Lecture 10

Just like the discrete case, the product group representations are important

Suppose: $|j_1 m_1\rangle_A$ and $|j_2 m_2\rangle_B$

somehow interact. (physics)

The new - generally reducible - product space spanned by

$$|j_1 j_2 m_1 m_2\rangle = |j_1 m_1\rangle_A |j_2 m_2\rangle_B$$

will be simultaneously eigenstates of

$$J^2(1), J^2(2), J_3(1), J_3(2)$$

which only operate in their appropriate subspaces.

eg.

$$J_3(1)\underbrace{|j_1 j_2 m_1 m_2\rangle}_{\substack{\text{hasn't got} \\ \text{a well-defined} \\ j}} = |j_1 j_2 m_1 m_2\rangle m_1$$

Need to decompose the product state into IRR's .. states with well-defined j's. (groups them)

Suppose we have $|\xi_n^{(\alpha)}\rangle$ and $|\xi_m^{(\beta)}\rangle$ which transform according to $\Gamma^{(\alpha)}$ and $\Gamma^{(\beta)}$

Then, the direct product $|\xi_n^{(\alpha)}\rangle|\xi_m^{(\beta)}\rangle$ transform according to the direct product rep. $\Gamma^{(\alpha\times\beta)}$

$$\Gamma(\gamma)|\xi_n^{(\alpha)}\rangle|\xi_m^{(\beta)}\rangle = \sum_i\sum_j |\xi_i^{(\alpha)}\rangle|\xi_j^{(\beta)}\rangle \Gamma^{(\alpha)}(\gamma)^i{}_n \Gamma^{(\beta)}(\gamma)^j{}_m$$

$$= \sum_{ij} |\xi_i^{(\alpha)}\rangle|\xi_j^{(\beta)}\rangle \Gamma^{(\alpha\times\beta)}(\gamma)^{ij}{}_{nm}$$

$$\Big\downarrow$$

$$\Gamma(\gamma)|\alpha\beta, nm\rangle$$

Further, $\Gamma^{(\alpha\times\beta)} = \sum_\delta m_\delta \Gamma^{(\delta)}$

By a <u>change of basis</u>, linear combinations can be formed

$$|\xi_h^{(\delta)}\rangle = \sum_{ij} |\xi_i^{(\alpha)}\rangle|\xi_j^{(\beta)}\rangle\, C(\alpha\beta\delta, ijh)$$

eigenstates of well defined δ and weight h

called Clebsch-Gordon coefficients for the groups.

For the rotation groups, $\Gamma^{(\alpha)}(\gamma) = D^{(j)}(\gamma)$ etc.

That is,

$$\Gamma(R)\,|j_1 j_2 m_1 m_2\rangle = |\psi^{(j_1)}_{n_1}\rangle|\psi^{(j_2)}_{n_2}\rangle D^{(j_1)}(R)^{n_1}{}_{m_1} D^{(j_2)}(R)^{n_2}{}_{m_2}$$

$$= |j_1 j_2 n_1 n_2\rangle D^{(j_1 \otimes j_2)}(R)^{n_1 n_2}{}_{m_1 m_2}$$

and as before.

"simply reducible" since $n_j = 1 \ \forall j$

$$D^{(j_1 \otimes j_2)}(R) = \sum_j n_j D^{(j)}(R)$$

well defined total $j\,m$

The action, though, is with the basis states –

$$|j\,m\rangle = \sum_{m_1 m_2} C(j_1 j_2 j, m_1 m_2 m)|\psi^{(j_1)}_{m_1}\rangle|\psi^{(j_2)}_{m_2}\rangle$$

just one basis set expressed in terms of another –

unitary trans.

Clebsch Gordon coefficients

Usually you see

$$|j_1 j_2\, j m\rangle = \sum_{m_1 m_2} \underbrace{|j_1 j_2 m_1 m_2\rangle}_{\text{completeness}} \underbrace{\langle j_1 j_2 m_1 m_2 | j_1 j_2\, j m\rangle}_{\text{C.G.C.}}$$

not eigenfunctions of $\vec{J}$

The operators in the product space are

$$\vec{J} = \vec{J}(1) + \vec{J}(2) \quad \Rightarrow \quad J_3 = J_3(1) + J_3(2)$$

$$J^2 = \left(\vec{J}(1) + \vec{J}(2)\right)^2 \qquad J_\pm = J_\pm(1) + J_\pm(2)$$

From

$$J_3 |j_1 j_2 j m\rangle = J_3 \sum_{m_1} \sum_{m_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

$$|j_1 j_2 m_1 m_2\rangle = |j_1 m_1\rangle |j_2 m_2\rangle$$

since $\quad J_3 |j_1 m_1\rangle |j_2 m_2\rangle = [J_3(1) + J_3(2)] |1\rangle |2\rangle$

$$= (m_1 + m_2) |1\rangle |2\rangle$$

so

$$J_3 |j_1 j_2 j m\rangle = \sum_{m_1} \sum_{m_2} (m_1 + m_2) \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle |1\rangle |2\rangle$$

also, $\quad J_3 |j_1 j_2 j m\rangle = m \sum_{m_1} \sum_{m_2} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle |1\rangle |2\rangle$

by linear independence

$$(m_1 + m_2 - m) \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle = 0$$

which only happens if $m_1 + m_2 = m$

or if $\neq$, then $\langle \; | \; \rangle = 0$

Standard approach is to force $m = m_1 + m_2$

$$m_2 = m - m_1$$

or

$$|j_1 j_2 jm\rangle = \sum_{m_1} \sum_{m_2} \delta_{m_1+m_2, m} \langle \quad | \quad \rangle |1\rangle |2\rangle$$

$$= \sum_{m_1} \langle j_1 j_2 m_1 m_2 | j_1 j_2 jm\rangle |j_1 m_1\rangle |j_2\, m-m_1\rangle$$

Define the C.G.C. to be real:

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 jm\rangle = \langle j_1 j_2 jm | j_1 j_2 m_1 m_2\rangle$$

orthogonal

$$\langle jm\, j_1 j_2 | j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j'' m'\rangle = \delta_{jj''} \delta_{mm'}$$

abbreviate $\qquad \langle j_1 j_2 jm| \rightarrow \langle jm|$

$$\langle j_1 j_2 m_1 m_2| \rightarrow \langle m_1 m_2|$$

The j of the product space is bounded

$$|j_1 - j_2| \leq j \leq j_1 + j_2$$

and the explicit calculation of the C.G.C.
can be done with the raising and lowering
operators

Construct:

$$\langle m_1 m_2 | \overset{(\rightarrow)}{J_\pm} | j m \rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, \langle m_1 m_2 | j\, m \pm 1 \rangle$$

since $J_+ = J_-^\dagger$

$$\langle m_1 m_2 | \overset{(\leftarrow)}{J_\pm} | j m \rangle = \sqrt{(j_1 \pm m_1)(j_1 \mp m_1 + 1)}\, \langle m_1 \mp 1\, m_2 | j m \rangle$$

$$+ \sqrt{(j_2 \pm m_2)(j_2 \mp m_2 + 1)}\, \langle m_1\, m_2 \mp 1 | j m \rangle$$

RHS

equating; a recursion relation:

LHS $\sqrt{(j \mp m)(j \pm m - 1)}\, \langle m_1 m_2 | j\, m \pm 1 \rangle = \text{RHS}$

sufficient.. up to a phase..

Choose $\langle j_1\, j - j_1 | j j \rangle$: real & positive

$\boxed{\text{THREE WAYS}}$

① Common sense

ex. $j_1 = j_2 = 1/2 \Rightarrow |1/2 - 1/2| \le j \le \frac{1}{2} + \frac{1}{2}$

$\Rightarrow j = 0, 1$

since $j = j_1 \oplus j_2 = 0, 1$

$m = 0$ $\qquad$ $m = -1, 0, +1$

What are the available C.G.C.?

$$\langle m_1 m_2 | j m\rangle \left(= \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle \right)$$

$$\downarrow$$

$$\langle \tfrac{1}{2}\, \tfrac{1}{2} | 1 1\rangle$$
$$\langle \tfrac{1}{2}\, -\tfrac{1}{2} | 1 0\rangle$$
$$\langle -\tfrac{1}{2}\, \tfrac{1}{2} | 1 0\rangle$$
$$\langle -\tfrac{1}{2}\, -\tfrac{1}{2} | 1\, -1\rangle$$
$$\langle \tfrac{1}{2}\, -\tfrac{1}{2} | 0 0\rangle$$
$$\langle -\tfrac{1}{2}\, \tfrac{1}{2} | 0 0\rangle$$

For $j=1$ states, the product representation comes from

$$|jm\rangle = \sum_{m_1} |j_1 m_1\rangle |j_2 m_2\rangle \langle m_1 m_2 | j m\rangle$$

$$|1\, m\rangle = \sum_{m_1=-1/2}^{1/2} |\tfrac{1}{2}\, m_1\rangle |\tfrac{1}{2}\, m-m_1\rangle \langle m_1\, m-m_1 | 1\, m\rangle$$

one at a time

nope, not there

$$|1 1\rangle = |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle |\tfrac{1}{2}\, \tfrac{3}{2}\rangle \cancel{\langle \tfrac{1}{2}\, \tfrac{3}{2} | 1 1\rangle} + |\tfrac{1}{2}\, \tfrac{1}{2}\rangle |\tfrac{1}{2}\, \tfrac{1}{2}\rangle \langle \tfrac{1}{2}\, \tfrac{1}{2} | 1 1\rangle$$

$$= |\tfrac{1}{2}\, \tfrac{1}{2}\rangle |\tfrac{1}{2}\, \tfrac{1}{2}\rangle \langle \tfrac{1}{2}\, \tfrac{1}{2} | 1 1\rangle$$

$$\downarrow$$

$$= 1$$ if real & positive

next:

$$|1\,0\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\langle -\tfrac{1}{2}\,\tfrac{1}{2}|1\,0\rangle$$

$$+\ |\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\langle \tfrac{1}{2}\,-\tfrac{1}{2}|1\,0\rangle$$

$$|1\,-1\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\langle -\tfrac{1}{2}\,\tfrac{1}{2}|1\,-1\rangle$$

with the recursion relation
$j=1$, $m=1$ & lower sign:

$$\sqrt{(1+1)(1-1+1)}\quad\langle m_1 m_2|1\,0\rangle =$$

$$\sqrt{(j_1-m_1)(j_1+m_1+1)}\ \langle m_1+1\ m_2|1\,1\rangle$$

$$+\ \sqrt{(j_2-m_2)(j_2+m_2+1)}\ \langle m_1\ m_2+1|1\,1\rangle$$

$$\text{LHS} = \sqrt{2}\ \langle m_1 m_1|1\,0\rangle$$

$$\hookrightarrow\ = m_1+m_2 = \quad m_1=\tfrac{1}{2}\ m_2=-\tfrac{1}{2}\ \text{(A)}$$

$$m_1=-\tfrac{1}{2}\ m_2=\tfrac{1}{2}\ \text{(B)}$$

(A) $$\sqrt{2}\langle \tfrac{1}{2}\,-\tfrac{1}{2}|1\,0\rangle = \sqrt{0}\ \langle \tfrac{3}{2}\,\tfrac{1}{2}|1\,1\rangle$$

$$+\ \sqrt{(1)(1)}\ \underbrace{\langle \tfrac{1}{2}\,\tfrac{1}{2}|1\,1\rangle}_{=1}$$

$$\Rightarrow\quad \langle \tfrac{1}{2}\,-\tfrac{1}{2}|1\,0\rangle = \sqrt{\tfrac{1}{2}}$$

(B) $$\langle -\tfrac{1}{2}\,\tfrac{1}{2}|1\,0\rangle = \sqrt{\tfrac{1}{2}}$$

So, $|11\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle$

$$|10\rangle = \sqrt{1/2}\,|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + \sqrt{1/2}\,|\tfrac{1}{2}\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$|1\,-1\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

For $j=0$ the only $m=0$

The recursion relation:

$$\sqrt{(0\mp 0)(0\pm 0+1)}\,\langle m_1 m_2|1\pm1\rangle = 0 \qquad \text{LHS}$$

$$\text{RHS} \quad = \sqrt{(j_1\pm m_1)(j_1\mp m_1+1)}\,\langle m_1\mp1\, m_2|00\rangle$$

$$+ \sqrt{(j_2\pm m_2)(j_1\mp m_1+1)}\,\langle m_1\, m_2\mp1|00\rangle$$

<u>since</u> $\langle n\,m|j\,k\rangle = 0$ unless $n+m=k$.

from 1^{st} on RHS $\quad m_1\mp 1 + m_2 = 0$

$$m_1+m_2 = \pm 1$$

$$\left.\begin{array}{l} \Rightarrow\; m_1 = 1/2 = m_2 \\ \text{or}\;\; m_1 = \tfrac{1}{2} = m_2 \end{array}\right\} m_1 = m_2$$

from 2nd term: same.

so, with top sign $m_1 = \frac{1}{2} = m_2$

$$0 = \sqrt{(1/2 + 1/2)(\tfrac{1}{2} - \tfrac{1}{2} + 1)}\ \langle -\tfrac{1}{2}\ \tfrac{1}{2} | 00 \rangle$$

$$+ \sqrt{(1/2 + 1/2)(\tfrac{1}{2} - \tfrac{1}{2} + 1)}\ \langle \tfrac{1}{2}\ -\tfrac{1}{2} | 00 \rangle$$

$$\Rightarrow \quad \langle -\tfrac{1}{2}\ \tfrac{1}{2} | 00 \rangle = - \langle \tfrac{1}{2}\ -\tfrac{1}{2} | 00 \rangle$$

need orthogonality:

$$\sum_{m_1} |\langle m_1 m_2 | j m \rangle|^2 = 1$$

or

$$\sum_{m_1 = -\frac{1}{2}}^{\frac{1}{2}} |\langle m_1\ 0 - m_1 | 00 \rangle|^2 = 1$$

$$= |\langle -\tfrac{1}{2}\ \tfrac{1}{2} | 00 \rangle|^2 + \underbrace{|\langle \tfrac{1}{2}\ -\tfrac{1}{2} | 00 \rangle|^2}_{\parallel\ |-\langle -\frac{1}{2}\ \frac{1}{2} | 00 \rangle|^2} = 1$$

$$\Rightarrow \quad 2\, |\langle -\tfrac{1}{2}\ \tfrac{1}{2} | 00 \rangle|^2 = 1$$

$$\langle -\tfrac{1}{2}\ \tfrac{1}{2} | 00 \rangle = \sqrt{1/2} = - \langle \tfrac{1}{2}\ -\tfrac{1}{2} | 00 \rangle$$

and the decomposition is

$$|00\rangle = \sum_{m_1} |\tfrac{1}{2}\ m_1\rangle\, |\tfrac{1}{2}\ 0 - m_1\rangle\, \langle m_1\ 0 - m_1 | 00 \rangle$$

$$= \sqrt{1/2}\ |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\, |\tfrac{1}{2}\ \tfrac{1}{2}\rangle - \sqrt{1/2}\ |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle\, |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

which is orthogonal to the $|10\rangle$.

So, the C.G. coefficients can be tabulated --

$j_1 \otimes j_2$	J M - - - - -
$m_1 m_2$	
$m_1 m_2$	C.G.C.
$\vdots$	

The "particle data book" always includes them --

http://pdg.lbl.gov/

You can also think of them as a matrix

for $1/2 \otimes 1/2$:

$$\begin{array}{ccc} |JM\rangle & & |m_1 m_2\rangle \\ \begin{pmatrix} |11\rangle \\ |10\rangle \\ |1-1\rangle \\ |00\rangle \end{pmatrix} & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \sqrt{1/2} & -\sqrt{1/2} & 0 \end{pmatrix} & \begin{pmatrix} |1/2\ 1/2\rangle \\ |1/2\ -1/2\rangle \\ |-1/2\ 1/2\rangle \\ |-1/2\ -1/2\rangle \end{pmatrix} \end{array}$$

evolution of notation:

$$|j_1 m_1\rangle_A |j_2 m_2\rangle_B \longrightarrow |j_1 j_2 m_1 m_2\rangle \longrightarrow |m_1 m_2\rangle$$