Lecture 15 where we've been.

Last time I reminded you of the originally empirical relationship

$$Q = \frac{Y}{2} + I_3 \qquad Y = B + S$$

which suggests a rank-2 group, if symmetry is behind the clustering of hadrons into masses and isospins

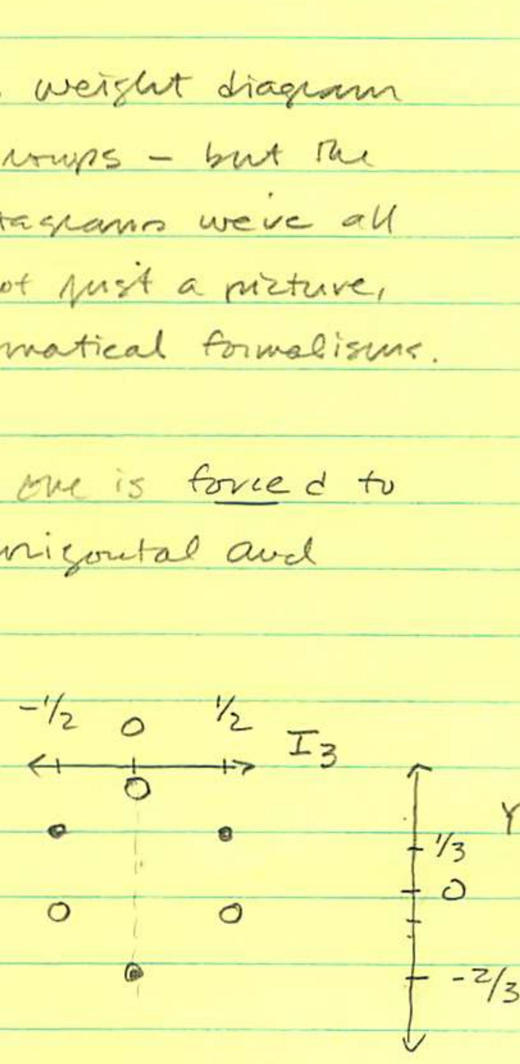
The simplest rank 2 group is SU(3) and that's what Gell-Mann focused on.

I showed you the root diagram, weight diagram approach to classifying Lie Groups – but the end results are the pretty diagrams we've all known and loved. They are not just a picture, they are honest to god mathematical formalisms.

So, specific to the quark model one is forced to the following, relating the horizontal and vertical weights:

The fundamental quark multiplet •

and the antiquark ∘



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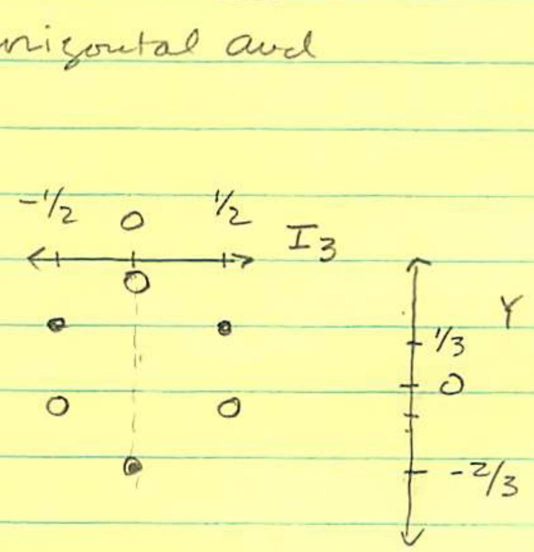


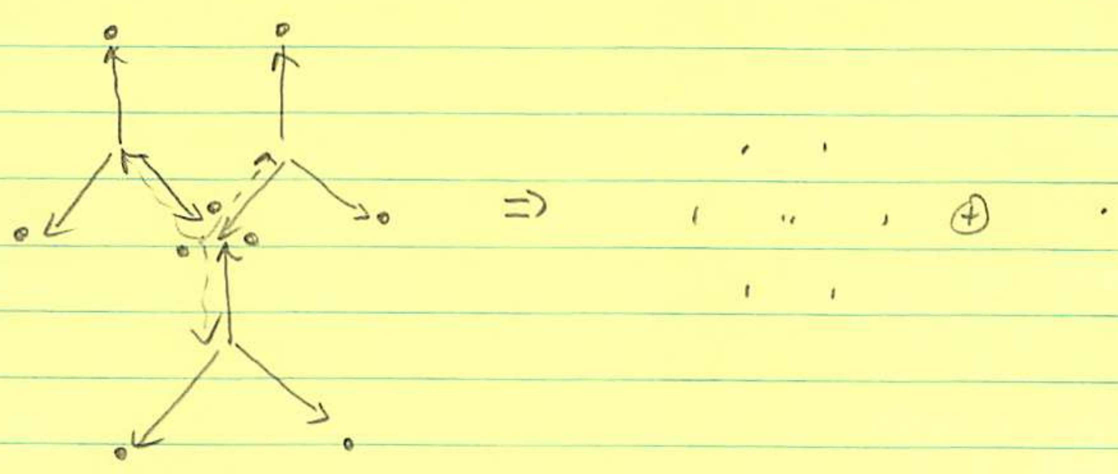
Then the product states, you get by, first using Young Tableaux to work out the IRR multiplicities and then the weight diagrams.

a) either by using the mechanics of the weights and roots

b) or, the cute cartoon of literally placing diagrams on top of one another

For example, the mesons. $q\bar{q} \rightarrow$

$\underline{3} \otimes \underline{3}^* \qquad \underline{8} \oplus \underline{1}$



So, I built up all of the diagrams for the $q\bar{q}$ and qqq product states showing that the

$q\bar{q}$ IRR $\underline{8} \oplus \underline{1}$

qqq IRR $\underline{8} \oplus \underline{10} \oplus \underline{8} \oplus \underline{1}$

By assigning the actually observed mesons and baryons to these patterns of weights – quantum numbers I_3 and Y – you have to conclude that

		I_3	Y	S	B	Q
u	q^1	$1/2$	$1/3$	0	$1/3$	$2/3$
d	q^2	$-1/2$	$1/3$	0	$1/3$	$-1/3$
s	q^3	0	$-2/3$	-1	$1/3$	$-1/3$

Then I showed how allowed or forbidden transitions can be calculated just with group theory.

The # of colors was assumed to be 3, and then confirmed in e^+e^- annihilation - (later).

Calling each color fundamental r (red)

y (yellow)

b (blue)

$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

and the same group theory is used for the generators and product representations.

We had:

$\underline{3} \otimes \underline{3}^* = \underline{8} \oplus \underline{1}$

$\underline{3} \otimes \underline{3} = \underline{3} \oplus \underline{6}$

$\underline{3} \otimes \underline{3} \otimes \underline{3} = \underline{1} \oplus \underline{8} \oplus \underline{8} \oplus \underline{10}$

$\underline{3} \otimes \underline{3} \otimes \underline{3}^* = \underline{3} \oplus \underline{3} \oplus \underline{6}^* \oplus \underline{15}$

etc. ONLY $\underline{3} \otimes \underline{3}$ and $\underline{3} \otimes \underline{3} \otimes \underline{3}$ combinations contain a $\underline{1}$ (singlet), so that's always been evidence that only $q\bar{q}$ and qqq states are possible.

Lecture 15

One more thing about the Quark Model.

Spin. This would be the combination of

$$SU(3) \otimes SU(2)$$

"flavor" spin

and the product group works. But the standard way is to actually use

$SU(6)$ and keep only the $SU(3) \otimes SU(2)$ states.

The $\underline{6}$ of $SU(6)$ is then

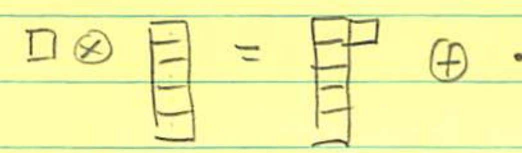
$$q = \begin{pmatrix} u\uparrow \\ d\uparrow \\ s\uparrow \\ u\downarrow \\ d\downarrow \\ s\downarrow \end{pmatrix}$$

and the conjugate $\bar{q}$

$q = \square$

$\bar{q} =$  so

Mesons:

$q\bar{q}$: 

$$\underline{6} \otimes \underline{6}^* = \underline{35} \oplus \underline{1}$$

Baryons:

$$\square \otimes \square \otimes \square = \yng(3) \oplus \yng(1,1,1) \oplus \yng(2,1) \oplus \yng(2,1)$$

$$\underline{6} \otimes \underline{6} \otimes \underline{6} = \underline{56} \oplus \underline{20} \oplus \underline{70} \oplus 70$$

Designate the spin and flavor content like this -

$$(\underline{F}, \underline{S})$$

SU(3) SU(2)

So, the $\underline{6} \rightarrow (\underline{3}, \underline{2})$

Mesons now:

$$\underline{6} \otimes \underline{6}^* = (\underline{3}, \underline{2}) \otimes (3^*, 2)$$

$$= (3 \otimes 3 \ , \ 2 \otimes 2)$$

$$= (\underline{8} \oplus \underline{1} \ , \ \underline{3} \oplus \underline{1})$$

$$= (\underline{8}, \underline{3}) \oplus (\underline{8}, \underline{1}) \oplus (\underline{1}, \underline{3}) \oplus (\underline{1}, \underline{1})$$

Since this has to match the $\underline{35} \oplus \underline{1}$ and since obviously, $1 = (1, 1)$

$q\uparrow \bar{q}\uparrow$

vector mesons octet $\quad q\uparrow\bar{q}\downarrow$

we get that $\quad$ scalar mesons octet

$$\underset{SU(6)}{\underline{35}} \rightarrow (\underline{8}, \underline{3}) \oplus (\underline{8}, \underline{1}) \oplus (\underline{1}, \underline{3})$$

To get a spin $\quad$ singlet: $\quad 2S+1 \Rightarrow S=0$

triplet: $\quad 2S+1 \Rightarrow S=1$

So, the

$\underline{35}$ of $SU(6) \rightarrow$ $SU(3)$ octet, spin ϕ & spin 1

$SU(3)$ singlet, spin 1

$\underline{1}$ of $SU(6) \rightarrow$ $SU(3)$ singlet, spin ϕ

Likewise, for baryons: $\quad \uparrow\uparrow\uparrow \quad (\uparrow\uparrow\downarrow + \quad)$

$$(3,2)\otimes(3,2)\otimes(3,2) = (10,4)\oplus(10,2)\oplus(8,4)\oplus(8,2)$$
$$\oplus(1,4)\oplus(1,2)$$

Need to think in terms of S_3 ..

SU(3)			SU(2)		
$\underline{10}$		S	$\underline{4}$		S
8		M	$\underline{2}$		M
				no A	
1	.	A			

The $\underline{56}$ is totally symmetric, so it must come from

$S \otimes S$

$A \otimes A$

$M \otimes M$

We have available $(\underline{10}, \underline{4})$ 40 states

$(8, 2)$ $\underline{16}$ states.

56

The $\underline{20}$ is A., so it must come from

$A \otimes S \longrightarrow (\underline{1}, \underline{4}) = 4$ states

$M \otimes M \quad (\underline{8}, \underline{2}) = \underline{16}$

20

The rest go into the mixed $\underline{70}$

$M \otimes M$	$(8,2)$	16
$S \otimes M$	$(10,2)$	20
$A \otimes M$	$(1,2)$	2
$M \otimes S$	$(8,4)$	$\underline{32}$
		70

So, it works:

$\underline{56} \supset (\underline{10}, \underline{4}), \ (\underline{8}, \underline{2})$

$\underline{70} \supset (\underline{10}, \underline{2}), \ (\underline{1}, \underline{2}), \ (\underline{8}, \underline{4}), \ (\underline{8}, \underline{2})$

$20 \supset (\underline{1}, \underline{4}), \ (\underline{8}, \underline{2})$

$(\underline{10}, \underline{4})$ is simple -- the $3/2^+$ decuplet fits
$(\underline{8}, \underline{2})$ of the $\underline{56}$ could contain the $1/2^+$ baryon octet.

and so on ..

Wavefunctions are kind of complicated --

Take $\quad \alpha = |\uparrow\rangle \qquad \beta = |\downarrow\rangle$

The labels of the 6 then are

$$\begin{array}{cc} \begin{pmatrix} u\alpha \\ d\alpha \\ s\alpha \\ u\beta \\ d\beta \\ s\beta \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Let's construct the $\underline{10}$ baryons $3/2^+$

Remember

$(\underline{10}, \underline{4})$ and $\boxed{\;|\;|\;}$ so totally symmetric

The highest weight state is the Δ^{++}, which is $3/2, 3/2$:

$$uuu\alpha\alpha\alpha \Rightarrow u(1)\,u(2)\,u(3)\,\alpha(1)\,\alpha(2)\,\alpha(3)$$

$$= \Delta^{++}\chi^{3/2}_{3/2} \rightarrow \boxed{1\,|\,1\,|\,1}$$

another.. $\boxed{1|1|4} \Rightarrow (114 + 141 + 411)$

$$= (u\alpha u\alpha u\beta + u\alpha u\beta u\alpha + u\beta u\alpha u\alpha)$$
$$= uuu(\alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha)$$

$$= \Delta^{++}\chi^{3/2}_{1/2}$$

and so on.

The "regular" baryon octet is harder.

If it really belongs in the $\underline{56}$

$$M'_{SU(3)} \otimes M_{SU(2)} = S_{SU(6)}$$

can get this by

$$[(2\text{ quarks, A singlet}) \times (\text{A, spin singlet})] \times 3^{rd}$$

to get mixed symmetry

For example.. for a spin $\uparrow$ proton:

a spin singlet: $\frac{\alpha\beta - \beta\alpha}{\sqrt{2}}$

a flavor singlet: $\frac{ud - du}{\sqrt{2}}$

which gives $\frac{1}{2}(ud\alpha\beta - ud\beta\alpha - du\alpha\beta + du\beta\alpha)$

then combine with $\underline{u\alpha}$ to get $S = +1/2$

$$\underbrace{\underbrace{(ud\alpha\beta - ud\beta\alpha - du\alpha\beta + du\beta\alpha)}_{\text{symmetric}}u\alpha}_{\text{mixed}} = (151 - 421 - 241 + 511)$$

But need eventually the $\underline{56}$ $\boxed{\;|\;|\;}$ => S

So, permute things to totally symmetrize .. and then normalize

⋮

$$p_{1/2} = \sqrt{1/6}\; uud\,[2\alpha\alpha\beta - \alpha\beta\alpha - \beta\alpha\alpha]$$

$$p_{1/2} = \sqrt{1/6}\; p\, \chi^{1/2}_{1/2}$$

likewise

$$n_{1/2} = \sqrt{1/6}\; n\, \chi^{1/2}_{1/2}$$

Define a magnetic dipole moment operator:

$$\vec{\mu} = \sum_{i=1}^{3} \frac{1}{2m} Q_i \, \vec{\sigma}(i) \qquad i = \text{quarks } 1, 2, 3$$

So,

$$\sum_{i=1}^{3} \langle p | Q_i | p \rangle = \sum \langle uud\chi | Q_i | uud\chi \rangle$$

and

$$Q_1 | uud\chi \rangle = 2/3 \, | uud\chi \rangle$$
$$Q_2 | \quad \rangle = 2/3 \, | \quad \rangle$$
$$Q_3 | \quad \rangle = -1/3 \, | \quad \rangle$$

So, $\langle p | Q | p \rangle = 2/3 + 2/3 - 1/3 = 1$ as expected.

Choose the 3-axis for quantization

$$\langle p | \overset{2/3}{Q_1} \sigma_3(1) \, | uud(2\alpha\alpha\beta - \alpha\beta\alpha - \beta\alpha\alpha) \rangle +$$

$$\langle p | \overset{2/3}{Q_2} \sigma_3(2) \, | uud(2\alpha\alpha\beta - \qquad) \rangle +$$

$$\langle p | \overset{-1/3}{Q_3} \sigma_3(3)) \, | uud(2\alpha\alpha\beta - \qquad) \rangle$$

$$= 2/3 \cdot \langle p | uud(2\alpha\alpha\beta - \alpha\beta\alpha + \beta\alpha\alpha) \rangle +$$

$$2/3 \, \langle p | uud(2\alpha\alpha\beta - \alpha\beta\alpha + \beta\alpha\alpha) \rangle +$$

$$-1/3 \, \langle p | uud(-2\alpha\alpha\beta - \alpha\beta\alpha - \beta\alpha\alpha \rangle$$

1st term

$\frac{2}{3} \langle uud(2\alpha\alpha\beta - \alpha\beta\alpha - \beta\alpha\alpha) | (2\alpha\alpha\beta - \alpha\beta\alpha + \beta\alpha\alpha) uud\rangle$

$\frac{2}{3}(4) + \frac{2}{3} - \frac{2}{3} = 6$

Remember the normalization $\sqrt{1/6}$ so,

$\langle p | \mu_3 | p \rangle \propto 1$

same thing for neutron: $\langle n | \mu_3 | n \rangle \propto -\frac{2}{3}$

and $\frac{\mu_p}{\mu_n} = -\frac{3}{2}$

The measurement is ~ -1.46.

$\Rightarrow$ the $\frac{1}{2}^+$ baryons belong in the $\underline{56}$.