we all carried around with us everywhere and showed us a brand new product: a portable scientific calculator, that they called the electronic slide rule. This was 1972 and he showed us the first HP calculator, the HP-35. Needless to say, I couldn't afford it—it cost $400— but later in graduate school I bought my first scientific calculator, the HP-25, pictured in Fig. 2.11 along with the slide rule that I carried for four years. Today I've got more processing power in my watch then I had in that calculator. But I'll bet you've got something like it...calculators are nothing but electronic function machines. So in the spirit of Fig. 1.5, Fig. 2.12 shows the circuit board from the inside of the HP-25 with it's simple processor at the bottom.

%

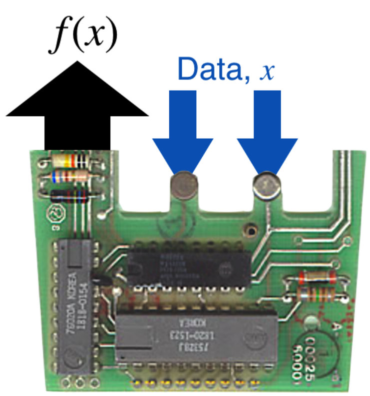


Figure 2.12: The AMI 1820-1523 Arithmetic, Control Timing processor: the heart of a function machine. Adapted for my silly purposes, but I'll bet you won't forget it! The tabs at the blue arrows are actually connected the processor to the keyboard. That's how data get in.

2.6.1 Functions: Mathematical Machines

Figure 2.12 shows what a function does: if you enter data through the keypad—a value of x—and hit the appropriate button, the display shows the value of the function. So if the function was the formula $f(x) = x^2$ and if I keyed in "4" and pushed the x^2 button, the display would read "16," the value of $f(2)$ for that particular function. Notice that it doesn't give you more than one result, and that's a requirement of a function: one result.

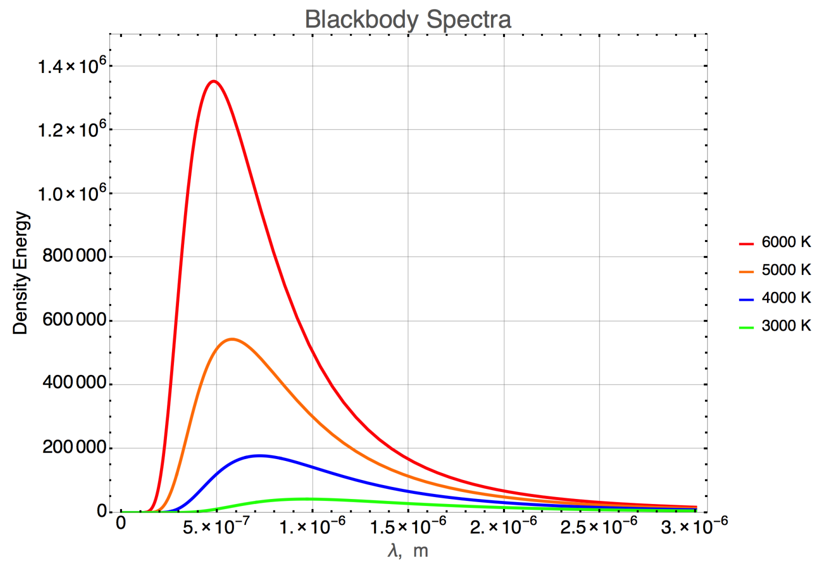


Figure 2.13: blackbodyvarious

So that's all a function is: a little mathematical machine that reports a single result for one or more inputs according to a rule. For us, functions can be represented by a formula, an algorithm, a table, or a graph. In all cases, it's one or more variables x or x & y... or x &y &z... *in*, a rule about what happens to them, and one numerical result *out*.

Your algebra teacher would have called the inputs (e.g., $x, y, ...$) the independent variables, which would have been members of the function's "Domain," and the output (e.g., $f(x, y, ...)$ or often y) the dependent variable, which would have been inside the "Range."

Nature seems to live by functions[28] and since in QS&BB we're all about Nature, we'll need to use functions. We'll solve actual formulas when they're simple functions and analyze plots of functions when they're complicated. For example, Fig. 2.13 is a function of two variables, a wavelength, λ and temperature (the units don't matter here). It's a messy formula which we'll admire, but not derive in Chapter **??**. But boy is it an important function. Here the little function machine calculates the value of the energy density of the radiation emitted by an object heated to a particular temperature. If you provide a wavelength and a temperature (in the figure, 3,000, 4,000, 5,000, or 6,000 degrees) to the function, then it reports back to you the value of the energy density that the body radiates. You can evaluate that function:

[28] Why? We don't know.

You Do It 2.5. Read Function

or copy the solution

What is the ratio of the value of the energy densities for one object at 4,000 degrees and another at 5,000 degrees at a wavelength of 1×10^{-6} meters?

There. You just evaluated a complicated function...twice.

2.6.2 Polynomials

Many of Nature's functions are in the form of polynomial equations, which are reminiscent of the quadratic equation:

$$f(x) = ax^2 + bx + c. \tag{2.10}$$

You may have "solved" this equation in a number of ways in your algebra classes. What solving means is finding the x's for which the value of the function is zero. There's also a geometrical interpretation of "solving" a polynomial and an algebraic rule for doing it. Notice that the quadratic has the form of the equation of a parabola, so let's look at an example:

$$f(x) = 2x^2 - 4x + 1.5. \tag{2.11}$$

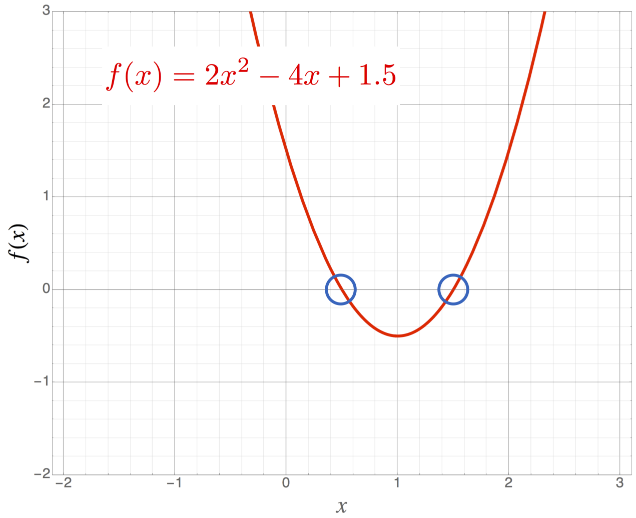


Figure 2.14: The quadratic function $f(x) = 2x^2 - 4x + 1.5$. plotted with blue circles at the points where $f(x) = 0$, the roots.

Remember that we can plot functions and Fig. 2.14 is a graphical representation of this function. When you solved a quadratic, you actually found the values of x for which the value of the function value—these are the “roots” of the function—of which there are two which I’ve called x_1 and x_2. So if we plug either into Eq. 2.10, then we will get $f = 0$..[29]

For quadratic equations, there is also a single formula to calculate the roots directly.[30] If we take Eq. 2.10 as the general form, then the “quadratic formula” you might remember from a former mathematics life is

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.12}$$

Of these two solutions: x_1 is for the + sign and x_2 is for the − sign.[31] So for our example in Eq. 2.11, $a = 2, b = -4$, and $c = 1.5$.

[29] Remember that the degree of a polynomial corresponds to the number of roots. For a quadratic, the degree is 2. For a cubic, it’s 3 and so on.

[30] For cubics, there is a procedure. For polynomials of higher degree, it’s complicated!

[31] Or the other way around—your choice.

You Do It 2.6. Function Root

For the example quadratic, use the quadratic formula, Eq. 2.12 to find the two roots of the function, Eq. 2.11. Do they match the "solution" you would get by looking at Fig. 2.14?

or copy the solution

A polynomial can be of any "degree," which is the highest power of x. Since the middle of the 16th century (Copernicus' time) mathematicians had figured out how to expand any such function for an arbitrary degree, like $(a+x)^n$, where n is a positive integer. This formula would save work since expanding $(a+x)^n$ if n was anything bigger than about 3 is a lot of calculating. Let's expand a quadratic polynomial, that is for $n = 2$:

$$(a+x)^2 = (a+x)(a+x) = a^2 + ax + xa + x^2 = x^2 + 2ax + a^2 \tag{2.13}$$

This old magic expansion formula is called the Binomial Expansion for polynomial of degree n—it has $n+1$ terms:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 \ldots + x^n \tag{2.14}$$

Until our hero, Isaac Newton came along, n was always a positive integer in this context.[32]

[32] Remember that the $n!$ notation stands for "n factorial." Which is $n! = n(n-1)(n-2)(n-3)\ldots 1$

Approximating Functions

Newton began inventing mathematics in the 17th Century and found a way to expand a formula for cases in which n could be anything: a positive integer, a negative integer, or even a fraction.[33] The result was an expansion that has an infinite number of terms! In contrast to how that sounds, it's actually very useful for many physics applications as we'll see.

[33] This was an essential step in the invention of the calculus. . . and the thing that Leibniz learned from Newton and used himself to invent a competing version of calculus. We'll touch on this in Chapter 5.4.1.

Let's take a particular case in which $a = 1$ and write it out Newton's idea in the same spirit as Eq. 2.14.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots \tag{2.15}$$

Here's where it will be interesting for physics. Look carefully at Eq. 2.15: each term is proportional to an increasing power of x, x^2, x^3, x^4 and so on. In physics, we can use this to make accurate approximations.[34] Suppose that $x < 1$. Then each term gets smaller and smaller since $x^3 < x^2$ and so on if $x < 1$...so each additional term *adds less and less* to the sum before it. Now we've got a little approximation-tool because many formulas that matter in physics look like

[34] While this sounds like just a work-saver, we'll see that it actually allows us to sometimes gain insight of some tricky physics. Be patient.

$$\frac{\text{something}}{(1 + \text{ something tiny})^{\text{ some power}}}$$

or can be rearranged to look like that.

Here's one that we'll use. Let's imagine the function

$$f(x) = (1+x)^{-1} = \frac{1}{1+x}.$$

Let's even plot it, which I've done in Fig. 2.15. Notice that this function becomes infinite when $x = -1$ and that it quickly falls until $x = 0$ and then slowly heads off towards zero as x becomes very large. That makes sense, right?

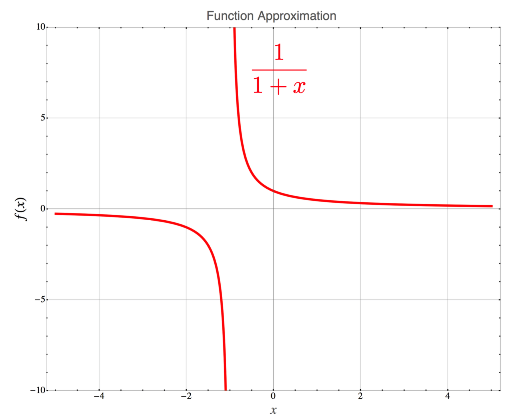


Figure 2.15: Our example function, $f(x) = \frac{1}{1+x}$.

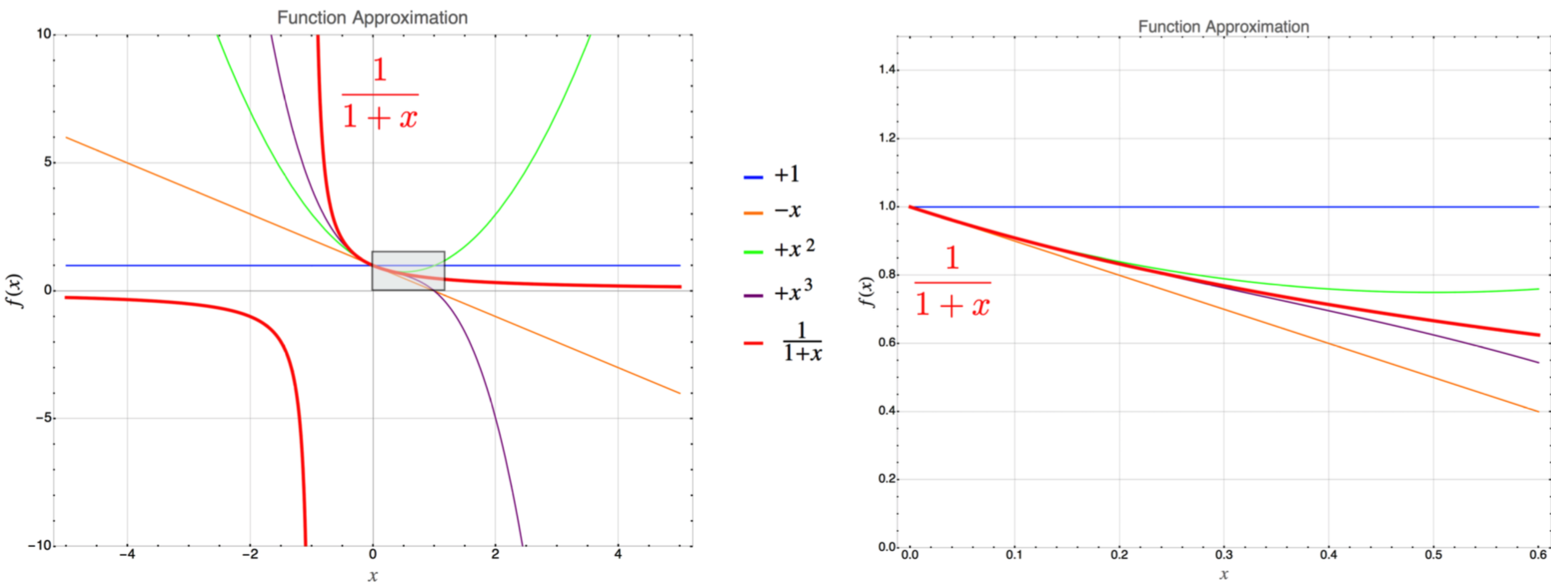
Now lets expand that function according to the approximation in Eq. 2.15. For this particular function, $n = -1$ and we will keep just the first four terms of the otherwise infinite number of terms:

$$f(x) = \frac{1}{1+x} \approx 1 - x + x^2 - x^3 \tag{2.16}$$

(By the way, the $\approx$ symbol in Eq. 2.16 stands for "almost equal to.") The right hand side of this equation is really the sum of four different, simple functions. When added together, we'll see that they get closer and closer to the original, depending on how many terms are included. Look at Fig. 2.16. The red curve in the left and right plots is our original function and the colored curves are each getting closer and closer to it.

The blue "curve" is the trivial function that's the first term in Eq. 2.16: $f = 1$. The orange curve takes the second term in Eq. 2.16 and adds it to the first, so it's $f(x) = 1 - x$. The green curve adds the third term, x^2

Figure 2.16: See the text for an explanation. The right plot is a blow-up of the left around the gray box.



to the orange curve and so on. The right plot is a blowup of the region in the gray box on the left. Notice that in the region of x which is very small, the few functions are a pretty good approximation to the red. The more terms we might add the further out in x that agreement would continue.

Remember this! It will become important later when we'll encounter functions and approximate them with a few terms of the expansion from Eq. 2.15. Here are the functions that we'll see in the pages ahead:

$$\sqrt{1+x} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \ldots \tag{2.17}$$

$$\frac{1}{\sqrt{1-x}} = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \ldots \tag{2.18}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \tag{2.19}$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \tag{2.20}$$

2.7 Euler's Number

You all know that π is an unusual number. It's simply the ratio of the circumference of a circle to its diameter (see Eq. 2.7) and, the Indiana Legislature[35] not withstanding, it's a number that has a decimal representation that never ends. It's "irrational" and has the (approximate!) value:

$$\pi = 3.1415926536\ldots \text{ forever!} \tag{2.21}$$

There is another irrational number that plays a big role in mathematics, but also in many other areas of "regular" life. It's called "Euler's Constant" although the prolific mathematician Euler didn't first discover it, he discovered many of its unique features and so his name is associated with it. We physicists tend to just call it "e" since that's the symbol that is used to represent it. It has the value:

$$e = 2.71828182845904523536\ldots \text{ forever!} \tag{2.22}$$

Euler first used e to understand compound interest. If you invest \$1 at a compounded interest of 100% per year, then at the end of the year your wealth would have been increased by a factor of e. While not many savings plans grant 100% interest, you get the point. It figures into the calculation of any interest rate. I'm going to try to convince you that it appears in many guises.

The importance of e in science comes from the fact that the rate at which e increases or decreases is proportional to *itself*. So if something increases by e^{ax} then the rate at which it increases is ae^{ax}. This leads directly (with some calculus) to the rule for how radioactive nuclei, atomic systems, or elementary particles decay. Suppose we start out with N_0 radioactive nuclei with a "lifetime" called τ at a time $t = 0$, then the number of left after a time t is equal to

$$N = N_0 e^{-t/\tau}. \tag{2.23}$$

So the fraction left is $\frac{N}{N_0} = e^{-t/\tau}$. Figure 2.17 shows two curves for both the exponential decay and exponential growth formulas.

But it's not only some sort of modern physics thing. Atmospheric pressure decreases the higher up you go...this is because there's less air above you. So home runs in Denver's Coors Field go further than in Chicago's Wrigley field since Denver is about a mile higher than Chicago. We could pretty closely calculate the density at any altitude using this same formula, but modified for the physical situation. Let's call the density of air at any height above sea-level (y) to be $\rho(y)$. Then if we let $\rho(0) \equiv \rho_0$ then the function that describes the density at any height turns out to be

$$\rho(y) = \rho_0 e^{-y/8000}. \tag{2.24}$$

[35] Yes, that story is true. In 1897 state legislature representative, Dr. Edward J. Goodwin, a physician who dabbled in mathematics, proposed changing the value of π to 3.2. The bill sailed through the House but was postponed indefinitely in the Senate. It seems that Professor C.A. Waldo at Purdue was horrified enough that he intervened and the bill died.

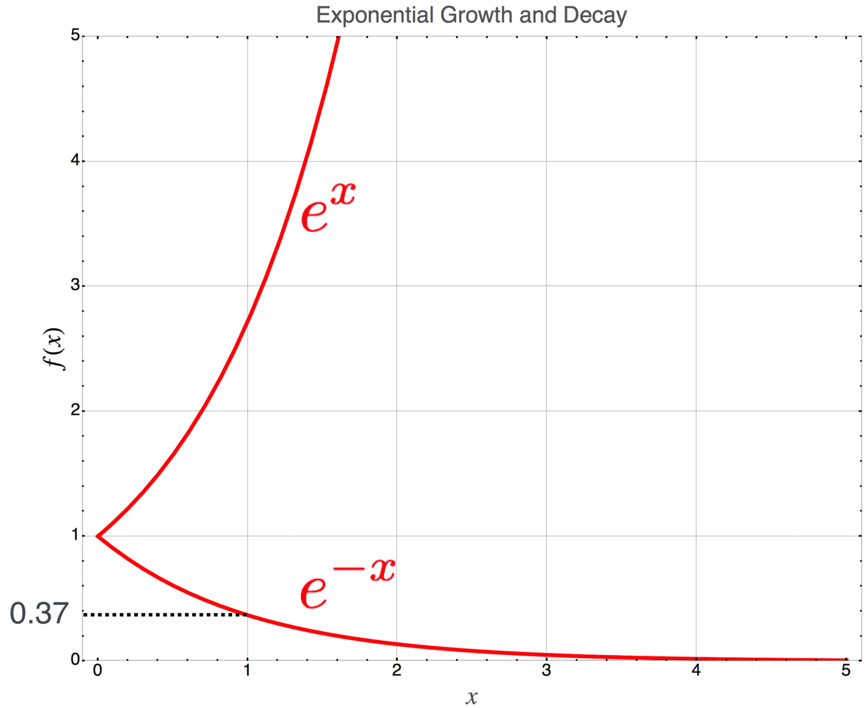


Figure 2.17: exponentials

where the distance above sea level, y is measured in meters. Let's do one more thing and then we can use our curves, even though the axes are just relative numbers. So we could directly ask the fractional change in density:

$$\frac{\rho(y)}{\rho_0} = e^{-y/8000} \tag{2.25}$$

Relative to sea level, then a mile high (1,609 m) makes the right side $e^{-(1609/8000)} = e^{-0.2}$ so we can use the general graph in Fig. 2.17 since we've determined that $y = 0.2$,[36] At that value, read across, we see that the density is reduced to about 80% of what it would be at $x = 0$. So,

$$\frac{\rho(y)}{\rho_0} = 0.8. \tag{2.26}$$

[36] Of course, we're using y in the formula for height, which is often a convention, but it's still playing the role of the x in the general graph.

Not everything in nature decays! Suppose you're a biologist studying bacterial growth. If a particular strain grows continuously at a rate of 5% per day, you could predict the size of the colony after some number of days.[37] The growth in the colony where t is measured in days is given by

$$F(\text{bacteria}) = F_0 e^{Rt} = F_0 e^{0.05t} \tag{2.27}$$

[37] Or, you could measure the increase and write the function that describes it.

where F(bacteria) is the number of bacteria after a time t and F_0 is the number that you started with. For a different bacterium, R would be a different number (a "rate"). If we waited patiently for about a month, say $t = 30$ days, we'd have

$$F(\text{bacteria in a month}/F_0 = e^{Rt} = e^{(0.05\times 30)} = e^{1.5} \tag{2.28}$$

Back to Fig. 2.17 with $x = 1.5$ the top graph reads about 4.4. So if we started with a population of 100, after 30 days it would have grown to $4.4 \times 100 = 440$.

This is what people mean when they refer to "exponential growth"—a very rapid increase in some phenomenon.

2.8 Vectors

We're about to talk about motion, but let's make an important point here that will be obvious. When you're driving on the highway and your (American) speedometer reads "60 mph," it's telling you the *speed* not your direction. Going 80 mph north is as much over the speed limit as going 80 mph east since speed is all the highway patrol radar cares about. (There isn't one speed limit for easterly travel and another for when the road bends north.)

The cops might not care, but you care a lot whether you're traveling north at 60 mph or east, since in order to get where you're going on schedule–your trip depends not only on how fast you go, but in what direction. The difference between *speed* and *velocity* is critical. Not all quantities are vectors...for example, what's the direction of a temperature? But, velocity, space coordinates, force, momentum, electric and magnetic fields, and many other physical quantities have directions as well as values.

A vector has both a magnitude and a direction *Key Concept 3*

There's an algebraic way to represent vectors, but we'll not need that. Instead we'll make use of the handy symbol of an arrow: $\rightarrow$. The length of the arrow represents the magnitude and of course the orientation and the head of the arrow represent the direction. Arrows can be $\longrightarrow$, or short $\rightarrow$, pointed in different directions, ↖, ←, ↗, etc. Very handy. The magnitude can mean many things, depending on the physical quantity being represented. Obviously, the simplest would be a distance in space, like an arrow on a map or a whiteboard during time-out. That's it.

Here's a way to think about them. Suppose you're in a strange city and you want to know how to get from your hotel to a particular restaurant. You go to the front desk and you're told that you need to walk for 7 blocks, Terrific. Now what? Seven blocks that way? Or, seven blocks the other way! Rather, "walk 4 blocks, east and then 3 blocks north" is more helpful, as you can see in Fig.~\ref{blocks}. (It's just like velocity.)

Now we can go around writing "four blocks east" (or "60 mph north") everywhere, but we need a better notation that packs both *directional* and *magnitude* information into a single symbol so that our hotel-restaurant stroll east is succinctly distinguished from one to the west (and so we don't need to use words in our equations). Traditionally, in print, a vector is represented as a bold letter.[38]

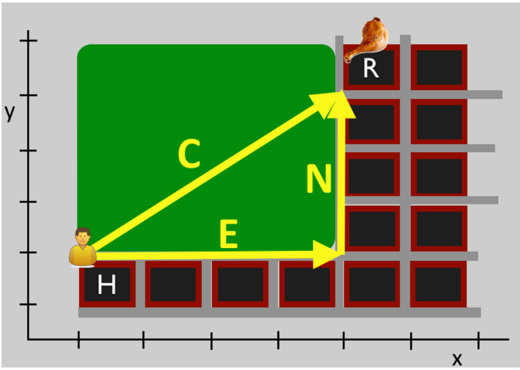


Figure 2.18: The layout showing my hotel (H), the restaurant (R) where there is fried chicken waiting, and the city block structure.

[38] There are at least three ways that I can think of to represent vectors. In print, the bold face **x** is most common. On a blackboard, usually people will draw an arrow over the top, $\vec{x}$. And, finally, some people put an underline when they write, $\underline{x}$.

Notation in equations is fine, but pictures of vectors are going to be most useful for us. It's easiest to think in terms of distance vectors. Just like "speed" and "velocity" are related, we can think of "distance" and "displacement" as analogs. So, our hotel tells us that the restaurant is a *distance* of 7 blocks away and that its *displacement* is "4 blocks, east and 3 blocks north" and we draw a picture to describe that instruction. Figure 2.18 shows two vectors that do that:

2.8.1 Vector Diagrams

Drawing arrows on a diagram represent a vector with its orientation representing the direction and its length representing the magnitude. Sometimes the length of the arrows are actual length dimensions (like meters, feet, and so on), since a displacement in regular-space is a vector. So, just like a scale on a map, a displacement can be represented as an arrow which is 3 inches long, but where each inch actually corresponds to 1 block (or feet, or miles, or furlongs). But, sometimes a vector doesn't represent a length in space, but some other physical quantity, like a force or a velocity. Now, this can be complicated since you're drawing an arrow that has a length, but you mean it to be something else, like a force. But, it still works geometrically (the arrow still points in space) and we just use a different scale: we might draw an arrow aimed at a box on a diagram that's 2 inches long where every inch corresponds to 2 pounds. So even though it's drawn on a diagram of an object, it represents the application of a force of 4 pounds applied at the point where the arrow is drawn. That's just a visual convenience since the length of the vector in pounds wouldn't have anything to do with any of the length scales in the picture that are lengths or heights.

For a couple of definitions, refer to Fig. 2.19. There are two basic ways to represent vectors, one for print and the other for blackboards (or pencils). The print version is to render the vector quantity as a bold letter. So in Fig. 2.19 the vector on the top is in print **A** and on paper we would write $\vec{A}$.

Two vectors, **A** and **B** are said to be equal if they are *both* the same length *and* point in the same direction. So, as shown **A** = **B**, but neither is equal to **D** even though the length of **D** is the same as that of **A**. Also, we say that **A** = −**C** if the vectors have the same length, but are pointing in exactly the opposite directions. This is shown in Fig.~2.19b. Another standard definition is to represent the magnitude of a vector–its length–using the symbol |**A**|. This quantity is a number, not a vector and so we would say that |**A**| = |**D**|.

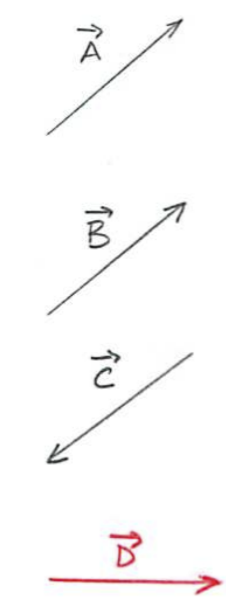


Figure 2.19: Vectors **A** and **B** are equal, and each is equal to −**C** and none are equal to **D**, even though the lengths are all same.

2.8.2 Combining Vectors

If you help me to push on my car, we're each applying a force. The whole reason for the two of us is not so we can bond in a shared accomplishment. That's not a guy thing. No, the reason we do it is that we each supply a force and the car then gets pushed with more force than either of us could supply by ourselves. That is, our forces add...and maybe we bond a little. So, vectors can be added both in symbols, and with pictures.

We can add vectors together by manipulating the arrows. If in our little moment together, I'm **A** and you're **B** then, the car gets pushed by our combined force as shown in Fig. 2.20(a). However, the car would not know the difference between being pushed by the two of us and by some brute who pushes with the force of our combined effort, which we'll call **C**.

$$\mathbf{C} = \mathbf{A} + \mathbf{B}. \tag{2.29}$$

Pencil 2.4.

To calculate this using pictures, you can place the tail of **B** to the head of **A** and then the displacement from the tail of **A** to the head of **B** is the sum, **C**. This is shown in Fig. 2.20(b), and the replacement of the two forces is shown as Fig. 2.20(c). It's important to realize that the situation (a) and (c) are identical, but you would not put both$ **C**$ and the two **A** and **B** on the same picture. It's one or the other.[39]

Notice, that for doing sums, we can translate vectors around our "space" if we don't change their orientation or length. I did that in the figure.

[39] Dare I carry my little story this far? It's as if I push on the car, and you push on me. If my arms hold up, we still push on the car with the combined force. But, I'd rather not do it that way, thanks.

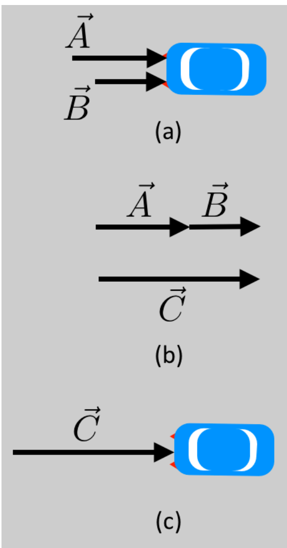


Figure 2.20: (a) Both of us pushing on a car; (b) the combination of our two force vectors; and (c) the replacement of our two independent forces with the combined force. The car doesn't know the difference between (a) and (c)!

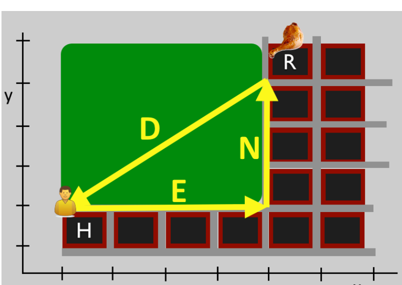


Figure 2.21: The same situation as before, but with the hotel-restaurant trip shown and the restaurant-hotel return shown on the same picture.

The car example was all in one dimension, but of course vectors are useful in 2, 3 or more dimensions. Let's go back to our trip to the restaurant from our hotel. What I didn't know, was that there was an open park just behind my hotel, and I could have cut across it to get to the restaurant. That is, an equivalent displacement would have been to follow **C** as shown in Fig. 2.18. That's all the adding of vectors says: a single vector that's equivalent to the operations of the first two. So my trip has two different paths (well, an infinite number):

$$\mathbf{C} = \mathbf{E} + \mathbf{N}$$

Notice that the two vectors don't point in the same direction, so it would be wrong to calculate the distance that **D** represents by just adding the lengths of **E** and **N**. That is, the magnitude of **D**, $|\mathbf{D}| \neq 4 + 3$. We have to keep the directions and the lengths pointing in their directions separate.

One more way to look at this trip–which resulted in a nice dinner, by the way–would be if we returned to the hotel across that field, then our trip would look like Fig. 2.21.

Notice, that it's different from Fig. 2.18 in that **D** points in the opposite direction from **C**. It's a "round trip" and so the total displacement in a round trip is: zero. In algebra, what this says is:

$$\mathbf{A} + \mathbf{B} + \mathbf{D} = 0$$

Any time you can rearrange a set of vectors to give a "round trip," you describe a situation in which there is no net displacement (we went from the hotel, back to the hotel), or if they are forces, no net force, or if they are velocities, no net velocity. It's a balance $\mathbf{A} + \mathbf{B}$ is balanced by its opposite, **D**. The other way to think of this is remembering that we could have gone to the restaurant across the field if we'd known about it. Notice, that then the vector describing that trip would be $-\mathbf{D}$. We replace $\mathbf{A} + \mathbf{B}$ with $-\mathbf{D}$. And, the balance is just the obvious: $-\mathbf{D} + \mathbf{D} = 0$. This balancing of vectors will be an important concept to us as we'll see in Chapter 6.5.

Finally, we can also subtract vectors graphically which is easiest to think about if we think about this almost silly statement:

$$a - b = d$$
$$a + (-b) = d$$

This says that the adding the negative of b to a is the same as subtracting it from a. With vectors, this is a little more meaningful. Referring to Fig. 2.21, let's create a vector subtraction.

$$\begin{aligned} \mathbf{C} &= \mathbf{E} + \mathbf{N} \\ \mathbf{D} &= -\mathbf{C} \\ -\mathbf{D} &= \mathbf{E} + \mathbf{N} = \mathbf{C} \end{aligned}$$

So, we change a subtraction of vectors into an addition of vectors by just turning the appropriate one around.

In order to make the negative of a vector, turn it around and reverse its direction. *Key Concept 4*

2.9 What To Take Away

> "…it is impossible to explain honestly the beauties of the laws of nature in a way that people can feel, without their having some deep understanding of mathematics. I am sorry, but this seems to be the case.
>
> "You might say, 'All right, then if there is no explanation of the law, at least tell me what the law is. Why not tell me in words instead of in symbols? Mathematics is just a language, and I want to be able to translate the language.' … I could convert all the symbols into words. In other words I could be kind to the laymen as they all sit hopefully waiting for me to explain something. Different people get different reputations for their skill at explaining to the layman in layman's language these difficult and abstruse subjects. The layman searches for book after book in the hope that he will avoid the complexities which ultimately set in, even with the best expositor of this type. He finds as he reads a generally increasing confusion, one complicated statement after another, one difficult-to-understand thing after another, all apparently disconnected from one another. It becomes obscure, and he hopes that maybe in some other book there is some explanation…The author almost made it—maybe another fellow will make it right.
>
> "But I do not think it is possible, because mathematics is not just another language. Mathematics is a language plus reasoning; it is like a language plus logic. Mathematics is a tool for reasoning."
>
> Feynman, R.P. (1965) *The Character of Physical Law* BBC. Reprinted by Penguin Books, 1992

Bibliography