Lecture 11

For SU(2), what we just did was:

$$\underbrace{\underline{2} \otimes \underline{2}}_{\substack{2,\ SU(2) \\ \text{fundamental} \\ \text{doublets}}} = \underbrace{\underline{3}}_{\substack{\text{an } SU(2) \\ \text{triplet}}} \oplus \underbrace{\underline{1}}_{\text{an } SU(2) \text{ singlet}}$$

Called a <u>Kronecher</u> <u>Decomposition</u>:

Create the IRR content of a reducible representation

Another notational simplicity

$$\psi^i(\underline{n})$$

$\underline{n}$ ← dimensionality

i ← relative weight, highest $\underline{=1}$ to lowest $= \underline{n}$

So, for the SU(2) triplet

$$\left.\begin{aligned} \psi^1(\underline{3}) &= \psi^{(11)} \\ \psi^2(\underline{3}) &= \psi^{(12)} \\ \psi^3(\underline{3}) &= \psi^{(22)} \end{aligned}\right\} \psi^i(\underline{3})$$

the triplet of SU(2)
or "the three" of SU(2)

lowering operation still "works"

$$\left.\begin{aligned} e(21)\,\psi^1(\underline{3}) &= \sqrt{2}\,\psi^2(\underline{3}) \\ e(21)\,\psi^2(\underline{3}) &= \sqrt{2}\,\psi^3(\underline{3}) \end{aligned}\right\} e(21)\,\psi^i(\underline{3}) = \sqrt{2}\,\psi^{i+1}(\underline{3})$$

Singlet: $\psi^1(\underline{1}) = \sqrt{1/2}\left(\xi^1\eta^2 - \xi^2\eta^1\right)$

How about 3 spin 1/2 states, 3 $SU(2)$'s = $\eta^i \xi^j \gamma^k$

algebra with the multiplicities

$$\underline{2} \otimes \underline{2} \otimes \underline{2} = (\underline{3} \oplus \underline{1}) \otimes \underline{2}$$
$$= \underline{3} \otimes \underline{2} \oplus \underline{1} \otimes \underline{2}$$
$$= \underbrace{\underline{3} \otimes \underline{2}}_{\text{something w/ 6 terms}} \oplus \underbrace{\underline{2}}_{\text{doublet}}$$

Highest weight state: $\phi^1 = \psi^1(\underline{3})\gamma^1 = \xi^1 \eta^1 \gamma^1$

start lowering $e(21)\phi^1 = [e(21)\psi^1(\underline{3})]\gamma^1 + \psi^1(\underline{3})[e(21)\gamma^1]$

$$\phi^2 = \sqrt{2}\, \psi^2(\underline{3})\gamma^1 + \psi^1(\underline{3})\gamma^2$$

$\psi^2(\underline{3}) = \psi^{(12)}$

$$= \sqrt{2} \tfrac{1}{\sqrt{2}} (\xi^2 \eta^1 \gamma^1 + \xi^1 \eta^2 \gamma^1) + \xi^1 \eta^1 \gamma^2$$
$$= \xi^2 \eta^1 \gamma^1 + \gamma^1 \eta^2 \gamma^1 + \xi^1 \eta^1 \gamma^2$$

normalized: $\phi^2 = \sqrt{1/3}\,\underbrace{(\xi^2 \eta^1 \gamma^1 + \xi^1 \eta^2 \gamma^1 + \xi^1 \eta^1 \gamma^2)}_{\text{totally symmetric}}$

all labels: 1, 1, 2 arranged in all symmetric ways.

112 121 211

In fact, the next lowering also arranges labels symmetrically: 122

and then 222

$\Rightarrow$ 3 objects with 2 labels, arranged in all symmetric ways: 111, 112, 122, 222 = 4

(111) highest weight state $\xi^1\eta^1\gamma^1 \equiv \psi^1(\underline{4})$

(112) just found $\psi^2(\underline{4})$

(122) next $\psi^3(\underline{4})$

(222) lowest $\psi^4(\underline{4})$

a quartet of SU(2).

$$\underline{2} \otimes \underline{2} \otimes \underline{2} = \underline{3} \otimes \underline{2} \oplus \underline{2}$$

$$\downarrow$$

$$\underline{4} \oplus \underline{2} \oplus \underline{2}$$

just found

The 1st doublet: from starting with $\psi^2(\underline{4})$ constructing orthogonal state and lowering

For example $\quad \psi^1(\underline{2}) = -\sqrt{1/3}\, \psi^2(\underline{3})\gamma^1 + \sqrt{2/3}\, \psi^1(\underline{3})\gamma^2$

and the remaining doublet

$$\psi^2(\underline{2}) = \sqrt{1/3}\, \psi^2(\underline{3})\gamma^2 - \sqrt{2/3}\, \psi^3(\underline{3})\gamma^1$$

THIS IS GENERAL...

For $SU(n)$, $n \geq 2$, there are conjugate states (which don't physically occur for spin $SU(2)$)

$$\xi^{*\prime i} = \xi'_i = \xi_j U^j{}_i$$

Define a permutation operator:

$$\psi^{ij} = \xi^i \eta^j \quad \text{such that} \quad P_{12} T^{ij} = T^{ji} = \xi^j \eta^i$$

Notice: $\quad \psi \to \psi' = U\psi$

$$U P_{12} \psi^{ij} = U P_{12} \xi^i \eta^j$$
$$= U \xi^j \eta^i = \psi^{ji\prime}$$

& $\quad P_{12} U \psi^{ij} = P_{12} \psi^{ij\prime} = \psi^{ji\prime}$ $\quad$ U & P_{12} commute

IMPORTANT!

We wrote product states as $\quad S + A$

Define

$$S^{ij} = \tfrac{1}{2}(\psi^{ij} + \psi^{ji}) \Rightarrow P_{12} S^{ij} = S^{ji}$$
$$A^{ij} = \tfrac{1}{2}(\psi^{ij} - \psi^{ji}) \Rightarrow P_{12} A^{ij} = -A^{ji}$$

An important Theorem:

If an m-order product state is an IRR of S_m and if it is constructed from basis vectors of a d-dimensional IRR of $SU(d)$, then the product state is also an IRR of $SU(d)$

$\Rightarrow$ the problem of studying combined states of vectors in $SU(n)$ becomes the well-understood problem of studying IRR of S_m.

Back to the Permutation Group.

Remember, for S_2 we had 2 elements

$$\begin{pmatrix} 12 \\ 12 \end{pmatrix} \text{ and } \begin{pmatrix} 12 \\ 21 \end{pmatrix}$$

which we wrote as <u>cycles</u> $(1)(2)$ and (12)

Each cycle is related to classes of S_n, as I'll show.

Remember, that a particular element of S_n can be written as a product of cycles.

Remember the big S_8 example I used.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 1 & 6 & 7 & 4 & 5 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 6 & 5 & 7 & 8 \\ 2 & 3 & 1 & 6 & 4 & 7 & 5 & 8 \end{pmatrix}$$

$$= (123)(46)(57)(8)$$

permute, right?

Each number appears <u>once</u> so the sum of the lengths of each cycle l_i must be equal to n for S_n

$$(123)(46)(57)(8)$$

$$l_i = \quad 3 \quad 2 \quad 2 \quad 1 \qquad \sum l_i = 8 = n$$

These lengths label the classes and are called <u>Partitions</u>.

$$[l_1\, l_2\, l_3 \ldots l_n] \qquad \text{where } l_i \geq l_{i+1}$$

so the partition for this particular element of S_8 is

$$[3\ 2\ 2\ 1]$$

Let's go back to S_3 -- manageable.

	cycles	partitions	
$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	$(1)(2)(3)$	$[111]$	$\mathcal{C}_1$
$p_a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	$(12)(3)$	$[21]$	
$p_b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	$(1)(23) = (23)(1)$	$[21]$	$\mathcal{C}_3$
$p_c = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \end{pmatrix}$	$(13)(2)$	$[21]$	
$p_d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}$	(132)	$[3]$	$\mathcal{C}_2$
$p_f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	(123)	$[3]$	

- A cycle of degree 2 is a transposition & any cycle can be written as a product of transpositions, P_{ij}

a) If the original cycle is even degree → odd # trans.
b) " odd " → even # trans.
c) If a given premutation factors into an even number of transpositions → "even perm"
d) " odd " → "odd perm"

e) The set of <u>even</u> permutations forms an invariant subgroup of S_n called the Alternating Group of order $n!/2$

Since # classes = # partitions – can calculate how many classes. Without proof:

$$\# \text{IRR of } S_n = \# \text{classes in } S_n$$

So, can calculate # IRR.

A partition is just the number of ways to take n numbers and make n by addition.

$n = 2$ $\quad 1+1 \;;\; 2$

$n = 3$ $\quad 1+1+1 \;;\; 2+1 \;;\; 3$

$n = 4$ $\quad 1+1+1+1 \;;\; 3+1 \;;\; 2+2 \;;\; 2+1+1 \;;\; 4$

$\vdots$

If a particular partition is:

$$n = \alpha \times 1 + \beta \times 2 + \gamma \times 3 + \cdots$$

Then the number of elements r_p in each class is (N_r)

$$r_p = \frac{n!}{1^\alpha \cdot 2^\beta \cdot 3^\gamma \cdots \alpha! \beta! \gamma!}$$

So, for S_3:

$\alpha = 3 \qquad n = 3 \times 1 \longrightarrow [111]$

$$r_{[111]} = \frac{3 \cdot 2}{1^3 \cdot 2^0 \cdots 3!} = \frac{6}{6} = 1$$

$\alpha = 1$, $\beta = 1$ $\qquad n = 1\times1 + 1\times2 = 3$

$$\rightarrow [1\,2] = [2\,1]$$

$$r_{[21]} = \frac{3\cdot2}{1^1 2^1 \cdots} = \frac{6}{2} = 3$$

$\alpha = 0$, $\beta = 0$, $\gamma = 1$ $\qquad n = 0\times1 + 0\times2 + 1\times3 = 3$

$$\rightarrow [3]$$

$$r_{[3]} = \frac{6}{3^1} = 2$$

There are 2 important operators for S_n

$$\left.\begin{aligned} S_n &= \frac{1}{n!}\sum_P \begin{pmatrix} 1 & 2 & \cdots & n \\ P_1 & P_2 & \cdots & P_n \end{pmatrix} \\ A_n &= \frac{1}{n!}\sum_P \varepsilon_P \begin{pmatrix} 1 & 2 & \cdots & n \\ P_1 & P_2 & \cdots & P_n \end{pmatrix} \end{aligned}\right\} \text{sum over all permutations}$$

ε_P: "parity" $= 1$ if even perm.
$= -1$ if odd perm.

For 2 objects

$$S_2 = \frac{1}{2}\left[e + \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right] = \frac{1}{2}\left[e + P_{12} \right]$$

P_{12}: transposition operator.

3 objects

$$S'_3 = \frac{1}{6}\left[e + P_{12} + P_{13} + P_{23} + \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} + \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \right]$$

$$\begin{pmatrix}2&1&3\\2&3&1\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} = P_{13}P_{12}$$

$$\begin{pmatrix}3&2&1\\3&1&2\end{pmatrix}\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix} = P_{12}P_{13}$$

so,

$$S'_3 = \frac{1}{6}\left[e + P_{12} + P_{13} + P_{23} + P_{13}P_{12} + P_{12}P_{13} \right]$$

also, $\leftarrow$ 1 transp → odd $\varepsilon_P = -1$

$$A_2 = \frac{1}{2}\left[e - P_{12} \right]$$

$$A_3 = \frac{1}{6}\left[e - P_{12} - P_{13} - P_{23} + P_{12}P_{13} + P_{13}P_{12} \right]$$

okay. Thinking ahead to the physics..

want to distinguish between the <u>state</u> of a particle and the particle <u>label</u>.

For example. $\alpha(1)\beta(2)\gamma(3)\delta(4)$

means: particle 1 in state α

" 2 " β

3 γ

4 δ

The symmetric group has clear applications to the physics problem of identical particles... so what happens under permutations:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \alpha(1)\beta(2)\gamma(3)\delta(4) = \alpha(2)\beta(1)\gamma(3)\delta(4)$$

label them this way:

$$| \; \boxed{} \; \boxed{} \; \boxed{} \; \boxed{} \; \rangle$$

slot # 1 2 3 4

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} |\alpha\beta\gamma\delta\rangle = |\beta\alpha\gamma\delta\rangle$$

For 2 objects

$$S_2' |\alpha\beta\rangle = \tfrac{1}{2}(e + P_{12})|\alpha\beta\rangle$$

$$= \tfrac{1}{2}|\alpha\beta\rangle + \tfrac{1}{2}|\beta\alpha\rangle \equiv |\Sigma_2\rangle \longrightarrow [2]$$

(symmetric)

$$\underline{A}_2 |\alpha\beta\rangle = \tfrac{1}{2}(e - P_{12})|\alpha\beta\rangle$$

$$= \tfrac{1}{2}|\alpha\beta\rangle - \tfrac{1}{2}|\beta\alpha\rangle \equiv |A_2\rangle \longrightarrow [11]$$

(antisymmetric)

For 3 objects.

$$S_3 |\alpha\beta\gamma\rangle = \frac{1}{6}\Big[|\alpha\beta\gamma\rangle + |\beta\alpha\gamma\rangle + |\gamma\beta\alpha\rangle + |\alpha\gamma\beta\rangle$$

$$+ P_{13}|\beta\alpha\gamma\rangle + P_{12}|\gamma\beta\alpha\rangle \Big]$$

$$= \frac{1}{6}\Big[|\alpha\beta\gamma\rangle + |\beta\alpha\gamma\rangle + |\gamma\beta\alpha\rangle + |\alpha\gamma\beta\rangle$$

$$+ |\gamma\alpha\beta\rangle + |\beta\gamma\alpha\rangle \Big] \equiv |\Sigma_3\rangle \to [3]$$

$$A_3 |\alpha\beta\gamma\rangle = \frac{1}{6}\Big[|\alpha\beta\gamma\rangle - |\beta\alpha\gamma\rangle - |\gamma\beta\alpha\rangle - |\alpha\gamma\beta\rangle$$

$$+ |\gamma\alpha\beta\rangle + |\beta\gamma\alpha\rangle \Big] \equiv A_3 \to [111]$$

Two basis vectors of 2 1d IRR ~ that's not all of them

In general, you need the character tables in order to figure out the basis states.

The thing about the Symmetric Group is that the permutation symmetries are either symmetric, antisymmetric or mixed. The remaining states here are mixed and 2 dimensional.

Calculation of and use of the character tables of S_n is very cumbersome

We can create some using successive permutation operations —

$$S_{ij} \equiv 1 + P_{ij}$$

$$A_{ij} \equiv 1 - P_{ij}$$

$$|M_3\rangle_a = A_{13} S_{12} |\alpha\beta\gamma\rangle$$
$$= A_{13} \left(|\alpha\beta\gamma\rangle + |\beta\alpha\gamma\rangle \right)$$
$$= |\alpha\beta\gamma\rangle - |\gamma\beta\alpha\rangle + |\beta\alpha\gamma\rangle - |\gamma\alpha\beta\rangle$$

and

$$|M_3\rangle_b = A_{23} S_{12} |\alpha\beta\gamma\rangle$$

$$|M_3\rangle_c = A_{23} S_{13} |\alpha\beta\gamma\rangle$$

$$|M_3\rangle_d = A_{12} S_{13} |\alpha\beta\gamma\rangle$$

(S_{23} states are not linearly independent.)

So, 6 states: $|\Sigma_3\rangle$, $|A_3\rangle$, $|M_3\rangle_{a,b,c,d}$

<u>not</u> 4d
BUT
2, 2d IRR's

A little vector space can be put together with basis states which are orthogonal & orthonormal –

$$|M_3\rangle_1 = \tfrac{1}{2}|M_3\rangle_a$$

$$|M_3\rangle_2 = \sqrt{1/3}\,\left[\,|M_3\rangle_b - \tfrac{1}{2}|M_3\rangle_a\right]$$

S_3 transforms these into one another.

another

$$|M_3\rangle_3 = \sqrt{1/3}\,\left(\,|M_3\rangle_c + \tfrac{1}{4}|M_3\rangle_a - \tfrac{1}{2}|M_3\rangle_b + \tfrac{1}{4}|M_3\rangle_a\right)$$

$$|M_3\rangle_4 = \tfrac{2}{3}\Big\{\,|M_3\rangle_d - \tfrac{1}{4}\Big[\,|M_3\rangle_a + 2\left(|M_3\rangle_b - \tfrac{1}{2}|M_3\rangle_a\right) + 2\left(\,|M_3\rangle_c + \tfrac{1}{4}|M_3\rangle_a - \tfrac{1}{2}\left(|M_3\rangle_b - \tfrac{1}{2}|M_3\rangle_c\right)\right)\Big]\Big\}$$

ditto about S_3.

There has to be an easier way...

and there is.