Lecture 12 what we did

Continued with the calculation of making product states. For the direct, or tensor product of 2 basis vectors of SU(2):

$\psi^{(11)} = \xi^1 \eta^1$

$\psi^{(12)} = \sqrt{1/2}\,(\xi^2\eta^1 + \xi^1\eta^2)$ SU(2)

$\psi^{(22)} = \xi^2 \eta^2$

$\psi^{[12]} = \sqrt{1/2}\,(\xi^1\eta^2 - \xi^2\eta^1)$

Kronecker decomposition: $\underline{2} \otimes \underline{2} = \underline{3} \oplus \underline{1}$

SU(2) triplet SU(2) singlet

For 3: $\eta^i \xi^j \chi^k$?

$\underline{2} \otimes \underline{2} \otimes \underline{2} = (\underline{3} \oplus \underline{1}) \otimes \underline{2}$

$= \underline{3} \otimes \underline{2} \oplus \underline{1} \otimes \underline{2}$

showed this to be

$= \underline{4} \oplus \underline{2} \oplus \underline{2}$

Back to the Permutation Group.

The cycle notation as standing for individual S_n group elements

eg S_3:

$$p_6 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \rightarrow \begin{matrix} (1)(23) \\ (23)(1) \end{matrix}$$

↓

partition notation .. the "length" of the cycles.

[21]

The partitions label the classes

For S_3 ...

[111]	(1)(2)(3)	1 element
[21]	(23)(1) ...	3 elements
[3]	(132) ...	2 elements

classes in S_n = # IRR in S_n
= # partitions.
↳ calculable

Invented a notation

$$|\alpha\ \beta\ \gamma\ \delta\rangle$$

↑

particle #	1	2	3	4
in state	α	β	γ	δ

A group element from S_4 operating on this basis vector.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} |\alpha\beta\gamma\delta\rangle = |\beta\alpha\gamma\delta\rangle$$

Can find the basis vectors in S_n by using

$$\underline{S}_n = \frac{1}{n!} \sum_p \begin{pmatrix} 1 & 2 & 3 \ldots & n \\ p_1 & p_2 & p_3 \ldots & p_n \end{pmatrix}$$

$$\underline{A}_n = \frac{1}{n!} \sum_p \left(\qquad\qquad \right) \varepsilon_p$$

For example, for S_3

$$\underline{S}_3 = \frac{1}{6} \Big[e + P_{12} + P_{13} + P_{23} + \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}}_{P_{13} P_{12}} + \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}}_{P_{12} P_{13}} \Big]$$

transposition operator.

can always take any S_n operation and turn it into a sum of products of transpositions.

So, for S_3 we found

$\underline{S}_3 |\alpha\beta\gamma\rangle \equiv |\Sigma_3\rangle$ completely symmetric [3]

$\underline{A}_3 |\alpha\beta\gamma\rangle \equiv |A_3\rangle$ completely antisymmetric [111]

Then, by selectively operating to construct linearly independent, orthogonal states..

we found 2, 2 dimensional states of
mixed symmetry -- neither
symmetric nor antisymmetric

Remember, decomposition of reducible representations goes according to the character tables.

$$\Gamma^{(red)}(\gamma_p) = \sum_i n_{[i]} \Gamma^{(i)}(\gamma_p)$$

times the i^{th} IRR appears.

further

$$\chi^{(red)}(\gamma_p) = \sum_i n_{[i]} \chi^{(i)}(\gamma_p)$$

$\vdots$

$$n_{[i]} = \frac{1}{g} \sum_r N_r \chi^{(i)}(\mathcal{C}_r) \chi^{(red)}(\mathcal{C}_r)$$

From that, I found

$$\Gamma^{(5)} = \Gamma^{(1)} \oplus \Gamma^{(3)}$$

The same is true of the Symmetric Group.

I didn't derive them, but here are the character tables for the first few S_n

S_2	(11)	(2)
$[2]$	1	1
$[11]$	1	-1
	$\begin{pmatrix}1&2\\1&2\end{pmatrix}$	$\begin{pmatrix}1&2\\2&1\end{pmatrix}$
N_r	1	1

$\leftarrow$ classes

S_3	(111)	(21)	(3)
$[3]$	1	1	1
$[21]$	2	0	-1
$[111]$	1	-1	1
N_r	1	3	2
	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$ $\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$ $\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$ $\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$

Look at the (21) elements – there are two kinds of permutations going on

$\begin{pmatrix}i&j\\j&i\end{pmatrix}$ simple permutations

$\begin{pmatrix}i\\i\end{pmatrix}$ nothing – e

These are S_2 subgroups of S_3

$$[11] \swarrow \quad \searrow [21]$$

So, we could say that

$$\Gamma^{[21]} = \underset{\substack{\uparrow \\ 1}}{\Gamma^{[2]}} \oplus \underset{\substack{\uparrow \\ 1}}{\Gamma^{[11]}}$$

There is no analog of [3].

This is always true...

IRR $S_n \longrightarrow$ IRR $S_{n-1} \longrightarrow$ IRR $S_{n-2} \longrightarrow \cdots$

From the character table we can see this.

$$n_{[11]} = \frac{1}{2!} \sum_r N_r \chi^{[x]} \chi^{[11]}$$

where

$$\Gamma^{[x]} = n_{[11]} \Gamma^{[11]} + n_{[2]} \Gamma^{[2]}$$

in $x = [111]$:

$$n_{[11]} = \frac{1}{2!} \left[N_{(11)} \chi^{(111)} \chi^{(11)} + N_{(2)} \chi^{(111)} \chi^{(2)} \right]$$

$$= \frac{1}{2} \left[1 \cdot 1 \cdot 1 + 1 \cdot (-1) \cdot (-1) \right] = 1$$

$$n_{[2]} = \frac{1}{2!} \left[1 \cdot 1 \cdot 1 + 1 \cdot (-1) \cdot (1) \right] = 0$$

only have $\binom{i}{i}$ changes in [111]

$$\Gamma^{[111]} = \Gamma^{[11]}$$

Likewise for $[21]$ you can show

$$\Gamma^{[21]} = \Gamma^{[11]} + \Gamma^{[2]}$$

all are $\begin{pmatrix} i & j & k \\ j & i & k \end{pmatrix}$ like

How about S_4 ? $\to S_3$?

S_4	(1111)	(211)	(22)	(31)	(4)
common $S_3 \to$	(111)	(21)		(3)	
$[4]$	1	1	1	1	1
$\to$ $[31]$	3	1	-1	0	-1
$[22]$	2	0	2	-1	0
$[211]$	3	-1	-1	0	1
$[1111]$	1	-1	1	1	-1

you can show in the same way, for example

$$\Gamma^{[31]} = \Gamma^{[3]} \oplus \Gamma^{[21]}$$

in $[31]$

$$m_{[3]} = \frac{1}{3!}\left(1\cdot 3\cdot 1 + 3\cdot 1\cdot 1 + 2\cdot(0)(1)\right] = 1$$

$$m_{[111]} = \frac{1}{3!}\left(1\cdot 3\cdot 1 + 3\cdot 1\cdot(-1) + 2\cdot 0\cdot 1\right) = 0$$

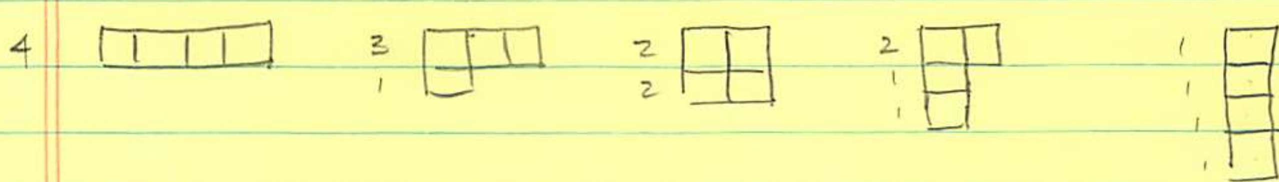
$$m_{[21]} = \frac{1}{3!}\left(1\cdot 3\cdot 2 + 3\cdot 1\cdot 0 + 2\cdot 0\cdot(-1)\right] = 1$$

Bingo

A visual picture of partitions is with Young Tableaux
(Alfred Young 1900)

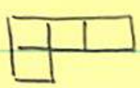
Example: S_4 rows $\rightarrow$ elements of the partition:

[4] [31] [22] [211] [1111]



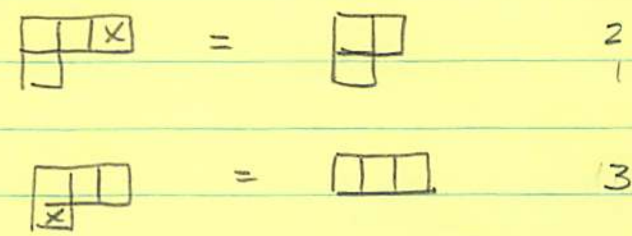
Lots of algebra goes away by following some rules.

For example the reduction I just did $S_4 \rightarrow S_3$ ..



[31] of S_4

loss of a box



3 boxes $\Rightarrow$ S_3 and specifically, the

[21] and [3]
IRR of S_3

Furthermore, since:

$\#$ partitions $= \#$ classes $= \#$ IRR

counting the diagrams tells you how many IRR

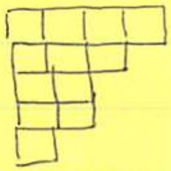
Need some topological rules

"Standard Arrangement" of Young Tableaux for S_n:

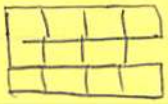
1) a Y.T. has n-boxes

2) A diagram with multiple rows:
- let q_i be the # boxes in the i^{th} row
- each row below it must have $q_j \leq q_i \quad j > i$

$\Rightarrow$ # boxes decreases with row #



standard



non-standard

3) To count the IRR for S_n fill the boxes with integers $1 \to n$ so that:

- they increase $L \to R$
- and $T \to B$
- count up the possible Tableaux.

So, for S_2 $\quad \boxed{1 \,|\, 2} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$ the only standard arrangements

the symmetric content

the antisymmetric content.

For S_3 $\quad \boxed{1 \,|\, 2 \,|\, 3} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}$

symm. $\qquad$ antisymm $\qquad$ mixed.

1d	1d	2d
[3]	[111]	[21]

Now, the connection with $SU(n)$ and finding their IRR. Still use Y.T., but with slightly different numbering and counting rules.

Because the same particle can be in more than one state $\xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$ $\xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$ ..

For SU(n) there is an additional freedom .. and different counting rules.

$$\xi^i \equiv \boxed{i} \qquad \eta^j \equiv \boxed{j}$$

} linking the basis states of IRRs to the diagrams of the IRR

product state $\xi^i \eta^j = \boxed{i} \otimes \boxed{j} = \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$

4. standard arrangement for SU(n)

put integer in each box so that they do not decrease $1 \to m$, L-R

\# possible states, not # particles

now ξ and η could be in the same state.

$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$ $\quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}$ $\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ 3 possible states, triplet.

$\xi^1\eta^2 + \eta^1\xi^2$

$\xi^1\eta^1 \qquad \xi^2\eta^2 \qquad = \underbrace{\xi^1\eta^2 + \xi^2\eta^1}$

which needs to be normalized to be useful for quantum mechanics

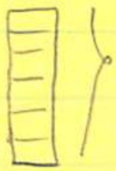
So,

$\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \qquad \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} \qquad \longrightarrow \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$ 1d rep SU(2)
1d rep S_2

symmetric triplet

antisymmetric singlet

5. Specific rule for $\underline{SU}(n)$ .. as opposed to $U(n)$

 $\underline{n}$ boxes $\rightarrow$ collapse to a singlet, "$\cdot$"

6. Tensor Product – combining states.
(Direct)

a) Draw all diagrams of 2 product IRR – put number in 2^{nd} one according to row.

b) Add each box of 2^{nd} one to the first to make a standard SU(n) tableaux

c) Draw path through all boxes crossing each row R $\rightarrow$ L, from top. Along the path, the number i must not occur more often than $i-1$ times

$$\boxed{} \otimes \boxed{1} = \boxed{\,|\,1} \oplus \begin{array}{|c|}\hline \\ \hline 1 \\ \hline \end{array}$$

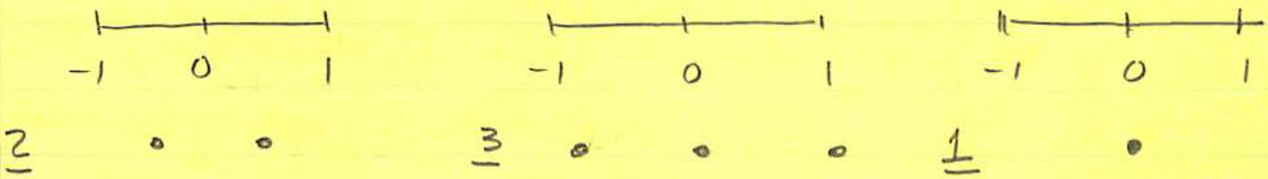
For SU(2) $= \boxed{\,|\,1} \oplus \cdot$

$\boxed{}$ – represents a doublet $\boxed{1}$ or $\boxed{2}$

$\boxed{\,|\,1}$ – represents a triplet

So, get $\underline{2} \otimes \underline{2} = \underline{3} \oplus \underline{1}$

This is represented graphically by the WEIGHT DIAGRAM



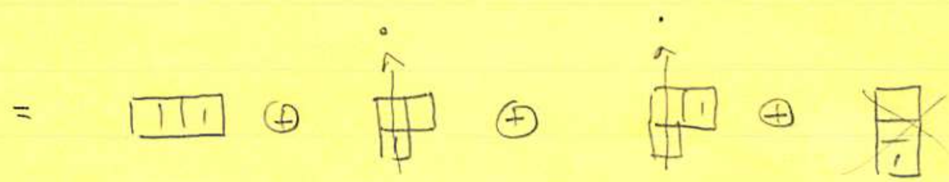
3 particles: $\xi^i \eta^j \gamma^k$

$$\square \otimes \square \otimes \square = (\square \otimes \square) \otimes \square$$

$$= \left(\square\square \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix}\right) \otimes \square$$

$$= \square\square \otimes \square \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix} \otimes \square$$

For $SU(2)$:



$$= \square\square\square \oplus \square \oplus \square$$

all symmetric

111
112 } 4, the $\underline{4}$ of $SU(2)$
122
222

So, $\underline{2} \otimes \underline{2} \otimes \underline{2} = \underline{4} \oplus \underline{2} \oplus \underline{2}$

7) The dimensionality formulae. for IRR of SU(n)

calculate $D = a/b$

(a)

fill

n	$n+1$	$n+2$	$n+3$	$n+4$
$n-1$	$n-1+1$			

a is product of all numbers.

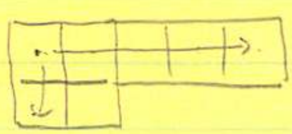
So, suppose SU(3)

3	4	5	6	7
2	3			

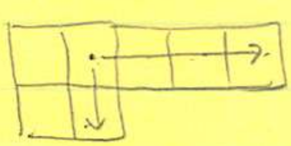
$a = 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 2 \cdot 3$

(b) count the "hooks"

place  at each box and count # boxes crossed by each arrow → put that number in that box.



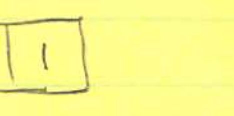
6				



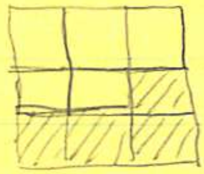
6	5	3	2	1
2	1			

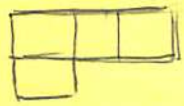
$b = \text{product} = 6 \cdot 5 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1$

$$D = \frac{\cancel{3} \cdot \cancel{4}^{2} \cdot \cancel{5} \cdot \cancel{6} \cdot 7 \cdot \cancel{2} \cdot 3}{\cancel{6} \cdot \cancel{5} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{2} \cdot 1} = 6 \cdot 7 = 42$$

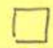
 $= \underline{42}$ of SU(3)

8) Conjugate states — the complementary tableaux necessary to get a singlet.

 in SU(3)

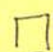
$\Rightarrow$  is the conjugate.

For SU(n), there are n-1 fundamental representations

SU(2): the $\underline{2}$:  states of $S = \pm 1/2$ or $I_z = \pm 1/2$

S(3): the $\underline{3}$ 

the $\underline{3}^*$ 

SU(4): the $\underline{4}$ 

$\underline{4}^*$ 

and  is a self-conjugate fundamental