Chapter 2

Everyone Needs Tools

A little math


René Descartes by Franz Hals, circa 1649

René Descartes (1596-1650)

"When I imagine a triangle, even though such a figure may exist nowhere in the world except in my thought, indeed may never have existed, there is nonetheless a certain nature or form, or particular essence, of this figure that is immutable and eternal, which I did not invent, and which in no way depends on my mind." *Meditations on First Philosophy* (1641)

It's always amazing to me , just how much we depend on the collaborative work of a handful of people from the 1600s. There must have been something in the water....in France, Italy, Britain, and Holland because this was a time of genius and courage. From people in this period—a number of whom we'll become familiar with—we received a way of thinking about, talking about, and poking at the world. René Descartes is one of my particular favorites. Let's learn a little bit about him.

2.1 Goals of this chapter:

- Understand:
  - Simple one-variable algebra.
  - Exponential notation.
  - Scientific notation.
  - Unit conversion.
  - Graphical vector addition and subtraction.
- Appreciate:
  - The approximation of complicated functions in an expansion.
- Be familiar with:
  - Descartes' life.
  - The importance of Descartes' merging of algebra and geometry.

2.2 A Little Bit of Descartes

The 17th century and just before saw a proliferation of "Fathers of –" figures: Galileo, the Father of Physics; Kepler, arguably the Father of Astrophysics, and Tycho Brahe, the Father of Astronomy. But the Granddaddy...um...Father was René Descartes (1596-1650), generally considered to be the Father of Western Philosophy and a Father of Mathematics.[1] If you've ever plotted a point in a coordinate system, you've paid homage to Descartes. If you've ever plotted a function, you've paid homage to Descartes. If you've ever looked at a rainbow? Yes. Him again. If you ever felt that the mind and the body are perhaps two different things, then you're paying homage to Descartes and if you were taught to be skeptical of authority and to work things out for yourself? Descartes. But above all—for us—René Descartes was the Father of analytic geometry.

He was born in 1596 in a little French village now called, Descartes.[2] By this time Galileo was a professor in Padua inventing physics and Caravaggio was in Rome inventing the Baroque. Across the Channel Shakespeare was in London inventing theater and Elizabeth had cracked the Royal Glass Ceiling and was reinventing moderate rule in England. This was a time of discovery and dangerous opinion when intellectuals began to think for themselves. That is, this is the beginning of the end of Aristotle's suffocating domination as The Authority on everything.[3]

[1] Who's your daddy, indeed.

[2] Coincidence? What do you think.

[3] After all, by the time St. Thomas absorbed Aristotle into Catholic dogma, he was called The Philosopher.

Descartes' mother died soon after childbirth when he was only a year old and he was raised by relatives. His' father was an upper-middle class lawyer who spent little time with his children.[4] He was sent to a prominent Jesuit school at the age of 10 and only a decade later emerged from the University of Poitiers with the family-expected law degree. Apart from his success in school, the most remarkable learned skill was his lifelong manner of studying. He was sickly as a child and had been allowed to spend his mornings in bed, a habit he retained until the last year of his life.[5]

One of the benefits of his schooling was a program to improve his physical conditioning, enough so that he became a proficient swordsman and soldier—he wore a sword throughout his life as befitting his status as a "gentleman."[6] And yes, he was essentially a soldier of fortune. During the decade following his graduation, he would alternate his time between combat assignments in various of the innumerable Thirty Year's War armies and raucous partying in Paris with friends.[7]

Somewhere in that period Descartes became serious and decided that he had important things to say. He wrote a handful of unpublished books and maintained a steady correspondence with intellectuals in Europe, becoming well-known through these letters. Catholic France and of course Italy, were becoming intolerant of challenges to Church doctrine and he moved to the relatively casual Netherlands in 1628. Mostly a good move: he'd been inspired by Galileo's telescopic discoveries and became a committed Copernican and in 1633 was completely spooked by the Italian's troubles with the Inquisition.[8] However, he had trouble with some evangelical protestant leaders in Holland.

Little did Descartes know that he was a mathematical genius. After study as a "mature" student at the University of Leiden, he found that he could solve problems in geometry that others could not. His devotion to mathematics and especially the rigor of the deductive method stayed with him and turned him into a new kind of philosopher. The logic of deduction and the certainty of mathematical demonstration were his philosophical touchstones.

Remember "deduction"? All squirrels are brown; that animal is a squirrel; therefore, that animal is brown kind of arguments? The important thing about this string of phrases is not that animal's color, but that the conclusion *cannot be doubted* if the two premises are true. Since Plato, "What can I know for sure?" was an essential question. For that particular Greek, things learned through your senses are untrustworthy. Only things you can trust are ideas which are eternal, outside of space and time. For other famous Greeks, you learn about the world through careful observation. Famously, Descartes convinced himself that he had discovered a method to truth: whatever cannot be logically doubted, is true.

[4] When Descartes' father died, his brother failed to notify him (he found out through one of his correspondents) and he decided he was too busy to attend the funeral. Not exactly a close family. The similarities with Newton's childhood are striking.

[5] There's a story there...

[6] He still worked in bed every morning until noon.

[7] He was a talented gambler, as befitting a mathematical mind.

[8] That year, one of his major books, *The World*, was ready for publication, but he delayed it until after his death. In *World*, he expounded Copernicanism, but also provided for a reason why the planets circled the sun. A mechanism that Newton demolished with gusto.

2.2.1 Descartes' Philosophy

This is not the place to teach the huge subject of Descartes' philosophy. But there are two aspects of his work that directly influence the development of physics: what can we know and what is the nature of the natural world.

Descartes believed he'd found the formula for determining what's true: when an idea is clear and distinct, which means incapable of being doubted, then you can believe it. His method was to keep doubting *everything* until you reach a point in this thought-process that can't be doubted.[9] The point he reached was the recognition that *he* was doing the doubting. Since that can't be doubted, then what he's learned that's true is: thought exists. One more step to *I exist*, because it is I who is doing that thinking: "Cogito ergo sum"[10] was his bumper sticker for truth.

[9] In this way you reduce a complex problem to a more manageable one. . . one of his essential components to his "analytic philosophy."

[10] "I think, therefore I am." Words to live by.

The rest of his argument is a little shaky but this is the beginning of dispassionately and vigorously analyzing a philosophical problem, setting a high bar for argument. Of course, Medieval thinking was not friendly to the idea that everything can be doubted. The Bible and pretty much all that Aristotle wrote was off-limits. In fact, under the rules of thought not only could neither source be doubted, those sources were the only authority used to determine truth and falsity. Descartes pretty much changed that in philosophy.

He called his method "analytic" and it's essentially applying mathematical problem solving strategies to philosophical questions. Hence, history's assignment of paternity to him for Western Philosophy.

He said later that he made this discovery about doubt while still a soldier and holed up on a snowy night alone in a remote cabin. Sometimes his military escapades were real combat, but mostly it seems like he had a lot of leisure time.

For our purposes, what he decided were that true things about the world could be obtained through pure thought. This is the "Rationalist" philosophy of which he is the king. This is in the spirit of Plato, but unlike Descartes, he gave up on the sensible world as simply a bad copy of the Real World, which is one of Ideas..."out there" somewhere. By contrast, Descartes asserted that there are two substances in the universe. One is mind and the other is matter. Understanding the universe means gaining knowledge of both by blending thinking with observing.

We'll see that physics takes some inspiration through Descartes' approach. Theoretical physicists are often motivated by knowledge gained through thought—and always mathematics—and many work as if those thoughts are representing the world.

Definition: Rationalism.
The only test of and source of knowledge is reason.

Definition: Empiricism.
All knowledge originates in experience—through experiment and observation.

This two-part universe is now called Cartesian Dualism and was all the rage when Newton was a student. But the important thing to take away from this is that Descartes is the proud proponent of the notion that true knowledge can be obtained purely through thought. The counter to this Rationalist belief is Empiricist belief, that knowledge can only be obtained through observation (and in modern form, experiment).

The other aspect of Descartes' philosophy that matters[11] is his notion of Mechanism. The Renaissance was saturated with ideas of nature that we'd consider magic. Nature was infused with occult properties,

[11] no pun intended. . . sort of.

that it is almost alive with "active principles," even human-like in ways. Of course, astrology, alchemy, signs and numerology, Cabala, black magic and white natural magic, and so on were aspects of organized occultism. But it went deeper. People lived lives, tended the sick, and found explanations for natural phenomena based on the assumption that what we would call inert natural objects were alive and possessed magical powers. This continued a long-standing philosophical discussion about Qualities. Is the boiling pot hot because it possess the innate quality of "hotness"?

Magical thinking was a threat to the Church and Descartes also subscribed to the growing program of ridding nature of these features. Things in the world are not possessed of innate features like hot or cold, blue or red, and so on. These for Descartes are attributes not innate qualities. "Things" possess...place. Now we'll think a bit later about what constitutes space, but for Descartes and others, space is determined by the extent of objects. In fact the only aspects of matter that are "clear and distinct" (and hence true) are that matter has the properties of spatial extent (length, width, height) and motion.

He needed to have a mechanism to explain everything in the material world. He explained motion as the point-to-point pushing of material objects that we see (planets) by innumerable, small-sized, varied atoms which are indivisible. This "plenum" of stuff is moving, initiated by God, and they preserve that motion as they transmit it to all moving material objects.[12] It's communicated to the planets, through vortices, as in Fig. 2.1 from *The World*.

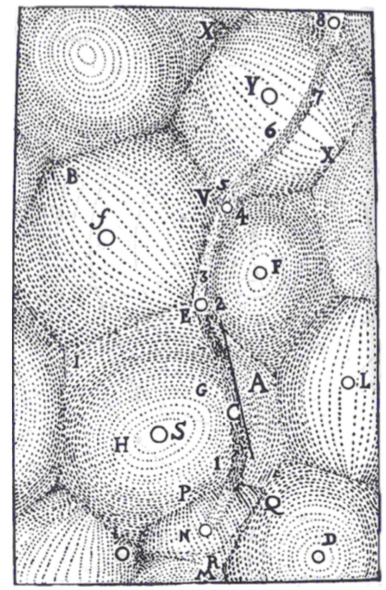


Figure 2.1: plenum

[12] Remember this when we get to momentum and energy!

Likewise magnetism. Boy, that's an occult-ish phenomenon if there ever was one. To Descartes magnetism was propagated by little, tiny left-handed screw-like object that find threaded holes in iron so as to attract or repel. Gravitation is another kind of material experience. First, Descartes hypothesized about a material cause for phenomena and then deduced the consequences.

Descartes paved the way for a reasoned approach to physics, that turns out to have been a part of the story. He motivated Newton and helped European thinkers to find their way to independent ideas, shedding the overbearing weight of Aristotelianism and Church dogma.

But this chapter is devoted to mathematics.

2.2.2 Descartes' Algebra-fication of Geometry

...or geometrification of algebra! Whatever. Descartes brought geometry and algebra together for the first time by reinterpreting the latter and inadvertently, rendering the former less important.[13]

[13] for a while.

Descartes pulled the very new, very unsophisticated new method of "algebra" to a role of supremacy over geometry. He did this by linking the solution of geometry problems—which would have been done with rule-obsessive construction of geometrical proofs—to solutions using symbols. He did this work in a

[14] *Geometry* can be considered an appendix to the *Discourse*.

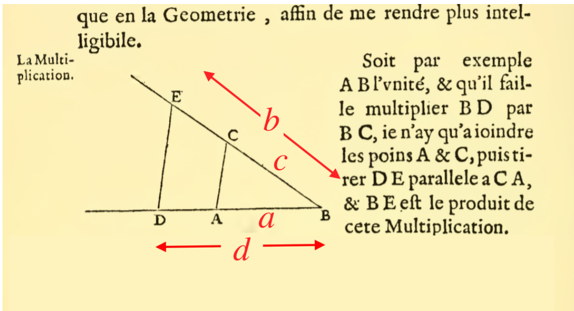


Figure 2.2: geometrymultiply

small book called *Le Géométrie* (*The Geometry*), which he published in 1637, the same year he published his *Discourse on Method.*[14]

He instituted a number of conventions which we use today. For example, he reserved the letters of the beginning of the alphabet $a, b, c, ...$ for things that are constants or which represent fixed lines. An important strategic approach was to assume that the solution of a mathematical problem may be unknown, but can still be found and he reserved the last letters of the alphabet $x, y, z...$ to stand for unknown quantities—variables. He further introduced the compact notation of exponents to describe how many times a constant or a variable is multiplied by itself.

Prior to Descartes, ab would be the product of a and b but explicitly refer to the area of a rectangle bounded by legs of lengths a and b. a^3 would be the volume of a cube. There would be no such thing as $abcd$ or a^4 because after all, nature has no more dimensions than 3. So the early algebra was confined to a strictly dimensional context. Descartes broke with that and explored equations of higher powers, even showing that equations of higher powers could be reduced to lower power equations and so on until a solution could be found. He did this algebraically and geometrically, side by side. In fact, *Le Géométrie* is just one example worked out after another: it's solutions-oriented. And it's abstract. There's no need to identify "things" to the variables, although one could do so if desired.

Just as arithmetic has addition, subtraction, multiplication, division, and square roots...so to he found geometrical interpretations of these operations. His geometrical description of multiplication—not referring to an area—is instructive of how he did things. FIgure 2.2 shows a figure from *Le Géométrie*. Using his notation, we immediately come upon a new "invention" of his: unity. A line of length "1" could be chosen arbitrarily, and then manipulated.

In Fig. 2.2 I've overlaid red letters in the fashion that Descartes would have, assigning a single letter to represent a line. The lines $\overline{DE}$ and $\overline{AC}$ are both parallel and so the triangles BED and BCA are similar. From elementary geometry, because of their similarity, we would have

$$\frac{b}{d} = \frac{c}{a}.$$

Now he does this clever thing with "1" and assigns the length $\overline{AB}$ to have length 1 so that we have

$$\frac{b}{d} = \frac{c}{1}.$$

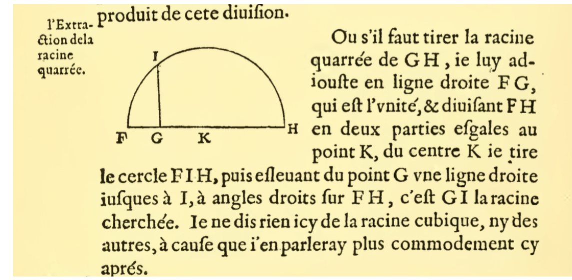
and so the product of $cd = b$. No areas. A brand new use of the brand new algebra!

Here's another example from *Le Géométrie*. Supposed you want to find the square root of a quantity. Figure 2.3 is again from his book. His trick here is to assign the distance $\overline{GH}$ to be an arbitrary length x[15] and the distance $\overline{GI}$ to be y. His goal is to compute the $\sqrt{y}$ for this abstract situation. Again, he uses the

[15] See? Algebra with unknowns.

"1 trick" and makes $\overline{FG} = 1$. The end result is that $y = \sqrt{x}$ and the problem is solved in general terms and in a way that could be measured with a ruler. Like Euclid would have liked.

The early translators of algebra considered equations in two unknowns—some $f(x, y) = 0$—to be impossible. Descartes actually found a way by treating the locus of points on a line as indeterminant, some abstract x. GIven any particular location along x however, another corresponding to the other unknown variable could be identified. He called such a point y and then worked to find solutions to particular problems that might be different depending on what the value of x was...but he did it in a way that was general for any x. This is the first example of what we'd now refer to as an axis. He didn't actually use two axes, but he still solved problems for an unknown y in terms of a parameter x. He called one of these the abscissa and the other, the ordinate.


l'Extraction dela racine quarrée.

produit de cete diuiſion.

Ou s'il faut tirer la racine quarrée de G H, ie luy adiouſte en ligne droite F G, qui eſt l'vnité, & diuiſant F H en deux parties eſgales au point K, du centre K ie tire le cercle F I H, puis eſleuant du point G vne ligne droite iuſques à I, à angles droits ſur F H, c'eſt G I la racine cherchée. Ie ne dis rien icy de la racine cubique, ny des autres, à cauſe que i'en parleray plus commodement cy aprés.

Figure 2.3: root

Mathematicians picked up on these ideas and extended them into the directions that we now love. One of those was John Wallis (1616-1703), a contemporary of Isaac Newton who learned from Wallis enough to construct the general Binomial Theorem.

The use of perpendicular axes, which we call x and y stems from Descartes' inspiration which is why they're called Cartesian Coordinates.

Descartes managed to get himself into a dispute with a Calvinist theologian, Gisbertus Voetius who wanted his university to officially condemn the teaching of "Cartesian Philosophy" as atheistic and bad for young people. Descartes responded by printing a reaction which was posted on public kiosks. This must have been quite a sight! In any case, Descartes began to imagine that his time in the Netherlands was coming to a close. An admirer, the Queen Christina of Sweden, was an intellectual of sorts and invited Descartes to Stockholm to work for her court and to teach her. She even sent a ship to Amsterdam to pick him up. He eventually accepted the position and this was the beginning of the end for him.

She required his presence at 4 AM for lessons. This, from the fellow who had spent every morning of his life in bed until noon! He caught a serious respiratory infection and died on February 11th, 1650 at the age of only 53.

We moderns owe an enormous debt to this soldier-philosopher-mathematician. Both for what he said that was useful and for what he said that was nonsense, but which stimulated productive reaction. In what follows from Section 2.5 there is a direct line from every word back to René Descartes.

2.3 Introduction

In this chapter we'll do some old things and some new things. Some of the old things will be mathematical in nature, while some of the new things will include some terminology and some techniques. I promise that the math will not be hard and we'll get through it together. We'll develop just a few of these tools that we'll return to repeatedly: simple algebra, exponents, unit conversions, and powers of ten. It will come back to you.

But I want to start with some topics which are timely and confusing to non-specialists. What are we doing when we "do" science?

2.4 It's Theory, All the Way Down

Coming.

2.5 The M Word

The language of physics is mathematics, so uttered Galileo a long time ago (although he said that the language of the *universe* is mathematics). Well, he was right and we have no idea why that seems to reliably be the case! So the importance of that realization will become clear as we go, which is partly why I don't want to avoid mathematics altogether. But it will be relatively simple. You've seen everything I'll ask you to do in high school, at the very least. It will be fine. Let me show you.

Wait. *I'm not a math person.*

Glad you asked. *Actually, nobody is. Really mathematics is a habit of mind and strategy for how you read. Certainly for what we're going to do. I promise you. Read with your pencil out. Read every line with a mathematics symbol. You'll get it.*

2.5.1 Some Algebra

Our algebraic experience here will be some simple solutions to simple equations. I'll need the occasional square root and the occasional exponent, but no trigonometry or simultaneous equation solving and certainly no calculus. I'll refer to vectors, but you'll not need to do even two-dimensional vector combinations.

Our Algebra will be pretty simple with basically one rule: Whatever you do to the left hand side of an equation, you must also do to the right side and visa versa. Words to live by.

Let me make my point by going back to the Gravitation law and asking a simple scientific question of it.

Wait. *Why bother doing this? Use your words!*

Glad you asked. *There's an economy in using equations, but also a hidden power. The form of an equation that describes something that nature does encodes new information that can be discovered by manipulation...information that would not be obvious in an English sentence.*

Here's what I mean. I keep coming back to Newton's Universal Law of Gravitation which I can indeed describe in a paragraph. Here goes:

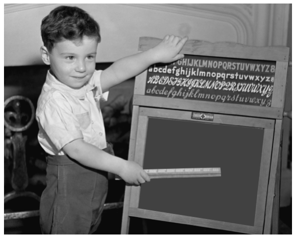
"The force of attraction experienced by two masses on one another is directly proportional to the product of those two masses and inversely proportional to the square of the distances that separate their centers. The constant of proportionality is called the Gravitational Constant which is $6.67408 \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$."

There. A perfectly good representation of Newton's Gravitational Law. Lots of writing, so it's inefficient. If I gave it a nickname, say Newton's Law and then used those two words every time I meant to refer to it, you might have to go back and re-read the paragraph again... and again. But what this doesn't do, besides allow you to quickly move through a gravity-narrative, is help you to find out new things about nature.

I mentioned that it's hard to measure G. Why is that? Does the paragraph enunciation of Newton's Law help you to estimate the ease or difficulty of making that measurement? I don't think so. But if we look at it as a formula, we can interrogate it and answer our question.

$$F = G\frac{mM}{R^2}$$

and then use the rules of algebra to ask about G and see what results. Let's do that:



You Do It 2.1.
/toolkit/SolvingNewton

Suppose we want to measure the Gravitational Constant, G. We expect it to be small...it's in the range of 10^{-11}. We have to use the tools available which include a climate and vibration-free lab area that's about 1 meter long and a dime-store spring scale that's incapable of measuring forces less than 0.1 Newtons. Can we make this measurement using any kind of reasonable masses? Does this experiment make sense?

You needed to literally touch equations and move the pieces around in order to gain insight.

Even if it's a part of the text, you should copy it out while you read. Remember, these parts are marked by

Our appetite for algebraic complexity in QS&BB will be limited. For example, we'll not encounter formulas that are much more complicated than these:

$$y = a \times x = ax \quad \text{solve for } x \text{ to get} \quad x = y/a$$
$$y = x + z \quad \text{solve for } x \text{ to get} \quad x = y - z$$
$$y = a \times x + b = ax + b \quad \text{solve for } x \text{ to get} \quad x = \frac{y-b}{a}$$
$$y = \sqrt{a+x} \quad \text{solve for } x \text{ to get} \quad x = y^2 - a$$

You can do this, right? That's about all that you'll need to remember of algebra. Just remember the rule. Then...it's merely a game—a puzzle to solve.

There's an important reason I have chosen to include some mathematics in QS&BB: I'd hate for you to miss...dare I say...a spooky feature of the universe. It behaves as if mathematics is an essential part of how it works.[16]

We'll take it slow with the math, but even a little will add a lot to your understanding. So let's spend the rest of this chapter reminding yourself of things that you would have learned in high school.

2.5.2 The Powers That Be

Once in a while, we'll need to multiply or divide terms that have exponents. There are simple rules for this, but let's figure them out by hand...so to speak. The first thing to remember about exponents is that in a term like x^n, a positive integer n tells you how many times you must multiply x by itself. So:

$$x^1 = x.$$

[16] There has been this eyes-open discussion in physics for a century now. Is mathematics invented or is it discovered? The former would suggest that it's in some sense, man-made. The latter would suggest that it's a deeply embedded feature of nature... to be found out. In 1960 the famous mathematical physicist Eugene Wigner wrote a paper that's still read today called the *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*. Ask Mr Google about it. Almost 30,000 hits, almost all of them "reprints."

Here, there's just one x, so: $x^1 = x$.
The second thing to remember is that $x^0 = 1$. There aren't any $x's$ in the product and so all that could be there is 1. Armed with that, let's kick it up a notch.

Suppose I have

$$x \times x$$

You'd be pretty comfortable calling that "x-squared"[17] and from the above, the number of $x's$ there are in that product is two. So

$$x \times x = x^2.$$

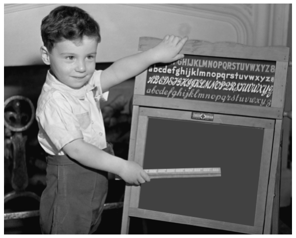
If I add another product, then I'd have $x \times x \times x = x^3$. Get it? Notice that what we've also got in this equation is:

$$x \times x \times x = x^2 \times x^1 = x^3$$

and we've just developed our first rule on combining exponents:

$$x^n \times x^m = x^{n+m}.$$

Now you try it.



What is $x^2 x^1 x^4$?

You Do It 2.2. /toolkit/Exponents

One more time, but different. Another rule:

$$x^{-n} = \frac{1}{x^n}.$$

If the same rule for adding exponents works—and it does—then we can multiply factors with powers by keeping track of the positive and negative signs of the exponents.

[17] From the discussion of Descartes, you can see why the word "squared" is used since this is a legacy of the early linking of algebra with geometry. Ditto for "cubed."

So here's an easy one, first by multiplying everything out:

$$\frac{x \times x \times x}{x \times x} = x$$

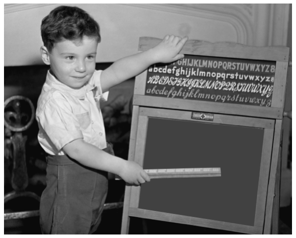
and now by using the powers and the rule:

$$\frac{x \times x \times x}{x \times x} = \frac{x^3}{x^2} = x^3 \times x^{-2} = x^{3-2} = x.$$

One more thing. The powers don't have to be integers.

Perhaps you'll remember that square roots can be written:

$$\begin{aligned}
\sqrt{x} &= x^{0.5} = x^{1/2} \\
\text{so:} & \\
\sqrt{9} &= 3 \quad = 9^{0.5} \\
\text{or:} & \\
\sqrt{\frac{1}{9}} &= \left(\frac{1}{9}\right)^{0.5} = \frac{1}{\sqrt{9}} = \frac{1}{9^{0.5}} = 9^{-0.5} \\
&= \frac{1}{3}
\end{aligned}$$



You Do It 2.3.
/toolkit/ExponentsAgain

What is $x^{-2}x^{1}x^{4}$?

That's it. Now we have everything we need to turn numbers into sizes of...stuff.

2.5.3 Units Conversions

Numbers are just numbers without some label that tells you what they refer to. Now not all numbers have to refer to something, pure number is a respectable object of mathematical research—prime numbers for example have been a topic of research for centuries. Irrational numbers–those that can't be expressed as a ratio of whole numbers, like π, –are likewise objects with no necessary relationship to..."stuff" in our world.

We're concerned with numbers that measure a parameter or count physical things and they come with some reference ("foot") unit that is a customary way to compare one thing with another.[18] Of course not everyone agrees on the units that should be used. Wait. There's *the world*, that agrees on one set and then there's the United States that marches to its own set of units. Thinking of you, feet.

I'll not use Imperial units (feet, inches, pounds, etc.) very much, except to give you a feeling for something that you've got an instinct for. . . like the average height of a person. We'll use the metric system, in particular the MKS units[19] in which the fundamental length unit is the meter (about a yard).

Just like an exchange rate in currency, so many euros per dollar, we'll need to be able to convert, among many different units. All the time.

[18] "Apples and Oranges" is a phrase that refers to units...you need to keep your fruit straight.

[19] This stands for meter-kilogram-second, as the basic units of length, mass, and time. It's a dated designation as the real internationally regulated system is now the International System of Units (SI) which stands for *Le Système International d'Unités*. The French have always been at the forefront of this.

Understand conversions! Conversions are a part of life! At least in QS&BB.

Let's get our bearings. What's a common sort of size in life? How about the height of an average male. Mr Google tells me that's about 5'10". How many inches tall is our average male? Here's the thought-process you'd use to calculate this.

Pencil 2.1.

Three steps:

1. A single foot is 12 inches.
2. So, 5 feet is $5 \times 12 = 60$ inches
3. and the combination is $60 + 10 = 70$ inches.

...which you could do in your head I'll bet. But this simple, almost intuitive calculation uses a more general conversion from one unit to another through the use of a conversion factor. All unit manipulations use a conversion factor, which is just a number,[20] which will be expressed as a ratio or fraction, of the conversion of one set of units ("from") to the new set ("to"). It will appear like this:

[20] . . . a number that's actually like a fancy way to write "1" since it's really relating one thing in a set of units to the same thing in a different set of units.

$$\text{where you're going to} = \left(\frac{\text{to}}{\text{from}}\right) \times \text{where you're coming from}$$

The action is in that bracketed term. It's arranged so that the "from" in the denominator cancels the *units* of the right hand "coming from" term. What's left in the numerator you intentionally set up to have the units of what you are going to, here in step 2 above...we're going from feet to inches. In this case, step 1 defines the bracket and step 2 uses it and in symbols, step 1 says:

$$\left(\frac{\text{to}}{\text{from}}\right) = \frac{\text{number of inches in a foot}}{\text{a foot}} = \frac{12}{1}$$

So armed with this, we can do the conversion of feet to inches.

$$\begin{aligned}\text{five feet in inches} &= \frac{\text{number of inches in a foot}}{\text{a foot}} \times 5 \text{ ft} = \frac{12}{1} \times 5 \text{ ft} \\ &= \frac{12 \text{ inches}}{1 \text{ ft}} \times 5 \text{ ft} = \frac{60}{1} \text{inches} \\ &= 60 \text{ inches.}\end{aligned}$$

There's another way to think about this (which is identical, but just spun differently) which might be useful. You know that you can always multiple any number times 1 and get that back. So in the inches-feet world, we could write:

$$\begin{aligned}1 \text{ foot} &= 12 \text{ inches} \\ 1 &= \frac{12 \text{ inches}}{1 \text{ foot}}\end{aligned}$$

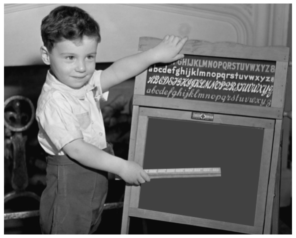
It's that "1" that I want to use to convert 5 feet to inches. We'd do that by writing:

$$5 \text{ foot} = x \text{ inches (looking for } x \text{ here)}$$
$$5 \text{ foot} \times 1 = x \text{ inches (haven't done anything with "}\times 1\text{")}$$
$$5 \text{ foot} \times \frac{12 \text{ inches}}{1 \text{ foot}} = x \text{ inches (used what "1" is here)}$$
$$5\,\cancel{\text{foot}} \times \frac{12 \text{ inches}}{1\,\cancel{\text{foot}}} = 5 \times 12 \text{ inches} = 60 \text{ inches} = x \text{ inches}$$

Notice that we treat units like algebraic terms and can cancel them as if they were symbols or numbers: the "feet" cancel above. That's the neat thing. If you set up the conversion factor right, the units will multiply and divide along with numbers so you can always see that you get what you want. While this is a particularly simple conversion, sometimes we'll need to do some which are either more complicated, or use units that maybe you're not very familiar with. I won't be so pedantic usually, but hopefully you get the point!

Let's do a harder one. *If a furlong is 201.2 meters, how furlongs are there in a mile?* What we know — the "1" as in the above discussion is that 1 furlong = 201.2 m. Then we have to think about it since *miles* is where we start from, not meters. More conversions. How you do this might depend on what you remember. For me[21] what is stuck in my head is that a mile is 5,280 feet and that a foot is 12 inches and that an inch is 2.54 centimeters and that a meter is 100 cm. So I always start there. You might do it differently. So for me, that's 4 conversions, or four brackets along with my fancy "1" that I would use to do this conversion. It's kind of fun. Really.

[21] ...for some reason



You Do It 2.4. /toolkit/FurlongMi

How many furlongs in a mile if there are 201.2 meters in a single furlong?

Did you get that there are 8 furlongs in a mile? If not, click on the little guy and watch me do it. I've collected a number of the useful conversions into graphs which you can use later.

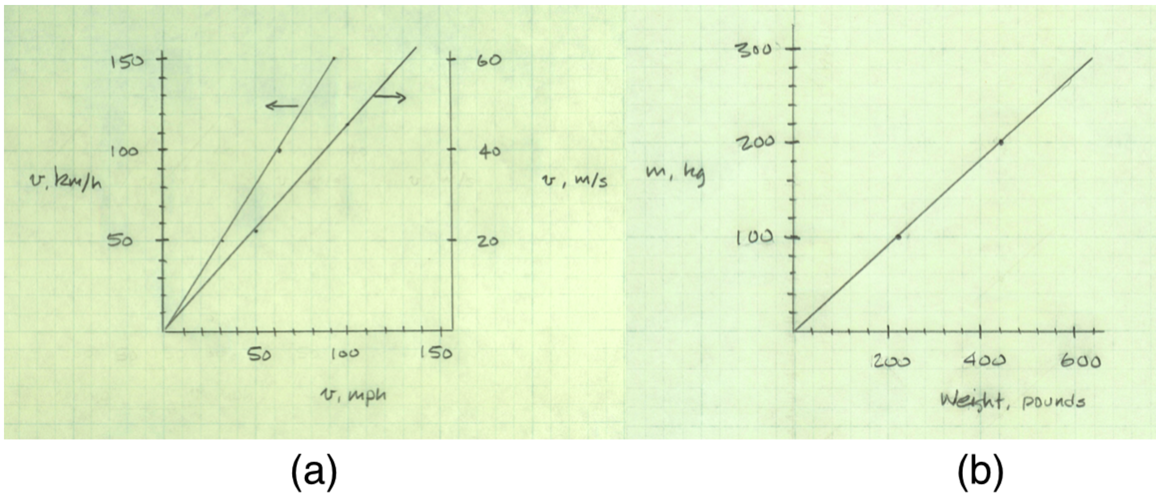


Figure 2.4: The right hand curve shows a constant speed of 4 m/s, holding steady for 10 s. The left hand curve shows the distance that an object will travel at that constant speed as a function of time.

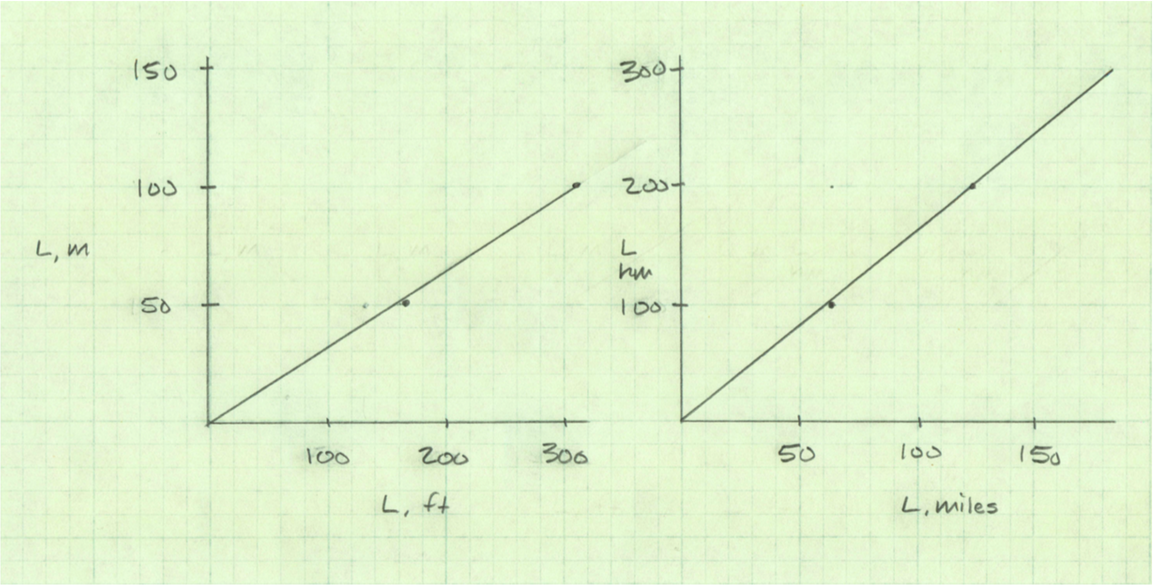


Figure 2.5: The right hand curve shows a constant speed of 4 m/s, holding steady for 10 s. The left hand curve shows the distance that an object will travel at that constant speed as a function of time.

2.5.4 The Big 10: "Powers Of," That Is

One of the more difficult things for us to get our heads around will be the sizes of things, the speeds of things, and the masses of things that fill the pages of QS&BB. Lots of zeros means lots of mistakes, but it also means a complete loss of perspective on relative magnitudes. Big and small numbers are really difficult to process for all of us.

As we think of things that are bigger and bigger and things that are smaller and smaller, where do you start to loose track and one is the same as another? Keep in mind our average-guy height of about a meter and half–for this purpose, thing... "about a couple of meters"–and here is a ranked list of big and small things with approximate sizes:

1. African elephant, 4 m
2. Height of a six story hotel, 30 m
3. Statue of Liberty, 90 m
4. Height of Great Pyramid of Giza, 140 m
5. Eiffel Tower, 300 m
6. Mount Rushmore 1700 m

7. District of Columbia, 16,000 m square
8. Texas, East to West, 1,244,000 m
9. Pluto, 2,300,000 m diameter
10. Moon, 3,500,000 m diameter
11. Earth, 12,800,000 m diameter
12. Jupiter, 143,000,000 m diameter
13. Distance Earth to Moon, 384,000,000 m
14. Sun, 1,390,000,000 m diameter
15. distance, Sun to Pluto, 5,900,000,000 m
16. Distance to nearest star (Alpha Centuri), 41,300,000,000,000,000,000 m
17. diameter of the Milky Way Galaxy, 950,000,000,000,000,000,000 m
18. Distance to the Andromeda Galaxy, 24,000,000,000,000,000,000,000 m
19. Size of the Pisces–Cetus Supercluster Complex, our supercluster, 9,000,000,000,000,000,000,000,000 m
20. Distance to UDFj-39546284, the furthest object observed, 120,000,000,000,000,000,000,000,000 m

Do I need to go any further? Given what I know from my life, I have a pretty good idea of how big #1-8 are. Beyond that, I have no idea how much bigger the Milky Way Galaxy is than the size of Jupiter. It all blends together.

But there's a way: exponential notation... using our power rules and the number 10. It's easy.

A number expressed in exponential notation as:

$$\text{a number} \times 10^{\text{power}}$$

Let's think about this in two parts. First, the 10-power part.

The rules above work for 10 just like any number, so 10^n is shorthand for the number that you get when you multiply 10 by itself n times. This has benefits because of the features of 10-multiples, that we count in base-10, and how you can just count zeros. So for example:

$$10^3 = 10 \times 10 \times 10 = 1,000.$$

The power counts the zeros, or more specifically, the position to the right of the decimal point from 1. So if you have any number, you can multiply it by the 10-power part and have a compact way of representing big and small numbers. So, following through:

$$3 \times 10^3 = 3 \times 10 \times 10 \times 10 = 3 \times 1000 = 3000.$$

We can do the same thing with numbers less than 1, by using negative exponents for the 10-power part.

$$0.03 = \frac{3}{100} = \frac{3}{10^2} = 3 \times 10^{-2}.$$

So you just move the decimal place the power-number to the right to go from 3×10^{-2} to 0.03.

The second thing is the number in front that multiplies the power of 10. It's called the "mantissa" and that's all it is... a number.

Now that confusing list above can be written in a way that's more likely to allow your brain to compare one with the other, since now you'll immediately see that one thing is 10 or 1000 or so-on times another.

1. African elephant, 4 m
2. Height of a six story hotel, 30 m, 3.0×10^2 m
3. Statue of Liberty, 90 m, 9.0×10^2 m
4. Height of Great Pyramid of Giza, 140 m, 1.4×10^2 m
5. Eiffel Tower, 300 m, 3.0×10^2 m
6. Mount Rushmore 1700 m, 1.7×10^3 m
7. District of Columbia, 16,000 m square, 16.0×10^3 m, or 1.6×10^4 m
8. Texas, East to West, 1,244,000 m, 1.244×10^6 m
9. Pluto, 2,300,000 m diameter, 2.3×10^6 m
10. Moon, 3,500,000 m diameter, 3.5×10^6 m
11. Earth, 12,800,000 m diameter, 12.8×10^6 m, or 1.28×10^7 m
12. Jupiter, 143,000,000 m diameter, 143.0×10^6 m, or 1.43×10^8 m
13. Distance Earth to Moon, 384,000,000 m, 384.0×10^6 m, or 3.84×10^8 m
14. Sun, 1,390,000,000 m diameter, 1.39×10^9 m
15. Distance, Sun to Pluto, 5,900,000,000 m, 5.9×10^9 m
16. Distance to nearest star (Alpha Centuri), 41,300,000,000,000,000,000 m, 41.3×10^{18} m, or 4.13×10^{19} m
17. diameter of the Milky Way Galaxy, 950,000,000,000,000,000,000 m, 950×10^{18} m, or 9.5×10^{19} m
18. Distance to the Andromeda Galaxy, 24,000,000,000,000,000,000,000 m, 24.0×10^{21} m, or 2.4×10^{22} m
19. Size of the Pisces–Cetus Supercluster Complex, our supercluster, 9,000,000,000,000,000,000,000,000 m, 9.0×10^{24} m
20. Distance to UDFj-39546284, the furthest object observed, 120,000,000,000,000,000,000,000,000 m, 120×10^{24} m or 1.2×10^{26} m

So now you can compare and see that the distance from the Earth to the Moon is only a little more than three times the diameter of Jupiter. Now your "mind's eye" springs into action since you can sort of imagine three Jupiters between us and the Moon. With all of those zeros, I couldn't do that!

Powers of 10 have nicknames...Is "a google" really a power of ten?[22] Here's an official table of the names, size, and abbreviation for most of them:

Let's work out an example. Something you can use at a party. I first worked this out for a class when I was in Geneva, Switzerland working at CERN. It was July 4, 2010, which was just another Sunday over there. The United States came into existence on July 4, 1776[23] which was 2010 – 1776 = 234 years ago.

So how many seconds had the United States been around if we start from midnight on July 4, 1776?

Pencil 2.2.

[22] No. The word is Googol and it's $10^{1}00$. The rumor is that the Google founders misspelled it when they incorporated.

[23] Actually, the Declaration of Independence wasn't fully signed until August 2, 1776—my birthday! The day, not the year.

$$\begin{aligned}
\text{234 year per U.S.} &= 2.34 \times 10^2 \ \frac{\text{years}}{\text{U.S.}} \\
\text{86,400 seconds per year} &= 8.64 \times 10^4 \ \frac{\text{seconds}}{\text{year}} \\
\text{So:} & \\
\text{seconds per U.S.} &= 2.34 \times 10^2 \ \frac{\cancel{\text{year}}}{\text{U.S.}} * 8.64 \times 10^4 \ \frac{\text{seconds}}{\cancel{\text{year}}} \\
&= (2.34) * (8.64) \times 102 * 10^4 = (2.34) * (8.64) \times 10^6 \\
\text{seconds per U.S.} &= 20.218 \times 10^6 \\
\text{seconds per U.S.} &= 2.0218 \times 10^7
\end{aligned}$$

Wait. *You mean I treat the words of units as if they were algebraic variables?*

Glad you asked. *Yes. You can do that and even catch mistakes when the products and cancellations don't lead to what you expect. Had I gotten miles times hours, I'd know my actual formula was wrong even before doing it. No charge for this hint. Use it wisely.*

There are a few of things to notice here. First, that's a lot of seconds! Second (get it?), to multiply two numbers together, you separate the mantissas, and multiply them, and the exponents, and add them...separately.[24] Please understand these operations by doing them over by hand. The obvious thing happens when there are negative exponents involved. For example, convince yourself that 15% of the lifetime of the U.S. is 3,032,700 seconds, and do it by treating 15% as

$$15\% = 0.15 = 1.5 \times 10^{-1}.$$

Finally, notice that I canceled the units of "year." You can always do that with units—set them up right, keep them in your equations, and you can quickly find mistakes. Here, the units on the right have to give you the units on the left, which we wanted: "seconds/U.S."

2.5.5 Graphs and Geometry

One of the amazing mathematical discoveries of the 17th century was that geometry could be tied to algebra through the use of the growing notion of a function. This is almost entirely due to Rene Descartes and Leonhard Euler (1707-1783)[25]

[24] Remember? The "mantissa" in $X \times 10^y$ is X and the exponent is the y.

[25] Euler was one of the most amazing mathematicians in history. He did so much that his work is still being analyzed and cataloged today. To him we owe the notion of a function. But he also worked in physical problems like hydrodynamics, optics, astronomy, and even musical theory. While Swiss, Euler lived and worked most of his life in St. Petersburg, Russia.

We will deal with some functions that would be very hard to evaluate on your calculator. But Descartes' gift is that I can show you the graph and evaluation can be done by eye, which is in effect solving the equation. We'll use some simple geometrical relations which I'll summarize here.

Table 2.1: More powers of ten than you ever wanted to know. Except that many of them we need to know.

septillionth	yocto-	y	0.000000000000000000000001	10^{-24}
sextillionth	zepto-	z	0.000000000000000000001	10^{-21}
quintillionth	atto-	a	0.000000000000000001	10^{-18}
quadrillionth	femto-	f	0.000000000000001	10^{-15}
trillionth	pico-	p	0.000000000001	10^{-12}
billionth	nano-	n	0.000000001	10^{-9}
millionth	micro-	μ	0.000001	10^{-6}
thousandth	milli-	m	0.001	10^{-3}
hundredth	centi-	c	0.01	10^{-2}
tenth	deci-	d	0.1	10^{-1}
one			1	10^{0}
ten	deca-	da	10	10^{1}
hundred	hecto-	h	100	10^{2}
thousand	kilo-	k	1,000	10^{3}
million	mega-	M	1,000,000	10^{6}
billion	giga-	G	1,000,000,000	10^{9}
trillion	tera-	T	1,000,000,000,000	10^{12}
quadrillion	peta-	P	1,000,000,000,000,000	10^{15}
quintillion	exa-	E	1,000,000,000,000,000,000	10^{18}
sextillion	zetta-	Z	1,000,000,000,000,000,000,000	10^{21}
septillion	yotta-	Y	1,000,000,000,000,000,000,000,000	10^{24}

Formulas From Your Past

I know that you've seen most of this somewhere in your past! So return with us now to those thrilling days of yesteryear.[26]

[26] Google it!

Equation of a Straight Line

A straight line with a slope of m and a y intercept of b is described by the equation:

$$y = mx + b. \tag{2.1}$$

Figure 2.6 shows such a straight line.

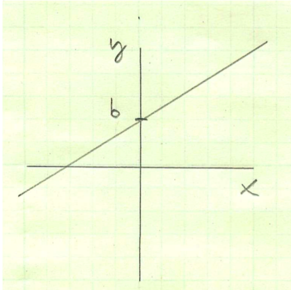


Figure 2.6: straight

Equation of a Circle

A circle of radius R in the $x - y$ plane centered at a (a, b) is described by the equation:

$$R^2 = (x - a)^2 + (y - b)^2. \tag{2.2}$$

Of course if the circle is centered at the origin, then it looks more familiar as

$$R^2 = x^2 + y^2. \tag{2.3}$$

is described by the formula Figure 2.7 shows such a circle.

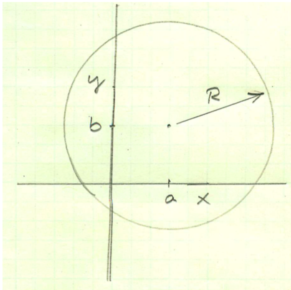


Figure 2.7: circle

Equation of a Parabola

A parabola in the $x - y$ plane with vertex at (a, b)

$$y = C(x - a)^2 + b \tag{2.4}$$

where C is a constant. Figure 2.8 shows a parabola.

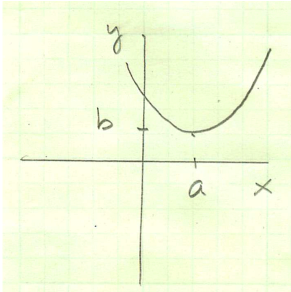


Figure 2.8: parabola

Area of a Rectangle

A rectangle with sides a and b has an area, A of

$$A = ab \tag{2.5}$$

Area of a Right Triangle

A right triangle (which means that one of the angles is 90 degrees) with base of a and height of b has an area, A of

$$A = 1/2ab. \tag{2.6}$$

For a right triangle, the base and height are equal to the two legs. But the formula works for any triangle. Figure 2.9 shows how that works.

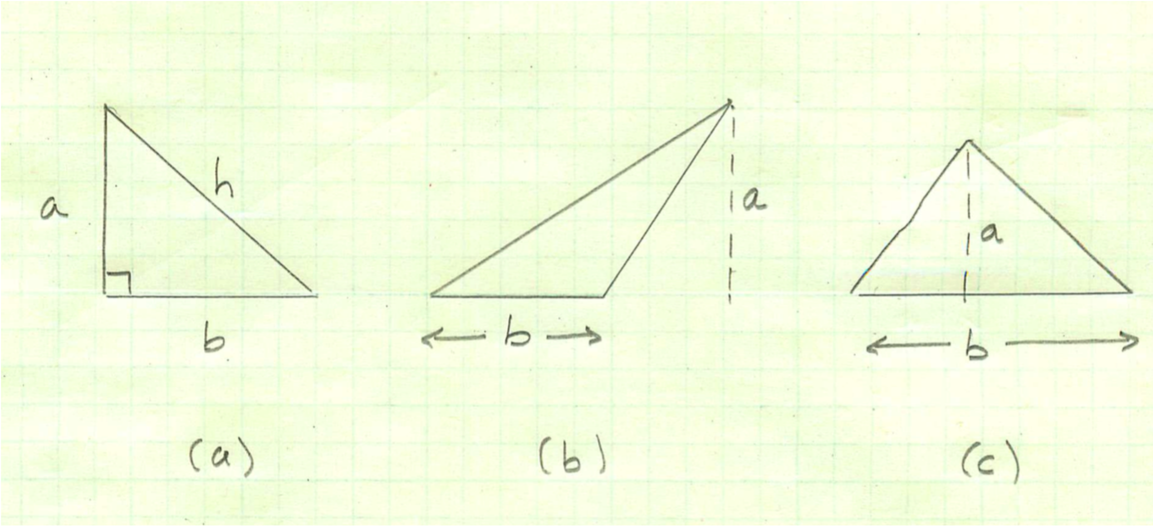


Figure 2.9: triangles

Figure 2.10: You realize that two pizzas is a "circumference"? Because...wait for it...it's "2 pie are." You're welcome. (papajohns)

Area and Circumference of a Circle

For a circle of radius R, the area, A is

$$A = \pi R^2 \tag{2.7}$$

and the circumference, C is

$$C = 2\pi R. \tag{2.8}$$

Pythagoras' Theorem

For a right triangle, the hypotenuse, h is related to the lengths of the two sides a and b by the Theorem of Pythagoras:

$$h^2 = a^2 + b^2. \tag{2.9}$$

2.6 Shapes of the Universe

One of the remarkable consequences of the mathematization of physics that began with Descartes is that we've come to expect that our descriptions of the universe will be in the language of mathematical *func-*

tions. Do you remember what a function is? The fancy definition of a function can be pretty involved, but you do know about function machines and I'll remind you how.

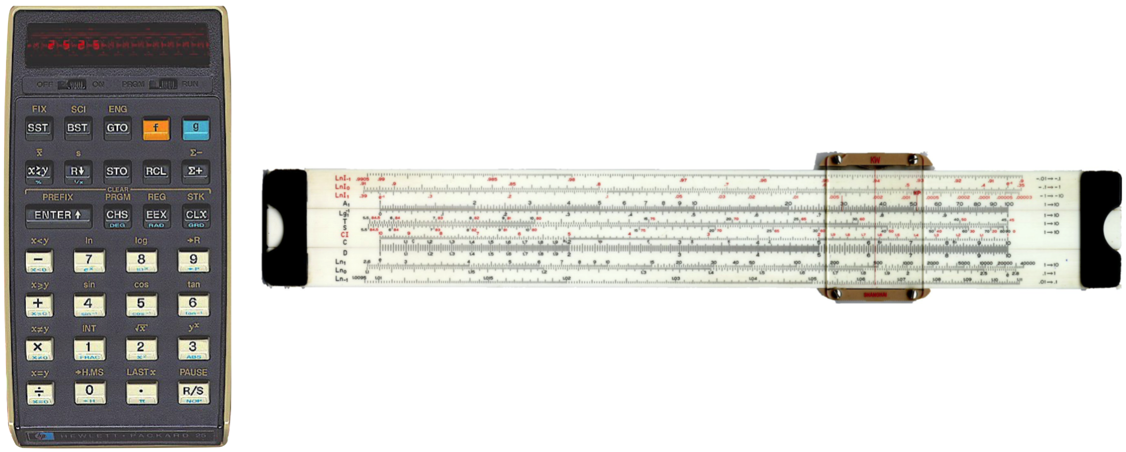


Figure 2.11: Left: the venerable HP-25 programable (!) scientific calculator. Right: a slide rule used for all calculations until the early 1970's. It was not programmable (although it was wireless).

When I was a senior in college, finishing my electrical engineering degree, our department had a visitor from the Hewlett Packard Company. It was either Bill Hewlett or Dave Packard, I can't remember which. But they promised to do away with the slide rule that we all carried around with us everywhere and showed us a brand new product: a portable scientific calculator, that they called the electronic slide rule. This was 1972 and he showed us the first HP calculator, the HP-35. Needless to say, I couldn't afford it—it cost $400— but later in graduate school I bought my first scientific calculator, the HP-25, pictured in Fig. 2.11 along with the slide rule that I carried for four years. Today I've got more processing power in my watch then I had in that calculator. But I'll bet you've got something like it...calculators are nothing but electronic function machines. So in the spirit of Fig. 3, Fig. 2.12 shows the circuit board from the inside of the HP-25 with it's simple processor at the bottom.

%

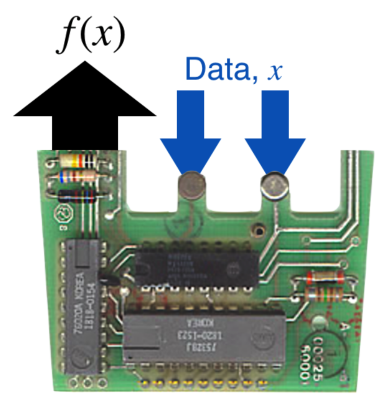


Figure 2.12: The AMI 1820-1523 Arithmetic, Control Timing processor: the heart of a function machine. Adapted for my silly purposes, but I'll bet you won't forget it! The tabs at the blue arrows are actually connected the processor to the keyboard. That's how data get in.

2.6.1 Functions: Mathematical Machines

Figure 2.12 shows what a function does: if you enter data through the keypad—a value of x—and hit the appropriate button, the display shows the value of the function. So if the function was the formula $f(x) = x^2$ and if I keyed in "4" and pushed the x^2 button, the display would read "16," the value of $f(2)$ for that particular function. Notice that it doesn't give you more than one result, and that's a requirement of a function: one result.

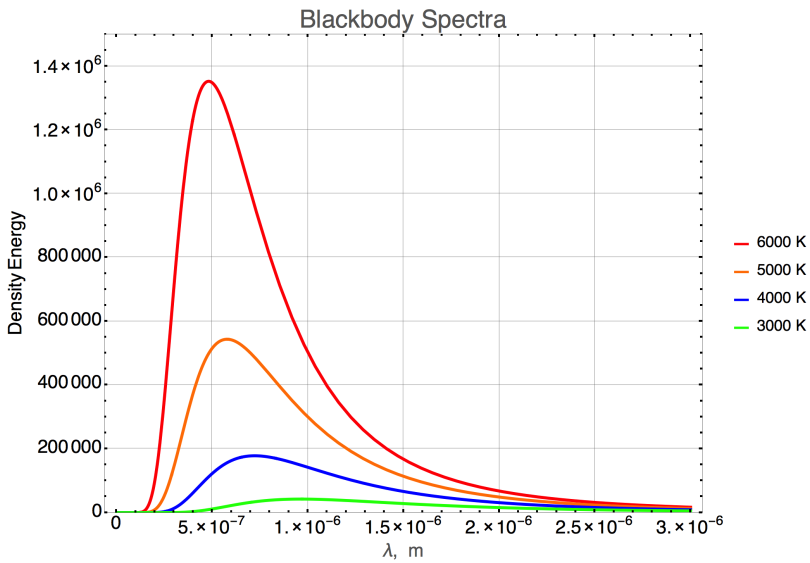


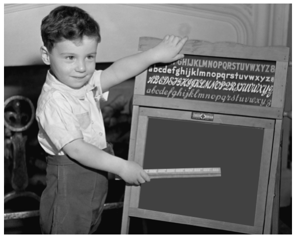
Figure 2.13: blackbodyvarious

So that's all a function is: a little mathematical machine that reports a single result for one or more inputs according to a rule. For us, functions can be represented by a formula, an algorithm, a table, or a graph. In all cases, it's one or more variables x or x & y... or x &y &z... *in*, a rule about what happens to them, and one numerical result *out*.

Your algebra teacher would have called the inputs (e.g., $x, y, ...$) the independent variables, which would have been members of the function's "Domain," and the output (e.g., $f(x, y, ...)$ or often y) the dependent variable, which would have been inside the "Range."

Nature seems to live by functions[27] and since in QS&BB we're all about Nature, we'll need to use functions. We'll solve actual formulas when they're simple functions and analyze plots of functions when they're complicated. For example, Fig. 2.13 is a function of two variables, a wavelength, λ and temperature (the units don't matter here). It's a messy formula which we'll admire, but not derive in Chapter **??**. But boy is it an important function. Here the little function machine calculates the value of the energy density of the radiation emitted by an object heated to a particular temperature. If you provide a wavelength and a temperature (in the figure, 3,000, 4,000, 5,000, or 6,000 degrees) to the function, then it reports back to you the value of the energy density that the body radiates. You can evaluate that function:

[27] Why? We don't know.



What is the ratio of the value of the energy densities for one object at 4,000 degrees and another at 5,000 degrees at a wavelength of 1×10^{-6} meters?

You Do It 2.5. /toolkit/GraphRead

There. You just evaluated a complicated function...twice.

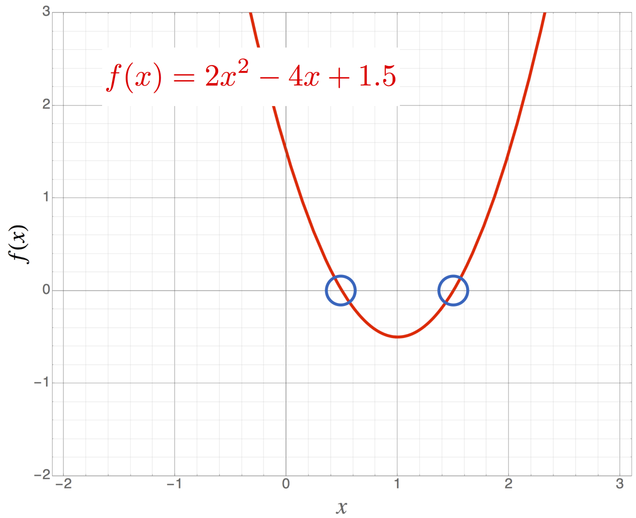


Figure 2.14: The quadratic function $f(x) = 2x^2 - 4x + 1.5$. plotted with blue circles at the points where $f(x) = 0$, the roots.

2.6.2 Polynomials

Many of Nature's functions are in the form of polynomial equations, which are reminiscent of the quadratic equation:

$$f(x) = ax^2 + bx + c. \tag{2.10}$$

You may have "solved" this equation in a number of ways in your algebra classes. What solving means is finding the x's for which the value of the function is zero. There's also a geometrical interpretation of "solving" a polynomial and an algebraic rule for doing it. Notice that the quadratic has the form of the equation of a parabola, so let's look at an example:

$$f(x) = 2x^2 - 4x + 1.5. \tag{2.11}$$

Remember that we can plot functions and Fig. 2.14 is a graphical representation of this function. When you solved a quadratic, you actually found the values of x for which the value of the function value—these are the "roots" of the function—of which there are two which I've called x_1 and x_2. So if we plug either into Eq. 2.10, then we will get $f = 0$..[28]

For quadratic equations, there is also a single formula to calculate the roots directly.[29] If we take Eq. 2.10 as the general form, then the "quadratic formula" you might remember from a former mathematics life is

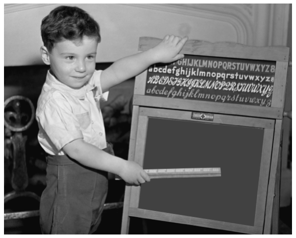
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.12}$$

Of these two solutions: x_1 is for the + sign and x_2 is for the − sign.[30] So for our example in Eq. 2.11, $a = 2$, $b = -4$, and $c = 1.5$.

[28] Remember that the degree of a polynomial corresponds to the number of roots. For a quadratic, the degree is 2. For a cubic, it's 3 and so on.

[29] For cubics, there is a procedure. For polynomials of higher degree, it's complicated!

[30] Or the other way around—your choice.



For the example quadratic, use the quadratic formula, Eq. 2.12 to find the two roots of the function, Eq. 2.11. Do they match the "solution" you would get by looking at Fig. 2.14?

You Do It 2.6. toolkit/Quad

A polynomial can be of any "degree," which is the highest power of x. Since the middle of the 16th century (Copernicus' time) mathematicians had figured out how to expand any such function for an arbitrary degree, like $(a+x)^n$, where n is a positive integer. This formula would save work since expanding $(a+x)^n$ if n was anything bigger than about 3 is a lot of calculating. Let's expand a quadratic polynomial, that is for $n=2$:

$$(a+x)^2 = (a+x)(a+x) = a^2 + ax + xa + x^2 = x^2 + 2ax + a^2 \tag{2.13}$$

This old magic expansion formula is called the Binomial Expansion for polynomial of degree n—it has $n+1$ terms:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 \ldots + x^n \tag{2.14}$$

Until our hero, Isaac Newton came along, n was always a positive integer in this context.[31]

[31] Remember that the $n!$ notation stands for "n factorial." Which is $n! = n(n-1)(n-2)(n-3)\ldots 1$

Approximating Functions

Newton began inventing mathematics in the 17th Century and found a way to expand a formula for cases in which n could be anything: a positive integer, a negative integer, or even a fraction.[32] The result was an expansion that has an infinite number of terms! In contrast to how that sounds, it's actually very useful for many physics applications as we'll see.

[32] This was an essential step in the invention of the calculus. . . and the thing that Leibniz learned from Newton and used himself to invent a competing version of calculus. We'll touch on this in Chapter **??**.

Let's take a particular case in which $a=1$ and write it out Newton's idea in the same spirit as Eq. 2.14.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots \tag{2.15}$$

Here's where it will be interesting for physics. Look carefully at Eq. 2.15: each term is proportional to an increasing power of x, x^2, x^3, x^4 and so on. In physics, we can use this to make accurate approximations.[33] Suppose that $x < 1$. Then each term gets smaller and smaller since $x^3 < x^2$ and so on if $x < 1$...so each

[33] While this sounds like just a work-saver, we'll see that it actually allows us to sometimes gain insight of some tricky physics. Be patient.

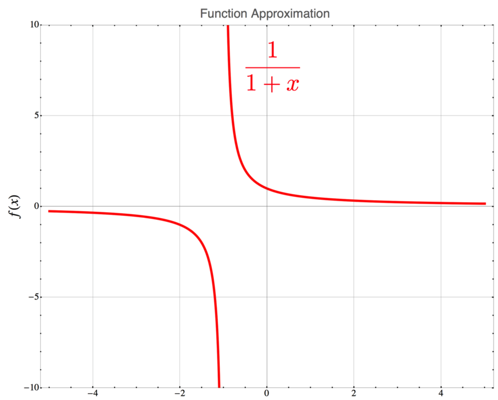
additional term *adds less and less* to the sum before it. Now we've got a little approximation-tool because many formulas that matter in physics look like

$$\frac{\text{something}}{(1+\text{ something tiny})^{\text{ some power}}}$$

or can be rearranged to look like that.

Here's one that we'll use. Let's imagine the function

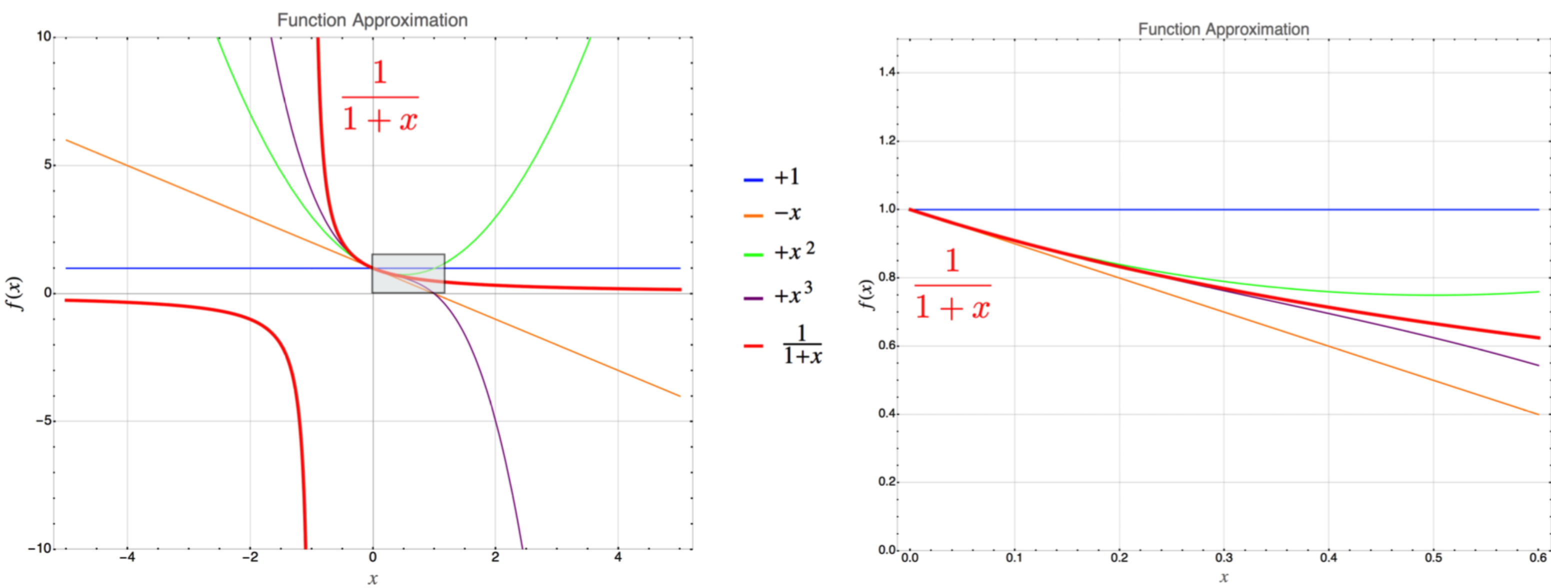
$$f(x) = (1+x)^{-1} = \frac{1}{1+x}.$$



Let's even plot it, which I've done in Fig. 2.15. Notice that this function becomes infinite when $x = -1$ and that it quickly falls until $x = 0$ and then slowly heads off towards zero as x becomes very large. That makes sense, right?

Now lets expand that function according to the approximation in Eq. 2.15. For this particular function, $n = -1$ and we will keep just the first four terms of the otherwise infinite number of terms:

Figure 2.16: See the text for an explanation. The right plot is a blow-up of the left around the gray box.



$$f(x) = \frac{1}{1+x} \approx 1 - x + x^2 - x^3 \tag{2.16}$$

(By the way, the ≈ symbol in Eq. 2.16 stands for "almost equal to.") The right hand side of this equation is really the sum of four different, simple functions. When added together, we'll see that they get closer and closer to the original, depending on how many terms are included. Look at Fig. 2.16. The red curve in the left and right plots is our original function and the colored curves are each getting closer and closer to it.

The blue "curve" is the trivial function that's the first term in Eq. 2.16: $f = 1$. The orange curve takes the second term in Eq. 2.16 and adds it to the first, so it's $f(x) = 1 - x$. The green curve adds the third term, x^2 to the orange curve and so on. The right plot is a blowup of the region in the gray box on the left. Notice that in the region of x which is very small, the few functions are a pretty good approximation to the red. The more terms we might add the further out in x that agreement would continue.

Remember this! It will become important later when we'll encounter functions and approximate them with a few terms of the expansion from Eq. 2.15. Here are the functions that we'll see in the pages ahead:

$$\sqrt{1+x} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \ldots \tag{2.17}$$

$$\frac{1}{\sqrt{1-x}} = 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^3 + \ldots \tag{2.18}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \tag{2.19}$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \tag{2.20}$$

2.7 Euler's Number

You all know that π is an unusual number. It's simply the ratio of the circumference of a circle to its diameter (see Eq. 2.7) and, the Indiana Legislature[34] not withstanding, it's a number that has a decimal representation that never ends. It's "irrational" and has the (approximate!) value:

$$\pi = 3.1415926536\ldots \text{ forever!} \tag{2.21}$$

There is another irrational number that plays a big role in mathematics, but also in many other areas of "regular" life. It's called "Euler's Constant" although the prolific mathematician Euler didn't first discover it, he discovered many of its unique features and so his name is associated with it. We physicists tend to just call it "e" since that's the symbol that is used to represent it. It has the value:

$$e = 2.71828182845904523536\ldots \text{ forever!} \tag{2.22}$$

[34] Yes, that story is true. In 1897 state legislature representative, Dr. Edward J. Goodwin, a physician who dabbled in mathematics, proposed changing the value of π to 3.2. The bill sailed through the House but was postponed indefinitely in the Senate. It seems that Professor C.A. Waldo at Purdue was horrified enough that he intervened and the bill died.

Euler first used e to understand compound interest. If you invest \$1 at a compounded interest of 100% per year, then at the end of the year your wealth would have been increased by a factor of e. While not many savings plans grant 100% interest, you get the point. It figures into the calculation of any interest rate. I'm going to try to convince you that it appears in many guises.

The importance of e in science comes from the fact that the rate at which e increases or decreases is proportional to *itself*. So if something increases by e^{ax} then the rate at which it increases is ae^{ax}. This leads directly (with some calculus) to the rule for how radioactive nuclei, atomic systems, or elementary particles decay. Suppose we start out with N_0 radioactive nuclei with a "lifetime" called τ at a time $t = 0$, then the number of left after a time t is equal to

$$N = N_0 e^{-t/\tau}. \tag{2.23}$$

So the fraction left is $\frac{N}{N_0} = e^{-t/\tau}$. Figure 2.17 shows two curves for both the exponential decay and

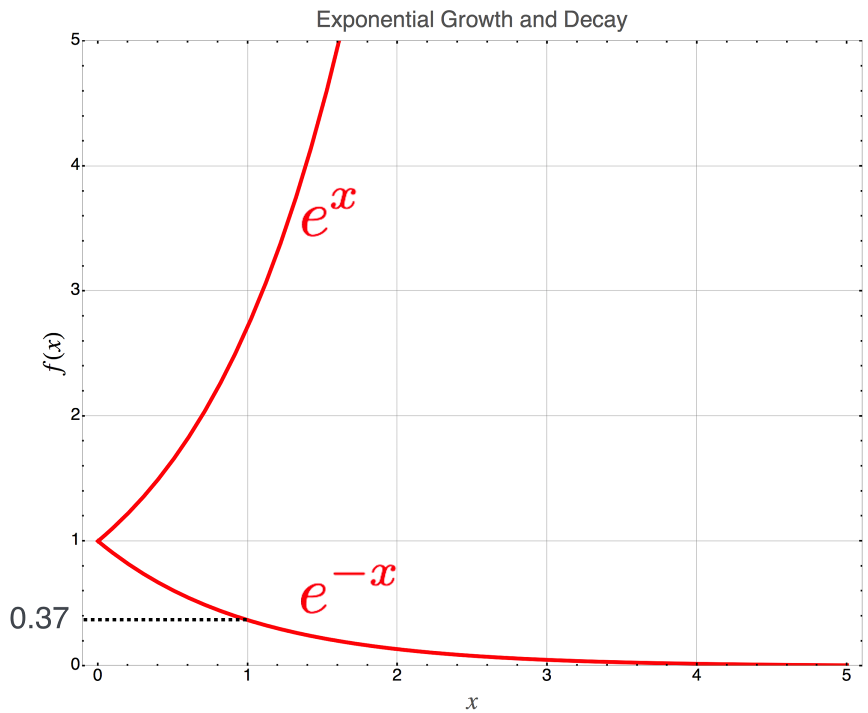


Figure 2.17: exponentials

exponential growth formulas.

But it's not only some sort of modern physics thing. Atmospheric pressure decreases the higher up you go...this is because there's less air above you. So home runs in Denver's Coors Field go further than in Chicago's Wrigley field since Denver is about a mile higher than Chicago. We could pretty closely calculate the density at any altitude using this same formula, but modified for the physical situation. Let's call the density of air at any height above sea-level (y) to be $\rho(y)$. Then if we let $\rho(0) \equiv \rho_0$ then the function that describes the density at any height turns out to be

$$\rho(y) = \rho_0 e^{-y/8000}. \tag{2.24}$$

where the distance above sea level, y is measured in meters. Let's do one more thing and then we can use our curves, even though the axes are just relative numbers. So we could directly ask the fractional change in density:

$$\frac{\rho(y)}{\rho_0} = e^{-y/8000} \tag{2.25}$$

Relative to sea level, then a mile high (1,609 m) makes the right side $e^{-(1609/8000)} = e^{-0.2}$ so we can use the general graph in Fig. 2.17 since we've determined that $y = 0.2$,[35] At that value, read across, we see that the density is reduced to about 80% of what it would be at $x = 0$. So,

$$\frac{\rho(y)}{\rho_0} = 0.8. \tag{2.26}$$

Not everything in nature decays! Suppose you're a biologist studying bacterial growth. If a particular strain grows continuously at a rate of 5% per day, you could predict the size of the colony after some number of days.[36] The growth in the colony where t is measured in days is given by

$$F(\text{bacteria}) = F_0 e^{Rt} = F_0 e^{0.05t} \tag{2.27}$$

where F(bacteria) is the number of bacteria after a time t and F_0 is the number that you started with. For a different bacterium, R would be a different number (a "rate"). If we waited patiently for about a month, say $t = 30$ days, we'd have

$$F(\text{bacteria in a month}/F_0 = e^{Rt} = e^{(0.05\times 30)} = e^{1.5} \tag{2.28}$$

Back to Fig. 2.17 with $x = 1.5$ the top graph reads about 4.4. So if we started with a population of 100, after 30 days it would have grown to $4.4 \times 100 = 440$.

This is what people mean when they refer to "exponential growth"—a very rapid increase in some phenomenon.

[35] Of course, we're using y in the formula for height, which is often a convention, but it's still playing the role of the x in the general graph.

[36] Or, you could measure the increase and write the function that describes it.

2.8 Vectors

We're about to talk about motion, but let's make an important point here that will be obvious. When you're driving on the highway and your (American) speedometer reads "60 mph," it's telling you the *speed* not your direction. Going 80 mph north is as much over the speed limit as going 80 mph east since speed is all the highway patrol radar cares about. (There isn't one speed limit for easterly travel and another for when the road bends north.)

The cops might not care, but you care a lot whether you're traveling north at 60 mph or east, since in order to get where you're going on schedule–your trip depends not only on how fast you go, but in what direction. The difference between *speed* and *velocity* is critical. Not all quantities are vectors...for example, what's the direction of a temperature? But, velocity, space coordinates, force, momentum, electric and magnetic fields, and many other physical quantities have directions as well as values.

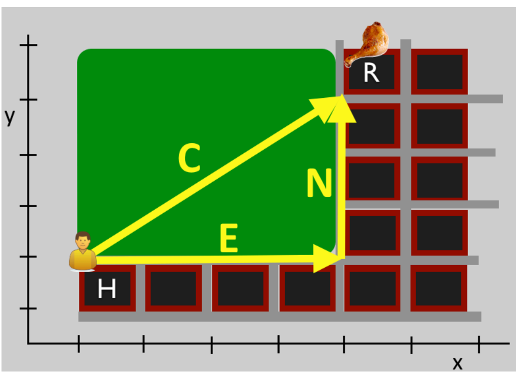


Figure 2.18: The layout showing my hotel (H), the restaurant (R) where there is fried chicken waiting, and the city block structure.

A vector has both a magnitude and a direction *Key Concept 3*

There's an algebraic way to represent vectors, but we'll not need that. Instead we'll make use of the handy symbol of an arrow: $\rightarrow$. The length of the arrow represents the magnitude and of course the orientation and the head of the arrow represent the direction. Arrows can be $\longrightarrow$, or short $\rightarrow$, pointed in different directions, $\nwarrow$, $\leftarrow$, $\nearrow$, etc. Very handy. The magnitude can mean many things, depending on the physical quantity being represented. Obviously, the simplest would be a distance in space, like an arrow on a map or a whiteboard during time-out. That's it.

Here's a way to think about them. Suppose you're in a strange city and you want to know how to get from your hotel to a particular restaurant. You go to the front desk and you're told that you need to walk for 7 blocks, Terrific. Now what? Seven blocks that way? Or, seven blocks the other way! Rather, "walk 4 blocks, east and then 3 blocks north" is more helpful, as you can see in Fig.~\ref{blocks}. (It's just like velocity.)

Now we can go around writing "four blocks east" (or "60 mph north") everywhere, but we need a better notation that packs both *directional* and *magnitude* information into a single symbol so that our hotel-restaurant stroll east is succinctly distinguished from one to the west (and so we don't need to use words in our equations). Traditionally, in print, a vector is represented as a bold letter.[37]

Notation in equations is fine, but pictures of vectors are going to be most useful for us. It's easiest to think in terms of distance vectors. Just like "speed" and "velocity" are related, we can think of "distance" and "displacement" as analogs. So, our hotel tells us that the restaurant is a *distance* of 7 blocks away

[37] There are at least three ways that I can think of to represent vectors. In print, the bold face **x** is most common. On a blackboard, usually people will draw an arrow over the top, $\vec{x}$. And, finally, some people put an underline when they write, $\underline{x}$.

and that its *displacement* is "4 blocks, east and 3 blocks north" and we draw a picture to describe that instruction. Figure 2.18 shows two vectors that do that:

2.8.1 Vector Diagrams

Drawing arrows on a diagram represent a vector with its orientation representing the direction and its length representing the magnitude. Sometimes the length of the arrows are actual length dimensions (like meters, feet, and so on), since a displacement in regular-space is a vector. So, just like a scale on a map, a displacement can be represented as an arrow which is 3 inches long, but where each inch actually corresponds to 1 block (or feet, or miles, or furlongs). But, sometimes a vector doesn't represent a length in space, but some other physical quantity, like a force or a velocity. Now, this can be complicated since you're drawing an arrow that has a length, but you mean it to be something else, like a force. But, it still works geometrically (the arrow still points in space) and we just use a different scale: we might draw an arrow aimed at a box on a diagram that's 2 inches long where every inch corresponds to 2 pounds. So even though it's drawn on a diagram of an object, it represents the application of a force of 4 pounds applied at the point where the arrow is drawn. That's just a visual convenience since the length of the vector in pounds wouldn't have anything to do with any of the length scales in the picture that are lengths or heights.

For a couple of definitions, refer to Fig. 2.19. There are two basic ways to represent vectors, one for print and the other for blackboards (or pencils). The print version is to render the vector quantity as a bold letter. So in Fig. 2.19 the vector on the top is in print **A** and on paper we would write $\vec{A}$.

Two vectors, **A** and **B** are said to be equal if they are *both* the same length *and* point in the same direction. So, as shown $\mathbf{A} = \mathbf{B}$, but neither is equal to **D** even though the length of **D** is the same as that of **A**. Also, we say that $\mathbf{A} = -\mathbf{C}$ if the vectors have the same length, but are pointing in exactly the opposite directions. This is shown in Fig.~2.19b. Another standard definition is to represent the magnitude of a vector–its length–using the symbol $|\mathbf{A}|$. This quantity is a number, not a vector and so we would say that $|\mathbf{A}| = |\mathbf{D}|$.

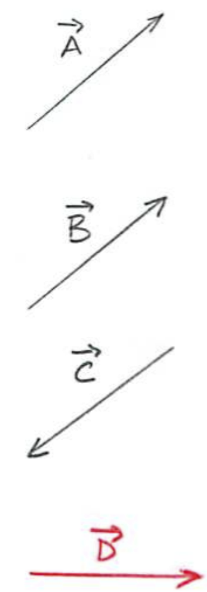


Figure 2.19: Vectors **A** and **B** are equal, and each is equal to $-\mathbf{C}$ and none are equal to **D**, even though the lengths are all same.

2.8.2 Combining Vectors

If you help me to push on my car, we're each applying a force. The whole reason for the two of us is not so we can bond in a shared accomplishment. That's not a guy thing. No, the reason we do it is that we each supply a force and the car then gets pushed with more force than either of us could supply by ourselves.

That is, our forces add...and maybe we bond a little. So, vectors can be added both in symbols, and with pictures.

We can add vectors together by manipulating the arrows. If in our little moment together, I'm **A** and you're **B** then, the car gets pushed by our combined force as shown in Fig. 2.20(a). However, the car would not know the difference between being pushed by the two of us and by some brute who pushes with the force of our combined effort, which we'll call **C**.

$$\mathbf{C} = \mathbf{A} + \mathbf{B}. \tag{2.29}$$

Pencil 2.3.

To calculate this using pictures, you can place the tail of **B** to the head of **A** and then the displacement from the tail of **A** to the head of **B** is the sum, **C**. This is shown in Fig. 2.20(b), and the replacement of the two forces is shown as Fig. 2.20(c). It's important to realize that the situation (a) and (c) are identical, but you would not put both$ **C**$ and the two **A** and **B** on the same picture. It's one or the other.[38]

[38] Dare I carry my little story this far? It's as if I push on the car, and you push on me. If my arms hold up, we still push on the car with the combined force. But, I'd rather not do it that way, thanks.

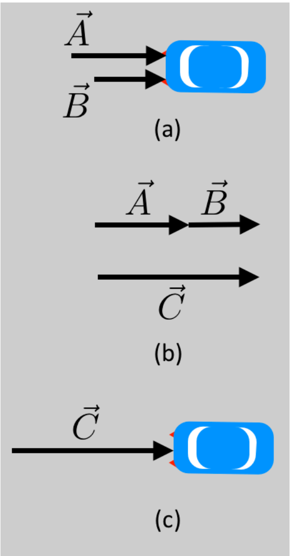


Figure 2.20: (a) Both of us pushing on a car; (b) the combination of our two force vectors; and (c) the replacement of our two independent forces with the combined force. The car doesn't know the difference between (a) and (c)!

Notice, that for doing sums, we can translate vectors around our "space" if we don't change their orientation or length. I did that in the figure.

The car example was all in one dimension, but of course vectors are useful in 2, 3 or more dimensions. Let's go back to our trip to the restaurant from our hotel. What I didn't know, was that there was an open park just behind my hotel, and I could have cut across it to get to the restaurant. That is, an equivalent displacement would have been to follow **C** as shown in Fig. 2.18. That's all the adding of vectors says: a single vector that's equivalent to the operations of the first two. So my trip has two different paths (well, an infinite number):

$$\mathbf{C} = \mathbf{E} + \mathbf{N}$$

Notice that the two vectors don't point in the same direction, so it would be wrong to calculate the distance that **D** represents by just adding the lengths of **E** and **N**. That is, the magnitude of **D**, $|\mathbf{D}| \neq 4+3$. We have to keep the directions and the lengths pointing in their directions separate.

One more way to look at this trip–which resulted in a nice dinner, by the way–would be if we returned to the hotel across that field, then our trip would look like Fig. 2.21.

Notice, that it's different from Fig. 2.18 in that **D** points in the opposite direction from **C**. It's a "round trip" and so the total displacement in a round trip is: zero. In algebra, what this says is:

$$\mathbf{A} + \mathbf{B} + \mathbf{D} = 0$$

Any time you can rearrange a set of vectors to give a "round trip," you describe a situation in which there is no net displacement (we went from the hotel, back to the hotel), or if they are forces, no net force, or if they are velocities, no net velocity. It's a balance $\mathbf{A} + \mathbf{B}$ is balanced by its opposite, **D**. The other way to think of this is remembering that we could have gone to the restaurant across the field if we'd known about it. Notice, that then the vector describing that trip would be $-\mathbf{D}$. We replace $\mathbf{A} + \mathbf{B}$ with $-\mathbf{D}$. And, the balance is just the obvious: $-\mathbf{D} + \mathbf{D} = 0$. This balancing of vectors will be an important concept to us as we'll see in Chapter **??**.

Finally, we can also subtract vectors graphically which is easiest to think about if we think about this almost silly statement:

$$\begin{aligned} a - b &= d \\ a + (-b) &= d \end{aligned}$$

This says that the adding the negative of b to a is the same as subtracting it from a. With vectors, this is a little more meaningful. Referring to Fig. 2.21, let's create a vector subtraction.

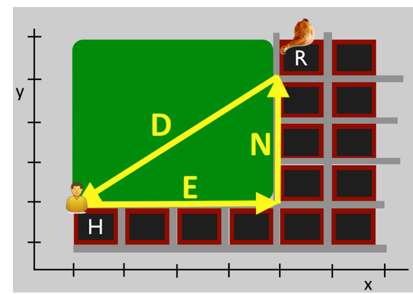


Figure 2.21: The same situation as before, but with the hotel-restaurant trip shown and the restaurant-hotel return shown on the same picture.

$$\begin{aligned} \mathbf{C} &= \mathbf{E} + \mathbf{N} \\ \mathbf{D} &= -\mathbf{C} \\ -\mathbf{D} &= \mathbf{E} + \mathbf{N} = \mathbf{C} \end{aligned}$$

So, we change a subtraction of vectors into an addition of vectors by just turning the appropriate one around.

In order to make the negative of a vector, turn it around and reverse its direction. *Key Concept 4*

2.9 What To Take Away

"...it is impossible to explain honestly the beauties of the laws of nature in a way that people can feel, without their having some deep understanding of mathematics. I am sorry, but this seems to be the case.

"You might say, 'All right, then if there is no explanation of the law, at least tell me what the law is. Why not tell me in words instead of in symbols? Mathematics is just a language, and I want to be able to translate the language.' ... I could convert all the symbols into words. In other words I could be kind to the laymen as they all sit hopefully waiting for me to explain something. Different people get different reputations for their skill at explaining to the layman in layman's language these difficult and abstruse subjects. The layman searches for book after book in the hope that he will avoid the complexities which ultimately set in, even with the best expositor of this type. He finds as he reads a generally increasing confusion, one complicated statement after another, one difficult-to-understand thing after another, all apparently disconnected from one another. It becomes obscure, and he hopes that maybe in some other book there is some explanation...The author almost made it—maybe another fellow will make it right.

"But I do not think it is possible, because mathematics is not just another language. Mathematics is a language plus reasoning; it is like a language plus logic. Mathematics is a tool for reasoning."

Feynman, R.P. (1965) *The Character of Physical Law* BBC. Reprinted by Penguin Books, 1992