Lecture 14

A word about the algebra...

SU(3) is a rank-2 group, so 2 diagonalizable generators. It will have $n^2-1 = 9-1 = 8$ total generators. In Gell-Mann's notation

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \underline{\underline{\lambda_8}} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\underline{\underline{\lambda_3}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

notice that λ_1, λ_4 & $\lambda_6 \sim \sigma_1$

$\lambda_2, \lambda_5 \sim \sigma_2$

$\lambda_3 \sim \sigma_3$

λ_8 like nothing in SU(2)

The Lie Algebra is

$$[\lambda_i, \lambda_j] = 2i \sum_k f_{ijk} \lambda^k$$

with the structure constants available on-line..

They are 1, or $\pm 1/2$ or $\sqrt{3}/2$

and the f_{ijk} are antisymmetric under permutation

There is also a set of anticommutation relations (nothing to do with the Lie Algebra)

$$\{\lambda_i, \lambda_j\} = 2 \sum_k d_{ijk} \lambda^k + \frac{4}{3}\delta_{ij}$$

constants on-line

Classification of Lie Algebras through diagrams is standard. Cartan and Dynkin are responsible for much of this and it's a big, big subject, since there are lots of complicated Lie Algebras

This is just to give you a feel

Suppose we have a group with

N total generators

l of them diagonalizable (like J_3) $\Rightarrow$ rank l

Call the diagonalizable ones H_i. Then

$$[H_i, H_j] = 0 \qquad i, j = 1, 2 \ldots l$$

This leaves $N-l$ which are left and can be arranged into 2 sets

$\frac{N-l}{2}$ called E_α

$\frac{N-l}{2}$ called $E_{-\alpha}$

Salam's Theorem #7: $\exists$ a basis of the Lie Algebra consisting of the N elements $H_i, E_{\pm\alpha}$ such that the following hold:

$$[H_i, H_j] = 0$$

for $\alpha = 1, 2 \ldots \frac{N-l}{2}$ $\qquad$ $r_i(\alpha)$ are ROOTS of the algebra

$$[H_i, E_\alpha] = r_i(\alpha) E_\alpha \quad \underline{\text{or}} \quad [\vec{H}, E_\alpha] = \vec{r}(\alpha) E_\alpha$$

$$[E_\alpha, E_{-\alpha}] = r_i(\alpha) H_i \quad \underline{\text{or}} \quad [E_\alpha, E_{-\alpha}] = \vec{r}(\alpha) \cdot \vec{H}$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_\gamma \qquad \alpha \neq -\beta$$

$N_{\alpha\beta}$ are non-zero real numbers iff $\vec{r}(\alpha) + \vec{r}(\beta)$ is another root.

Roots have the properties

$$\vec{r}(\alpha) = -\vec{r}(-\alpha)$$

$$\sum_\alpha r_i(\alpha)\, r_j(\alpha) = R\, \delta_{ij}$$

arbitrary constant

The roots allow for classification.

Solomon's Rule #8:

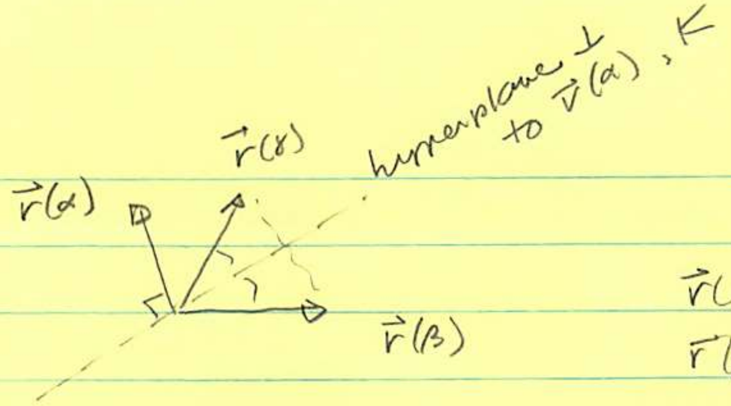
1. For a simple group of rank l $\exists$ l simple roots which are linearly independent.
2. If $\vec{r}(\alpha)$ and $\vec{r}(\beta)$ are 2 simple roots, the angle between them $\theta_{\alpha\beta}$ can take on values: $0, 30°, 60°, 90°, 120°, 135°, 150°$

A root is simple if it is positive and cannot be decomposed into the sum of two positive numbers.

Theorem: If $\vec{r}(\alpha)$ and $\vec{r}(\beta)$ are two roots

$$\vec{r}(\gamma) = \vec{r}(\beta) + 2\vec{r}(\alpha)\, \frac{[\vec{r}(\alpha) \cdot \vec{r}(\beta)]}{|\vec{r}(\alpha)|^2}$$

is also a root.



$\vec{r}(\gamma)$ is reflection of $\vec{r}(\beta)$ through K.

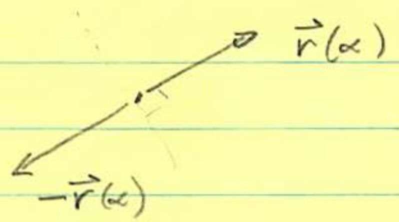
For groups of rank l, the root diagram is l-dimensional.

Example: rank 1

$l = 1$

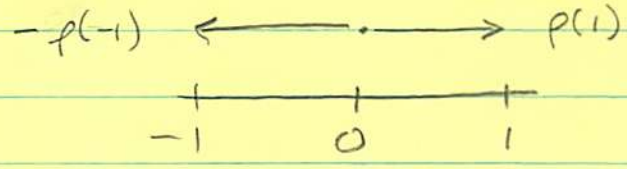
$N = n^2 - 1$ for $SU(n)$

$\alpha = 1, 2 \ldots \frac{n^2-1-l}{2} = 1$ for $SU(2)$



hyperplane = pt.

Generally, for all rank 1 groups, the normalized root diagram is



Example: rank 2

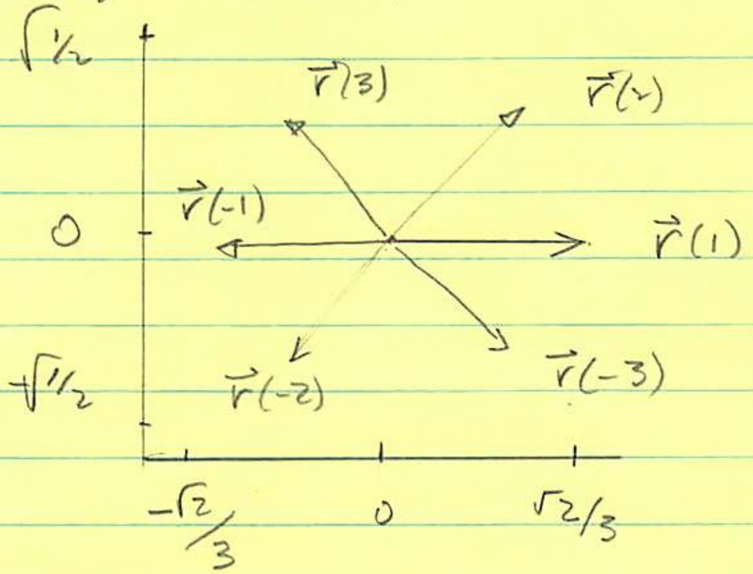
hyperplane = line

for SU(3), there are 2 roots

$$\alpha = 1, 2 \ldots \frac{9-1-2}{2} = 1, 2, 3$$

$$N = 3^2 - 1 = 8$$

Normalized root diagram:



It's nicely symmetrical. Other rank-2 groups have different weight diagrams.

A classification scheme called Dynkin Diagrams are built from the root diagrams. But, they are pretty trivial for SU(2) and SU(3), so I'm not going to go there

The physics is in the weights, gotten from the roots. The standard SU(3) generators combine to give the matrix representation needed for this

$$H_1 = \sqrt{1/6}\,\lambda_3 \qquad H_2 = \sqrt{1/6}\,\lambda_8$$

$$E_{\pm 1} = \frac{\lambda_1 \pm i\lambda_2}{2\sqrt{2}}$$

$$E_{\pm 2} = \frac{\lambda_4 \pm i\lambda_5}{2\sqrt{3}}$$

$$E_{\pm 3} = \frac{\lambda_6 \pm i\lambda_7}{2\sqrt{3}}$$

or.

$$H_1 = \sqrt{1/6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H_2 = \frac{1}{3\sqrt{2}}\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -2 \end{pmatrix}$$

$$E_1 = \sqrt{1/3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad E_{-1} = \sqrt{1/3}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so on...

yes... these look like raising and lowering operators

For the basis vectors $\xi^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\xi^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\xi^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

we have $\vec{H}\,\xi^i = \vec{m}\,\xi^i$

weights $\leftrightarrow$ roots

So, $\vec{m}(1) = \left(\sqrt{1/6},\ \frac{1}{3\sqrt{2}}\right)$ $\quad \vec{H}\xi^1 = \vec{m}(1)\xi^1$

$\vec{m}(2) = \left(-\sqrt{1/6},\ \frac{1}{3\sqrt{2}}\right)$ $\quad$ etc.

$\vec{m}(3) = \left(0,\ \sqrt{2}/3\right)$

for the $\underline{3}$. For the $\underline{3}^*$, you can get them by

$$\lambda_i' = -\lambda_i^* = -\lambda_i^T$$

since $H = H^T$, wts of the 2nd fundamental are negatives of the 1st

$\vec{m}'(1) = \left(0,\ \sqrt{2}/3\right)$

$\vec{m}'(2) = \left(\sqrt{1/6},\ -\frac{1}{3\sqrt{2}}\right)$

$\vec{m}'(3) = \left(-\sqrt{1/6},\ -\frac{1}{3\sqrt{2}}\right)$

Theorem – For any weight $\vec{m}$ and root $\vec{r}(\alpha)$, the quantity

$$h = \frac{2\,\vec{m}\cdot\vec{r}}{r^2}$$

is an integer and

$$\vec{m}' = \vec{m} - \vec{r}\,\frac{2\,\vec{m}\cdot\vec{r}}{r^2}$$

is another weight with the same multiplicity as $\vec{m}$.

So, $\vec{r}(1) = \left(\sqrt{2/3}, 0\right)$

$$h = 2\,\frac{\left[m_1(1)\,r_1(1) + m_2(1)\,r_2(1)\right]}{r^2(1)}$$

$$= \frac{2m_1(1)\sqrt{2/3}}{2/3} + 0$$

so $m_1(1) = h/\sqrt{6}$

Take $h = 1$, $m_1(1) = \sqrt{1/6}$ → smallest non-trivial value of $m_1(1)$

$\vec{r}(2) = \left(\sqrt{1/6}, \sqrt{1/2}\right)$

$h =$

$$2\left[\frac{m_1(1)\,r_1(2) + m_2(1)\,r_2(2)}{r^2}\right]$$

$$k = 2\left[\frac{-\sqrt{1/6}\,\sqrt{1/6} + m_2(1)\sqrt{1/2}}{1/6 + 1/2}\right]$$

or $$m_2(1) = \frac{\sqrt{2}}{3}k - \frac{1}{3\sqrt{2}}$$

$$k=1\,, \quad m_2(1) = \frac{1}{3\sqrt{2}}$$

$$k=0 \qquad m_2(1) = -1/3\sqrt{2}$$

so $\vec{m}(1) = \left(\sqrt{1/6}\,,\ \frac{1}{3\sqrt{2}}\right)$ is one value $= \vec{M}^{(1)}$

$\vec{m}(1) = \left(\sqrt{1/6}\,,\ -\frac{1}{3\sqrt{2}}\right)$ is another and belongs to the second fundamental representation

Definition and Theorem: The highest weight of a set of equivalent weights (weights related by reflection) is Dominant.

For a group of rank l, there are l fundamental weights $\vec{M}^{(i)}$ such that any other dominant wt is a linear combination.

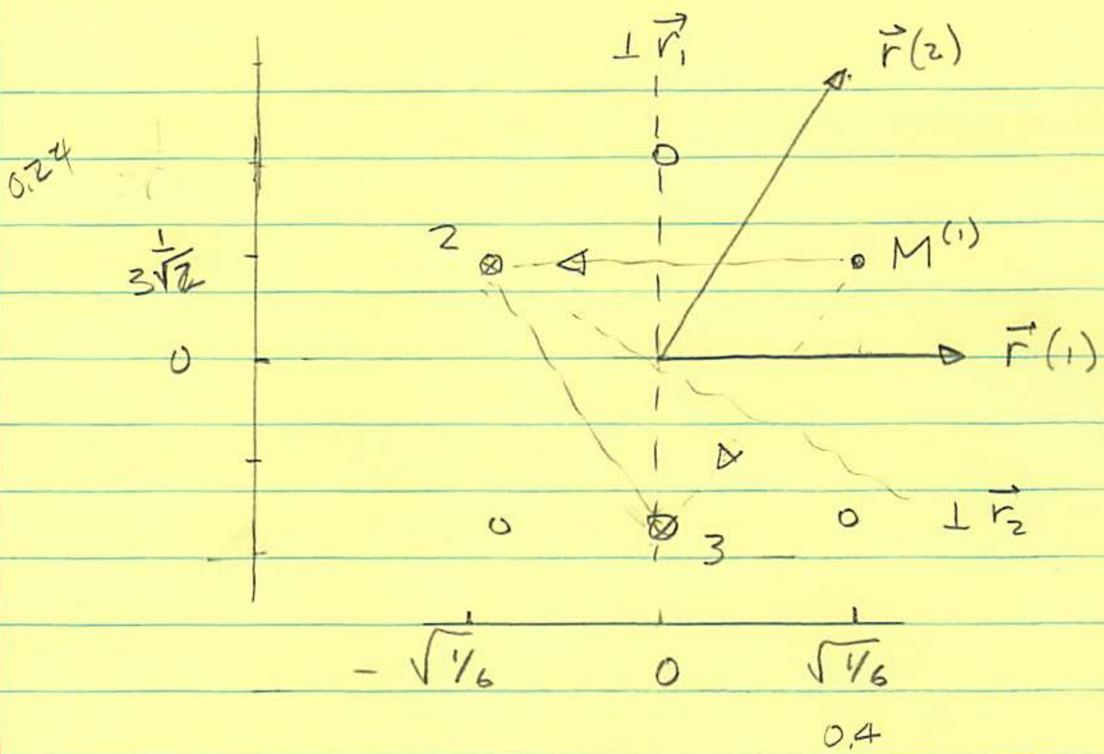
$$\vec{M} = \sum_{i=1}^{l} n_i \vec{M}^{(i)}$$

non-negative integers

There are l fundamental, IRR which have the l fundamental dominant weights as highest.

$\vec{m}(1)$ is dominant and fundamental $\equiv \vec{M}^{(1)}$

Starting from $M^{(1)}$, can use the reflection theorem twice to get $\otimes 2$ and $\otimes 3$



Doing the same thing to $\underline{3}^*$ gives the o weights above

Labeling $\vec{M} = n_1 \vec{M}^{(1)} + n_2 \vec{M}^{(2)}$

by ordered pairs (n_1, n_2)

$\bigtriangledown$ rep. is dominant & fundamental $(1\ 0) \rightarrow \underline{3}$

$\bigtriangleup$ rep is fundamental $(0\ 1) \rightarrow \underline{3}^*$

And, I began the quark model

$$\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix}$$

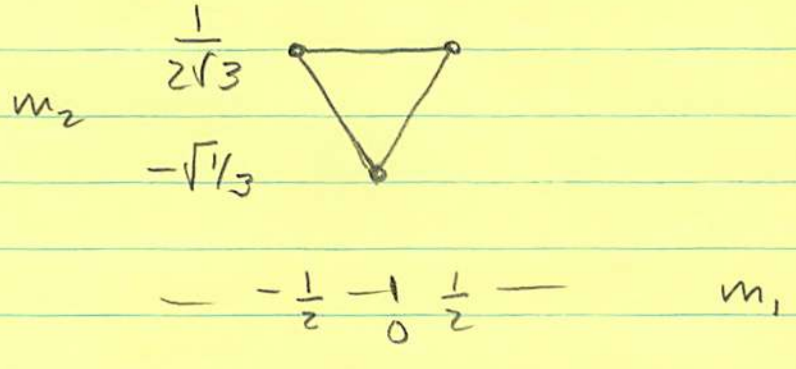
SU(3)
$\Rightarrow$ motivated as the simplest rank 2 Lie group — two diagonalizable generators $\rightarrow$ I_3 & Y

$$q^i = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

$$\underline{3} \otimes \underline{3} \otimes \underline{3} = \underline{10} \oplus \underline{8} \oplus \underline{3}^* + \underline{8}^*$$

$$\underline{3} \otimes \underline{3}^* = \underline{8} \oplus \underline{1}$$

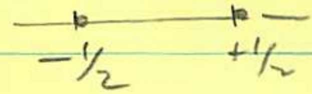
Normalizations are arbitrary — $\times \sqrt{3/2}$



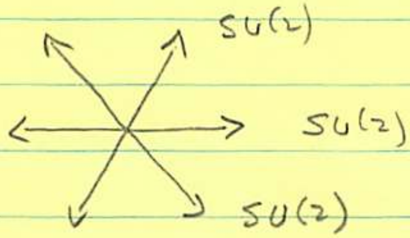
and relate them to the hadronically conserved quantum numbers

$$I_3 = m_1 \rightarrow \text{eigenvalue of } \tfrac{1}{2}\lambda_3 \equiv F_3$$

$$Y = \tfrac{2}{\sqrt{3}}\, m_2 \longrightarrow \text{eigenvalue of } \tfrac{1}{2}\lambda_8 \equiv F_8$$

Notice that the horizontal axis 

is the SU(2) subgroup of SU(3). But, the root diagram tells us that there are 2 others



Theorem - The highest weight for a particular IR will be SIMPLE (multiplicity = 1) while all other weights around the periphery will be simple as well.

Theorem - The highest weight multiplicity, ν, in any particular IR will be

$$\nu = \tfrac{1}{2}(n_1 + n_2) - \tfrac{1}{2}|n_1 - n_2| + 1$$

multiplicities increase by 1 along the I_3, U_3, V_3 axes

So, our fundamentals are now written_

$$\vec{M}^{(1)} = \left(\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}\right)$$

$$\vec{M}^{(2)} = \left(0, \sqrt{\tfrac{1}{3}}\right)$$

We can construct other SU(3) IR weight diagrams.

How about $(3\ 0) \Rightarrow n_1 = 3$
$n_2 = 0$

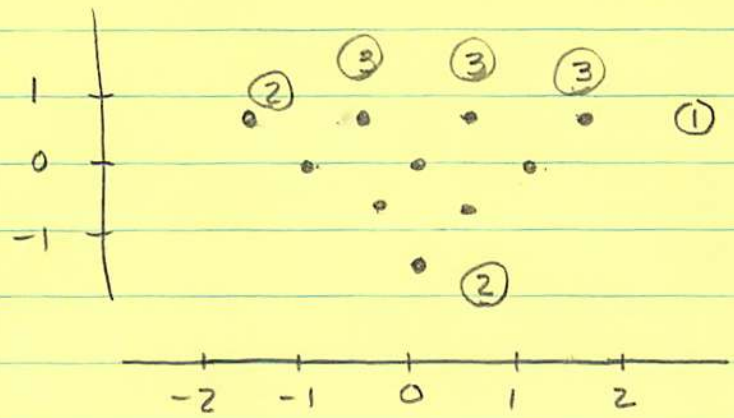
start:

$$\vec{M} = 3\,\vec{M}^{(1)} = \left(\tfrac{3}{2}, \tfrac{\sqrt{3}}{2}\right) \quad ①$$

(2) Using the symmetry of the root diagram of 120°

(3) Weights differ by 1 around the edge from the I_3, U_3, V_3 steps

(4) highest multiplicity $\nu = \frac{1}{2}(3) - \frac{1}{2}(3) + 1 = 1$
$\Rightarrow$ all are simple



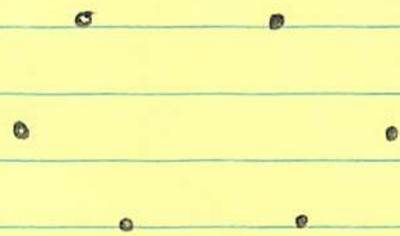
The dimensionality comes by counting the weights.
This is an SU(3) $\underline{10}$.

How about $(1\ 1)$

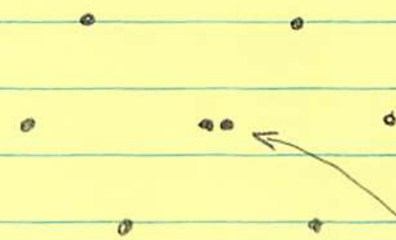
- Highest weight $\quad \vec{M} = 1 \cdot \vec{M}^{(1)} + 1\, \vec{M}^{(2)}$

$$\vec{M} = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right)$$

- $120°$ symmetry



- weights differ by 1



- highest multiplicity

$$\nu = \tfrac{1}{2}(2) - \left(\tfrac{1}{2}\right)(0) + 1 = 2 \rightarrow$$ which must go in the middle

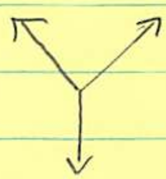
Count the weights. This is an SU(3) $\underline{8}$.

This can be gotten by

$$\ydiagram{1} \oplus \ydiagram{1,1} = \ydiagram{2,1} \oplus \cdot$$

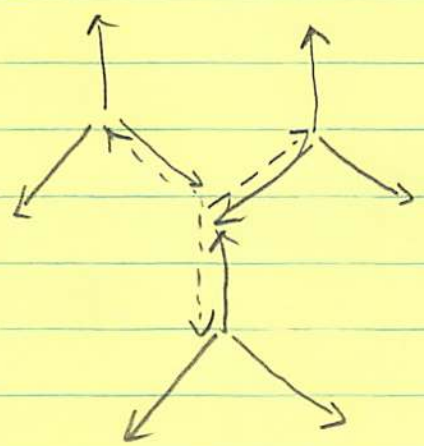
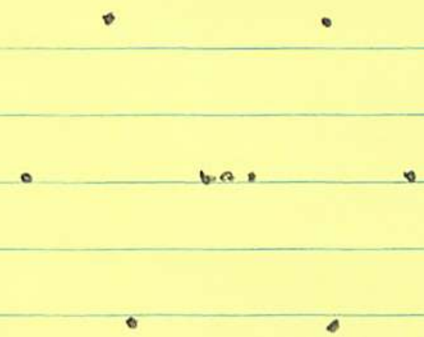
$$\underline{3} \oplus \underline{3}^* \qquad \underline{8} \oplus \underline{1}$$

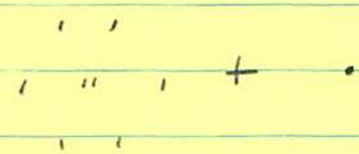
and rather a cute method

Take the $\square$, $\underline{3}$: 

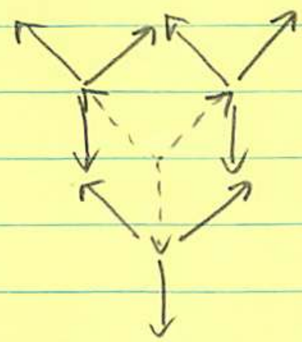
and put the $\ydiagram{1,1}$, $\underline{3}^*$ at each weight and 

count where the arrows end up

 $\Rightarrow$ 

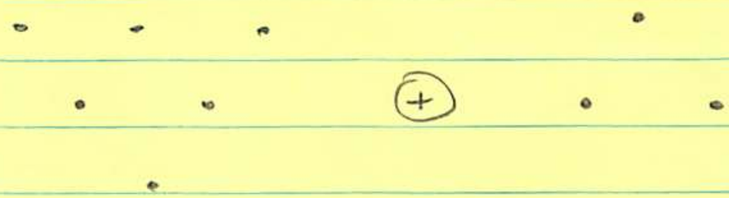
which can be decomposed into 

How about $\underline{3} \otimes \underline{3} = \underline{6} \oplus \underline{3}^*$



$\Rightarrow$

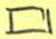
which can be decomposed

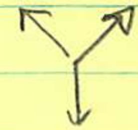


$\underline{6}$ $\qquad\qquad$ $\underline{3}^*$

So, this is how all of the SU(3) diagrams come about.

$I_3 = \ -\tfrac{1}{2} \quad 0 \quad \tfrac{1}{2} \qquad Y =$

(1 0)  $\underline{3}$



$\tfrac{1}{3}$

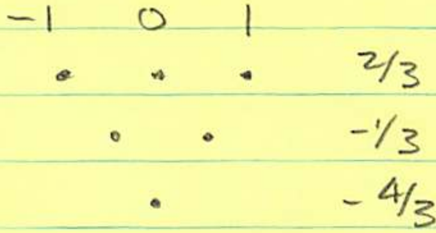
$-\tfrac{2}{3}$

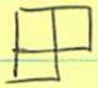
(0 1)  $\underline{3}^*$

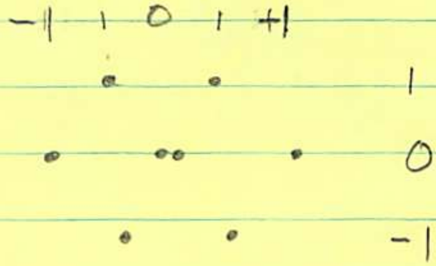
$\tfrac{2}{3}$

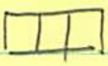
$-\tfrac{1}{3}$

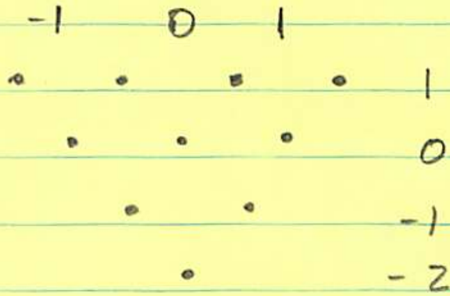
(2 0)  $\underline{6}$



(1 1)  $\underline{8}$

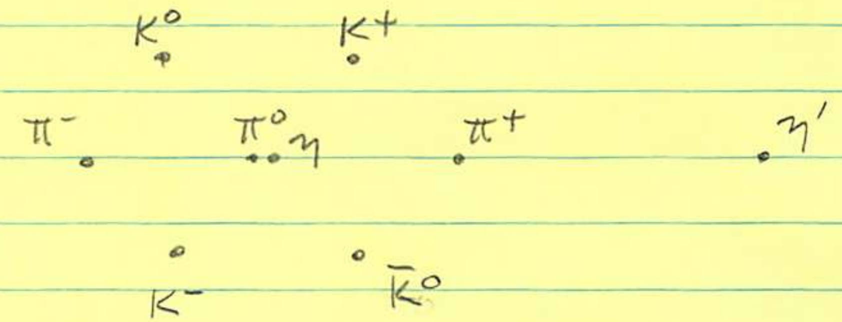


(3 0)  $\underline{10}$

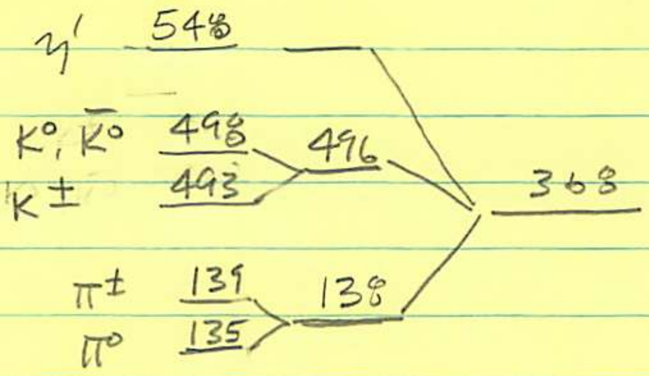


So, in this scheme, the fundamentals are combined together with the goal of finding product states which are the I_3 and Y homes for real particles.

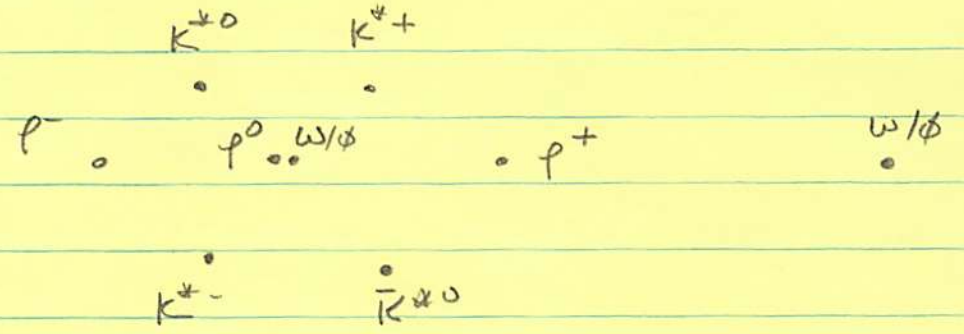
For example, for the spin 0^- mesons and the spin 1^- mesons, it works

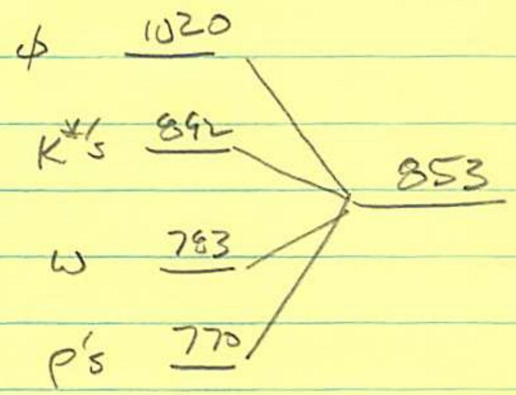


Their masses -- MeV/c^2



Likewise, there's another





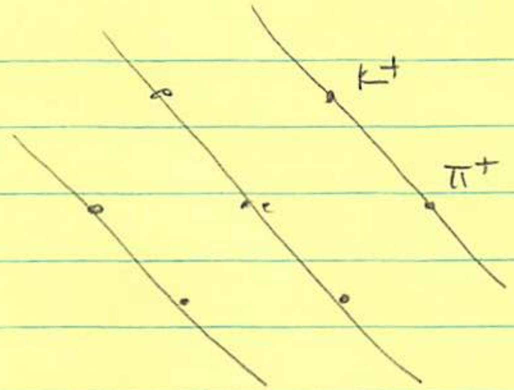
Remember, the octet in SU(3) had SU(2) content

$$\underline{8} \supset \underline{2} \oplus \underline{2} \oplus \underline{3} \oplus \underline{1}$$

and the masses sort of group this way

$\underline{2}$	$K^0 - K^+$
$\underline{2}$	$K^- - \bar{K}^0$
$\underline{3}$	π
$\underline{1}$	η

And for U-spin → lines of constant Q



etc.

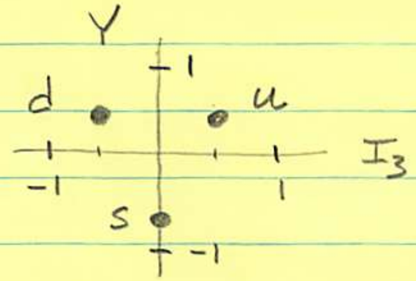
$\underline{2}$ $K^+ \pi^+$

$\underline{2}$ $\pi^- K^-$

$\underline{3}$ $K^0 - \pi^0/\eta - \bar{K}^0$

$\underline{1}$ η/π^0

So, if you buy this — then the composition & the GROUP THEORY force you into assigning.



and from $Q = I_3 + Y/2$ fractional charges.

	I_3	Y	S	B	Q
u	1/2	1/3	0	1/3	2/3
d	-1/2	1/3	0	1/3	-1/3
s	0	-2/3	-1	1/3	-1/3

Group theory (I_3, Y)

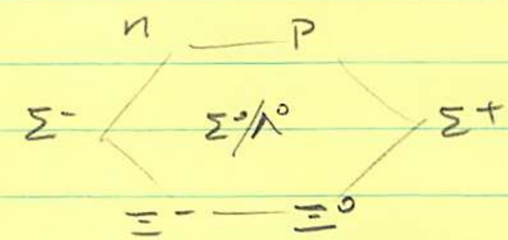
assign like Sakata (S)

consequences (B, Q)

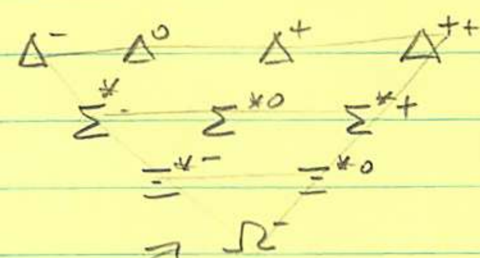
The Baryons work as well

$$\underline{3} \otimes \underline{3} \otimes \underline{3} = \underline{8} \oplus \underline{10} \oplus \underline{8} \oplus \underline{1}$$

Baryon Octet:



Baryon Decuplet



famously predicted ... and found at BNL

The quark-content assignments come from the following.

Start with $q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} = \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix}$

The general tensorial product:

$$\xi^i \eta_j = \left(\xi^i \eta_j - \tfrac{1}{n}\,\delta^i{}_j\, \xi^k \eta_k\right) + \tfrac{1}{n}\,\delta^i{}_j\, \xi^k \eta_k$$

So: $q \times \bar{q}$ is

$$q^i q_j = \underbrace{\left(\underset{9}{q^i q_i} - \tfrac{1}{3}\,\underset{1}{\delta^i{}_j\, \xi^k \eta_k}\right)}_{8} + \underbrace{\tfrac{1}{3}\,\underset{1}{\delta^i{}_j\, \xi^k \eta_k}}_{1}$$

Look at the $\underline{8}$:

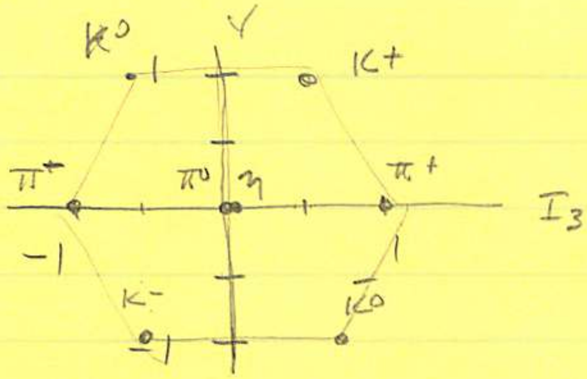
$$q^1 q_2 = q^1 q_2 + 0 \quad = u\bar{d} \longrightarrow \pi^+$$

$$q^1 q_1 = q^1 q_1 - \tfrac{1}{3}\left(q^1 q_1 + q^2 q_2 + q^3 q_3\right)$$
$$= u\bar{u} - \tfrac{1}{3} u\bar{u} - \tfrac{1}{3} d\bar{d} - \tfrac{1}{3} s\bar{s} = \tfrac{2}{3} u\bar{u} - \tfrac{1}{3}\left(d\bar{d} + s\bar{s}\right)$$

$$q^2 q_2 = \tfrac{2}{3} d\bar{d} - \tfrac{1}{3}\left(u\bar{u} + s\bar{s}\right)$$

$$\Rightarrow \quad q^1 q_2 - q^2 q_2 = u\bar{u} - d\bar{d}$$

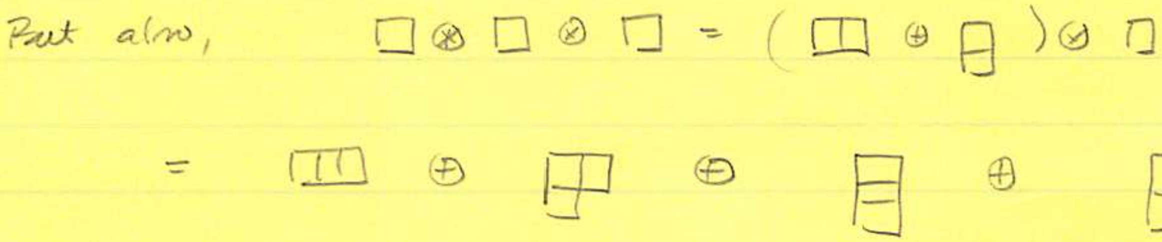
normalize (C.G.C.) $\sqrt{1/2}\,(u\bar{u} - d\bar{d}) \rightarrow \pi^0$



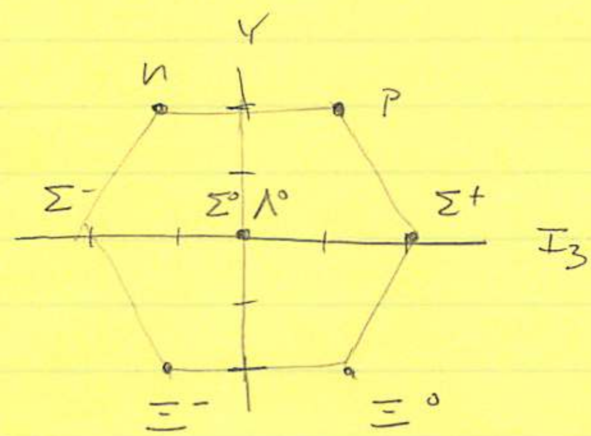
Meson Octet

$\underline{8}$

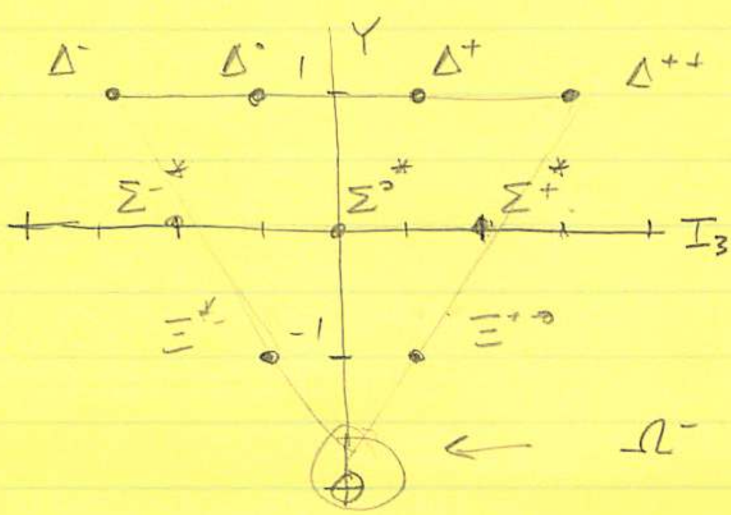
But also,



$\underline{10}$ $\underline{8}$ $\underline{1}$ $\underline{8}^*$



Baryon Octet - $\underline{8}$



Baryon Decuplet - $\underline{10}$

MESONS

$|\pi^+\rangle = |u\bar{d}\rangle \qquad \rightarrow |\rho^+\rangle_{770}$

$|\pi^0\rangle = \sqrt{1/2}(|d\bar{d}\rangle - |u\bar{u}\rangle) \qquad \rightarrow |\rho^0\rangle$

$|\pi^-\rangle = |\bar{u}d\rangle \qquad \rightarrow |\rho^-\rangle$

$|\eta\rangle_{549} = \sqrt{1/2}(|d\bar{d}\rangle + |u\bar{u}\rangle) \qquad \rightarrow |\omega\rangle_{783}$

${}^1S_0 \; J^P = 0^- \qquad {}^3S_1 \; J^P = 1^-$

$|K^-\rangle = |u\bar{s}\rangle \qquad \rightarrow |K^{*+}\rangle_{892}$

$|K^0\rangle = |d\bar{s}\rangle \qquad |K^{*0}\rangle$

$|K^-\rangle = |\bar{u}s\rangle \qquad |K^{*-}\rangle$

$|\bar{K}^0\rangle = |\bar{d}s\rangle \qquad |\bar{K}^{*0}\rangle$

$|\eta'\rangle_{958} = |s\bar{s}\rangle \qquad |\varphi\rangle_{1020}$

BARYONS

$J^P = 1/2^+ \qquad J^P = 3/2^+$

$|p\rangle = |uud\rangle \qquad \Delta^{++} = |uuu\rangle$

$|n\rangle = |udd\rangle \qquad \Delta^+ = |uud\rangle$

$\Delta^0 = |udd\rangle$

$\Delta^- = |ddd\rangle$

$|\Lambda^0\rangle = \sqrt{1/2}(|uds\rangle - |dus\rangle)$

$|\Sigma^+\rangle_{1189} = |uus\rangle \qquad |\Sigma^{+*}\rangle_{1383}$

$|\Sigma^0\rangle = \sqrt{1/2}(|uds\rangle + |dus\rangle) \qquad |\Sigma^{0*}\rangle$

$|\Sigma^-\rangle = |dds\rangle \qquad |\Sigma^{*}\rangle$

$|\Xi^0\rangle_{1315} = |uss\rangle \qquad |\Xi^{0*}\rangle_{1532}$

$|\Xi^-\rangle = |dss\rangle \qquad |\Xi^-\rangle$

$|sss\rangle \qquad |\Omega^-\rangle$

Group Theory can suggest a lot:

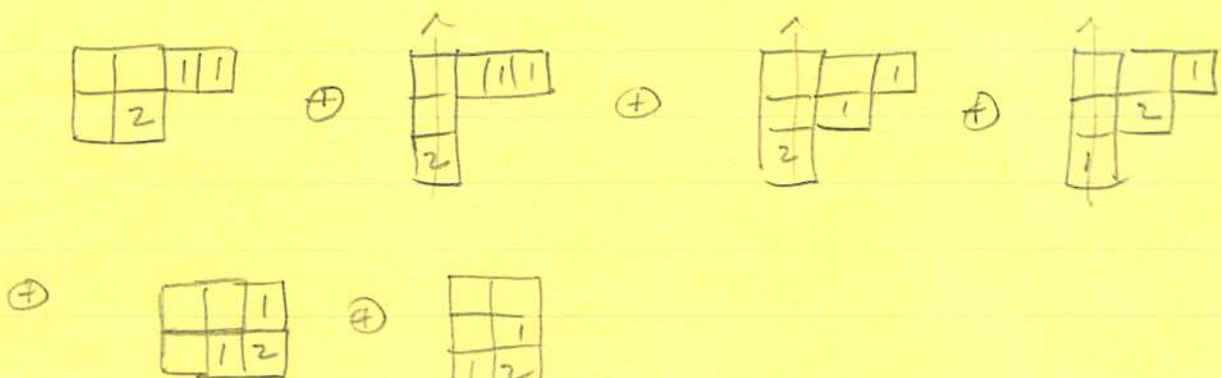
Can $\quad \Sigma^* \to \Lambda + \pi \quad$?

If this can happen, then the amplitude:

$$\langle \Lambda \pi | \Sigma^* \rangle \neq 0$$

So, the Kronecker product of $\Lambda\pi$ <u>has to contain the $\underline{10}$</u>

$\Lambda \otimes \pi \Rightarrow \underline{8} \otimes \underline{8}$ or 



$$\underline{27} \oplus \underline{10} + \underline{8} + \underline{8} + \underline{10}^* + \underline{1}$$

Okay, if SU(3) conserved