Lecture 16

Relativity -

not the place to teach relativity -

BUT - .

some of the earliest ideas can from symmetry considerations - not just Einstein's thought experiments - but Poincaré's ideas about multiple dimensions.

He came close to S.R.

Basic ideas:

In 3-d space, the primary invariant is the length

$$l^2 = x^2 + y^2 + z^2 \quad \Rightarrow \text{ invariance wrt } O(3)$$

Euclidean.

In Einstein-land, the primary invariant is the interval

$$s^2 = c^2t^2 - x^2 - y^2 - z^2 \quad \Rightarrow \text{ a new kind of invariance}$$

non-Euclidean

The Lorentz Transformations predate Einstein as Lorentz - "king of the electrodynamics" - was worried about the lack of Newtonian-like invariance in Maxwell's Equations. He asked what transformations would be required in order that M.E. would be invariant $\Rightarrow$ The Lorentz Transformations

~1893 or so

$$x' = \gamma(x - vt)$$

$$t' = \gamma\left(t - \beta\frac{x}{c}\right)$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = v/c$$

Just like $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$,

one can write

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

where $\tanh\xi = \beta$

$$\Rightarrow \qquad \cosh\xi = \gamma$$
$$\sinh\xi = \beta\gamma$$

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

The "angle" has a name:

$$\xi = \tanh^{-1}\beta = \tfrac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) \qquad \text{rapidity}$$

4. vector notation —

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}\, dx^\mu dx^\nu$$

"my" metric : $g_{00} = 1$

$g_{ii} = -1 \qquad i = 1, 2, 3$

$g_{\mu\nu} = 0 \qquad \mu \neq \nu$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

And spacetime coordinates are contravariant vectors

$$x^\mu = (x^0, x^1, x^2, x^3)$$
$$= (ct, x, y, z)$$

$$= [x^0, \vec{x}\,] \qquad c = 1 \text{ usually.}$$

$g_{\mu\nu}$ lowers (raises) indices.

$$x_\mu = g_{\mu\nu} x^\nu$$

Suppose we have two neighboring points

$B \quad \circ \; x^\mu + dx^\mu$

$A \quad \circ \; x^\mu$

and a coordinate system defined in terms of the "old" coordinates

$$x'^\mu = f^\mu(x^\nu)$$

Then, in the standard way

$$dx^{\mu'} = \frac{\partial f^\mu(x^\nu)}{\partial x^\nu} dx^\nu = \partial_\nu f^\mu(x^\mu) dx^\nu$$

$$= \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

$$dx^{\mu'} \equiv \Lambda^\mu{}_\nu \, dx^\nu$$

any quantity that transforms like this is a "4-vector"

$$A^\mu \rightarrow A^{\mu'} = \Lambda^\mu{}_\nu A^\nu$$

Suppose we have $\quad A_\nu \equiv \dfrac{\partial \phi(x^\mu)}{\partial x^\nu}$

$$\frac{\partial\phi(x^\mu)}{\partial x^{\nu\prime}} = \frac{\partial\phi(x^\mu)}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial x^{\nu\prime}} \qquad = \partial_\sigma \phi(x^\mu)\frac{\partial x^\sigma}{\partial x^{\nu\prime}}$$

$$= \frac{\partial x^\sigma}{\partial x^{\nu\prime}} A_\sigma = A'_\nu$$

$$\uparrow$$

not $\Lambda^\sigma{}_\nu$

transformation property of a gradient – a contravariant vector.

$$A_\mu B^\mu \longrightarrow A'_\mu B'^\mu \overset{\cdots}{=} A_\nu B^\alpha \frac{\partial x^\nu}{\partial x^\alpha}$$

but $\frac{\partial x^\nu}{\partial x^\alpha} = \delta^\nu{}_\alpha$

$$= A_\nu B^\nu$$

same $\Rightarrow$ scalar.

$\boxed{\Lambda\,?}$

From the invariance of the interval,

$$ds^2 = ds'^2$$

$$g_{\mu\nu}\, dx'^\mu dx'^\nu = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$g_{\mu\nu} \Lambda^\mu{}_\rho \underline{dx^\rho} \Lambda^\nu{}_\xi \underline{dx^\xi} = g_{\alpha\beta} \underline{dx^\alpha} \underline{dx^\beta}$$

so,

$$\Lambda^\mu{}_\rho\, g_{\mu\nu} \Lambda^\nu{}_\xi = g_{\rho\xi}$$

or

$$\Lambda^T{}_\rho{}^\mu\, g_{\mu\nu} \Lambda^\nu{}_\xi = g_{\rho\xi}$$

This is a constraint equation on the Λ

10 conditions on its 16 components

$\Rightarrow$ # independent parameters = 6

3 relative velocity.

3 angles to relate the orientation of x and x'

For zero-relative orientation: Pure Lorentz Transformation

$$\Lambda_{(1)}{}^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta_1\gamma & 0 & 0 \\ -\beta_1\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The inverse transformation

$$dx^{\alpha} = \Lambda^{-1\,\alpha}{}_{\mu}\, dx'^{\mu} \quad \Rightarrow \quad \Lambda^{-1\,\alpha}{}_{\mu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}$$

The homogeneous Lorentz Transformations leaving $ds^2 = g_{\mu\nu}\, dx^{\mu} dx^{\nu}$ invariant characterized by 4×4 Λ's satisfying

$\Lambda^T g \Lambda = g$

This set $\mathbb{L}$ has properties:

a. $\Lambda_1 \Lambda_2 = \Lambda_3$ ✓

b. $(\Lambda_1 \Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2\Lambda_3)$ ✓

c when $\Lambda = \mathbb{1}$ – $v=0$ and identity ✓

d. $\exists\, \Lambda^{-1} \ni \Lambda^{-1} g \Lambda = g$ inverse ✓

yup! a 6-parameter group – Lorentz Group. L.

As a matrix equation,

$$\Lambda^T g \Lambda = g$$

then

$$(\det \Lambda^T)\underbrace{(\det g)}_{=-1}(\det \Lambda) = \underbrace{\det g}_{=-1}$$

$$(\det \Lambda^T)(\det \Lambda) = 1$$

$$(\det \Lambda)(\det \Lambda) = 1$$

so

$$\det \Lambda = \pm 1$$

Look at $\Lambda^0{}_0$!

$$\Lambda^\mu{}_0\, g_{\mu\nu}\, \Lambda^\nu{}_0 = g_{00} = 1$$

$$\Lambda^0{}_0 \Lambda^0{}_0 - \Lambda^i{}_0 \Lambda^i{}_0 = 1$$

so, $(\Lambda^0{}_0)^2 \geq 1 \Rightarrow \Lambda^0{}_0 \geq +1$ or $\Lambda^0{}_0 \leq -1$

two disjoint regions

So, there are really 4 Lorentz Transformations -

$\det \Lambda$	$\Lambda^0{}_0$	name	Abelian subgroup	
1	≥ 1	$L_+^\uparrow$	$\mathbb{1}$	
-1	≥ 1	$L_-^\uparrow$	$\mathbb{I}_s$	space inv.
-1	≤ -1	$L_-^\downarrow$	$\mathbb{I}_t$	time inv
1	≤ -1	$L_+^\downarrow$	$\mathbb{I}_{st}$	spacetime inv.

$$I_s x^\mu = g_{\mu\nu} x^\nu = x_\nu = [x_0, -\vec{x}]$$

$$I_t x^\mu = -g_{\mu\nu} x^\nu = -x_\nu = [-x_0, \vec{x}]$$

$$I_{st} x^\mu = I_s I_t x^\mu = -x^\mu = [-x_0, -\vec{x}]$$

This is not a connected group -

also: only $L_+^\uparrow$ contains the identity

* Proper Orthochronous, homogeneous Lorentz Group

Note, for any $\Lambda \in \mathbb{L}$

$$\Lambda L_+^\uparrow \Lambda^{-1} = L_+^\uparrow$$

so $L_+^\uparrow$ is an invariant subgroup of $\mathbb{L}$

Remember the $\mathbb{H} = \vec{x}\cdot\vec{\sigma}$

when $\mathbb{H} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$?

which encrypted a coordinate transformation inside of an SU(2) transformation by

$$\mathbb{H} \to \mathbb{H}' = \underset{\substack{\uparrow \\ SU(2)}}{\mathbb{A}}\,\mathbb{H}\,\mathbb{A}^{-1}$$

Now, do it in 4-d.

$$\mathbb{H} \equiv \sigma_\mu x^\mu = \begin{pmatrix} x^0 + i x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

where

$$\sigma^\mu = [\sigma^0, \vec{\sigma}] \quad \text{and} \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{Tr}\,\sigma_\mu \sigma_\nu = 2g_{\mu\nu} \quad \text{so} \quad x^\mu = \tfrac{1}{2}\,\mathrm{tr}(\sigma^\mu \mathbb{H})$$

Induce a L.T. by transforming $\mathbb{H}$

$$\mathbb{H} \to \mathbb{H}' = \sigma_\mu x'^\mu = \begin{pmatrix} x'^0 + x'^3 & x'^1 - ix'^2 \\ x'^1 + ix'^2 & x'^0 - x'^3 \end{pmatrix}$$

and $\mathbb{H}' = \underset{\substack{\uparrow \\ \text{some } 2\times 2 \text{ matrix}}}{\mathbb{A}}\,\mathbb{H}\,\mathbb{A}^\dagger$

$$x^{\mu\prime}\sigma_\mu = \mathbb{A}\, x^\mu \sigma_\mu\, \mathbb{A}^+$$

$$\Lambda^\mu{}_\nu x^\nu \sigma_\mu = \mathbb{A}\, x^\nu \sigma_\nu\, \mathbb{A}^+$$ dummy indices

$\sigma^\rho \rightarrow$

$$\sigma^\rho \Lambda^\mu{}_\nu x^\nu \sigma_\mu = \sigma^\rho \mathbb{A}\, x^\nu \sigma_\nu\, \mathbb{A}^+$$

take trace in 2x2 space

$$\Lambda^\mu{}_\nu \underbrace{Tr\left[\sigma^\rho \sigma_\mu\right]}_{2\delta^\rho_\mu} x^\nu = Tr\left(\sigma^\rho \mathbb{A} \sigma_\nu \mathbb{A}^+\right) x^\nu$$

$$\underbrace{\left[2\Lambda^\mu{}_\nu \delta^\rho{}_\mu\right]}_{\text{operator}} x^\nu = \left[\qquad\right] x^\nu$$

So, $$\Lambda^\mu{}_\nu \delta^\rho{}_\mu = \frac{1}{2} Tr\left[\sigma^\rho \mathbb{A} \sigma_\nu \mathbb{A}^+\right]$$

$$\Lambda^\mu{}_\nu = \frac{1}{2} \underbrace{Tr\left[\sigma^\mu \mathbb{A} \sigma_\nu \mathbb{A}^+\right]}$$

so, the $\mathbb{A}$'s induce a L.T.

For $\mathbb{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, which can be complex,

with the condition $\det \mathbb{A} = \alpha\delta - \beta\gamma = 1$ unimodular

The $\mathbb{A}$'s form a group

$$SL(2, C)$$

special => $\det A = 1$ linear   2d   complex

Just like

$$\begin{tikzcd} SU(2) \arrow[dr] & \\ & SO(3) \\ -SU(2) \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} SL(2,c) \arrow[dr] & \\ & L \\ -SL(2,c) \arrow[ur] & \end{tikzcd}$$

same sort of homomorphism.

In fact, our definitions show immediately that

$$SU(2) \subset SL(2,c)$$

Look at $\quad x^\mu = [1, \vec{0}] \quad$ a time-like unit vector

$$\mathbb{H} = \sigma_0 x^0 = \mathbb{1}$$

} can show

$$\mathbb{H}' = \mathbb{A}\mathbb{A}^{+} \geq 0 \quad \text{or} \quad \sigma_\mu x^{\mu'} \geq 0$$

$$\text{or} \quad \begin{pmatrix} x^{0'} - x^{3'} & x^{1'} - i x^{2'} \\ x^{1'} + i x^{2'} & x^{0'} - x^{3'} \end{pmatrix} \geq 0$$

element by element

$$x^{0'} \geq x'^{3} \geq 0$$

so $x^{0'} \geq 0$

and

$\Lambda^0{}_0 \geq 0$ so the $\mathbb{A}$'s only induce $L_+^\uparrow$

Just like with SU(2), there can be two kinds of bases

1) The vector representation Λ with bases x^μ

2) a 2 dimensional representation, $\mathbb{A}$ with spinor bases

Look at

$$x^0 \to x^{0'} = x^0$$
$$x^3 \to x^{3'} = x^3$$
$$\left.\begin{aligned} x^1 &\to x^{1'} = x^1 - \delta\theta x^2 \\ x^2 &\to x'^2 = x^2 + \delta\theta x^1 \end{aligned}\right\} \text{spatial rotation in } z=0 \text{ plane}$$

Imagine a spinor rep that

$$\psi^\mu \xrightarrow{L.T.} \psi'^\mu = A^\mu{}_\nu \psi^\nu$$

and generally choose $\quad A = e^{iG\theta}$ ← generator

Infinitesimally,

$$A = \mathbb{1} + i\,\delta\theta\, G_3 \qquad \text{here.}$$

$$\mathbb{H}' = \sigma_\mu x^{\mu\prime} = \begin{pmatrix} x^0 + x^3 & x^1 - \delta\theta x^2 - i x^2 - i\delta\theta x^1 \\ x^1 - \delta\theta x^2 + i x^2 + i\delta\theta x^1 & x^0 - x^3 \end{pmatrix}$$

$$= \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{pmatrix} + \delta\theta \begin{pmatrix} 0 & -x^2 - i x^1 \\ -x^2 + i x^1 & 0 \end{pmatrix}$$

$$= \sigma_\mu x^\mu + \delta\theta \left[\begin{pmatrix} 0 & -i x^1 \\ i x^1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -x^2 \\ -x^2 & 0 \end{pmatrix} \right]$$

$$= \sigma_\mu x^\mu + \delta\theta \left[\sigma_2 x^1 - \sigma_1 x^2 \right]$$

Also,

$$\mathbb{H}' = \mathbb{A}\,\mathbb{H}\,\mathbb{A}^\dagger$$

$$= \mathbb{H} - i\,\delta\theta \left(G_3 \mathbb{H} - \mathbb{H} G_3^\dagger \right)$$

So,

$$-i\,\delta\theta \left(G_3 \mathbb{H} - \mathbb{H} G_3^\dagger \right) = \delta\theta \left(\sigma_2 x^1 - \sigma_1 x^2 \right)$$

$$G_3 \sigma_\mu x^\mu - \sigma_\mu x^\mu G_3^\dagger = i \left(\sigma_2 x^1 - \sigma_1 x^2 \right)$$

$$G_3(\sigma_0 x^0 - \sigma_1 x^1 + \sigma_2 x^2 - \sigma_3 x^3) - (\qquad\qquad) G_3^+ =$$

so

$$G_3 \sigma_0 x^0 - \sigma_0 x^0 G_3^+ = 0$$

$$-G_3 \sigma_1 x^1 + \sigma_1 x^1 G_3^+ = i \sigma_2 x^1$$
$$-G_3 \sigma_2 x^2 + \sigma_2 x^2 G_3^+ = -i \sigma_1 x^2$$

so, $$G_3 \sigma_2 - \sigma_2 G_3^+ = i \sigma_1$$

$\Rightarrow$ the G_3 satisfy an $SU(2)$ Lie Algebra.

$\rightarrow$ spin.

So, we can write $\quad G_3 = J_3 = \frac{1}{2}\sigma_3$

and $$A = e^{-i \vec{J} \cdot \vec{\theta}} = e^{-i\theta \vec{J} \cdot \hat{n}}$$

generates these transformations.

How about a pure Lorentz Transformation?

eg. along 3 axis.

Infinitesimally,

$$x'^0 = x^0 + \delta\xi\, x^3$$

$$x'^1 = x^1$$

$$x^{2\prime} = x^2$$

$$x'^3 = x^3 - \delta\xi\, x^3$$

$$\mathbb{H}' = \sigma_\mu x'^\mu = \begin{pmatrix} x^0 + \delta\xi x^3 + x^3 + \delta\xi x^0 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 + \delta\xi x^3 - x^3 - \delta\xi x^0 \end{pmatrix}$$

$$= \sigma_\mu x^\mu + \delta\xi \left[\begin{pmatrix} x^3 & 0 \\ 0 & x^3 \end{pmatrix} + \begin{pmatrix} x^0 & 0 \\ 0 & -x^0 \end{pmatrix} \right]$$

$$= \sigma_\mu x^\mu + \delta\xi \left[\sigma_0 x^3 + \sigma_3 x^0 \right]$$

also, $A = e^{iK\xi} \rightarrow \mathbb{1} + i\,\delta\xi\, K_3$

$\mathbb{H}' = A\mathbb{H}A^\dagger = \mathbb{H} - i\delta\xi\left(K_3\mathbb{H} - \mathbb{H}K_3^\dagger\right)$ infinitesimal.

so, $-i\left(K_3\sigma_\mu x^\mu - \sigma_\mu x^\mu K_3^\dagger\right) = \sigma_0 x^3 + \sigma_3 x^0$

just like before ... $K_3\sigma_0 - \sigma_0 K_3^\dagger = i\sigma_3$

or here

$$K_3 - K_3^\dagger = i\sigma_3$$

and $-K_3\sigma_3 + \sigma_3 K_3^\dagger = -i\sigma_0$ $\qquad$ $K_3\sigma_1 - \sigma_1 K_3^\dagger = 0$

$K_3\sigma_2 - \sigma_2 K_3^\dagger = 0$

$$K_3\sigma_3 - \sigma_3 K_3^\dagger = i$$

solve: $K_3 = \frac{i}{2}\sigma_3$

and $K_1 = \frac{i}{2}\sigma_1 \qquad K_2 = \frac{i}{2}\sigma_2$

<u>same</u> operators σ inducing very different transformations all within $SL(2,C)$!

Generalized Lorentz Transformation

$$\Lambda = RL$$

$$= e^{-i\theta \vec{J}\cdot\hat{n}}\, e^{-i\xi \vec{K}\cdot\hat{v}}$$

where $\vec{v} = \tanh \xi \qquad \hat{v} = \vec{v}/|v|$

$$\vec{K} = \pm i\frac{\vec{\sigma}}{2}$$

because both $\pm A$ and $\pm A^*$ satisfy the original

$$\Lambda = \frac{1}{2}\operatorname{Tr} \sigma A \sigma A^\dagger$$

$\Rightarrow$ 2 different non-equivalent bases for the spinor representations of the Lorentz Group.

How about the "real" coordinate representations?
Some, we know. like rotations about the 3 axis
by θ.

$$R_3(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

infinitesimally

$$R_3(\delta\theta) \to \mathbb{1} + \delta R$$

$$= \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta\theta & 0 \\ 0 & \delta\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \mathbb{1} + \delta\theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

but $\quad R_3 = e^{-i\theta J_3}$

$$\to 1 - i\,\delta\theta\, J_3$$

or $\quad J_3 = i \left.\dfrac{\partial R_3}{\partial\theta}\right|_{\theta\to 0} = (i)(-i) J_3 R_3 \Big|_{\theta\to 0}$

$$= J_3 \quad \checkmark$$

so, $$J_3 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Bigg|_{\theta\to 0}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Likewise

$$L_3(\xi) = \begin{pmatrix} \cosh\xi & 0 & 0 & -\sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix}$$

so

$$\delta L_3 = \begin{pmatrix} 0 & 0 & 0 & -\delta\xi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\delta\xi & 0 & 0 & 0 \end{pmatrix}$$

like the above

$$K_3 = i \frac{\partial L_3}{\partial \xi}\Bigg|_{\xi\to 0} \qquad \text{where } L_3 = e^{-i\xi K_3}$$

$$K_3 = \begin{pmatrix} \sinh\xi & 0 & 0 & -\cosh\xi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\cosh\xi & 0 & 0 & \sinh\xi \end{pmatrix} \Bigg|_{\xi\to 0}$$

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & & & 0 \\ 0 & & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$ (antihermitian)

The Lie Algebra of $L_+^\uparrow$ can be calculated from the matrix representations or the σ algebras.

$$[J_i, J_j] = i\,\varepsilon_{ijk} J^k$$

$$[K_i, K_j] = -i\,\varepsilon_{ijk} J^k$$

$$[J_i, K_j] = i\,\varepsilon_{ijk} K^k$$

don't transform among themselves pure L.T.
→ not a group – closure doesn't work

Define a second rank, antisymmetric tensor $M_{\mu\nu}$

$$\vec{J} = [M^{23}, M^{31}, M^{12}] \Rightarrow J^i = \frac{1}{2}\varepsilon^{ijk} M_{jk}$$

$$\vec{K} = [M^{01}, M^{02}, M^{03}] \Rightarrow K^i = M^{0i}$$

Then the entire L.T. can be written

$$\Lambda = e^{-\frac{i}{2}\omega^{\mu\nu} M_{\mu\nu}}$$ ← generators of L.T.

where $\omega^{\mu\nu}$ is a real, antisymmetric parameter matrix

The Lie Algebra is

$$[M^{\kappa\lambda}, M^{\mu\nu}] = -i\left(g^{\lambda\mu}M^{\kappa\nu} + g^{\kappa\nu}M^{\lambda\mu} - g^{\kappa\mu}M^{\lambda\nu} - g^{\lambda\nu}M^{\kappa\mu}\right)$$

$\Rightarrow$ 2 Casimir Operators can be constructed, rank 2.

$$\left.\begin{array}{l} \frac{1}{2} M_{\mu\nu}M^{\mu\nu} = J^2 - K^2 \\ \\ 1/4\ \epsilon^{\mu\nu\sigma\tau} M_{\mu\nu} M_{\sigma\tau} = -\vec{J}\cdot\vec{K} \end{array}\right\}$$ commute with all generators of the rank 2. Lorentz group

So, the basis states of the IRR are labeled by the eigenvalues of the Casimir Operators

For $L_+^\uparrow$, there are 2 operators $\Rightarrow$ 2 inequivalent bases

J^2? <u>no</u> $\Rightarrow$ spin can't be used to uniquely label relativistic basis states

To classify them,

Construct

$$\left.\begin{array}{l} \vec{A} = \frac{1}{2}(\vec{J} + i\vec{K}) \\ \\ \vec{B} = \frac{1}{2}(\vec{J} - i\vec{K}) \end{array}\right\} \quad \begin{array}{l} [A_i, A_j] = i\epsilon_{ijk}A^k \\ [B_i, B_j] = i\epsilon_{ijk}B^k \\ [A_i, B_j] = 0 \end{array}$$

$\uparrow$

2 SU(2) algebras

So, the L.G. is like an $SU(2) \otimes SU(2)$ product group.

So, label states by eigenvalues of A^2, B^2, A_3, B_3.

$$(j, j')$$

$$\underset{A}{\nearrow} \qquad \underset{B}{\nwarrow}$$

each with its own basis

$$j, j' = 0, \tfrac{1}{2}, 1, \tfrac{3}{2} \ldots \text{ etc.}$$

single valued

For $j + j' =$ integer $\rightarrow$ tensorial representation

$j + j' =$ half-integer $\rightarrow$ spinor representation

double valued.

The fundamental IRRs of $L_+^\uparrow$ are $(0, \tfrac{1}{2})$ & $(\tfrac{1}{2}, 0)$

Remember

$$R = e^{-i J \cdot \vec{\theta}} \longrightarrow R = e^{-i(\vec{A} + \vec{B}) \cdot \vec{\theta}}$$

$$L = e^{-i \vec{K} \cdot \vec{\xi}} \longrightarrow L = e^{-(\vec{A} - \vec{B}) \cdot \vec{\xi}}$$

The 2 fundamental representations are

Type I. $(\frac{1}{2}, 0)$ $\vec{J} = \vec{\sigma}/2$ & $\vec{K} = -i\frac{\vec{\sigma}}{2} = -i\vec{J}$: $\vec{B} = 0$, $\vec{A} = \vec{J}$

Type II $(0, \frac{1}{2})$ $\vec{J} = \vec{\sigma}/2$ & $\vec{K} = +i\frac{\vec{\sigma}}{2} = i\vec{J}$: $\vec{A} = 0$, $\vec{B} = \vec{J}$

The difference? the difference between $+\vec{v}$ and $-\vec{v}$.
So, it's a parity difference. So,

$$(j, 0) \xrightarrow{P} (0, j)$$

In the spinor representation, define a matrix rep.

I. $D^{(j)}(R) = e^{-i\vec{J}^{(j)}\cdot\vec{\theta}}$ $\quad D^{(j)}(L) = e^{-\vec{J}^{(j)}\cdot\vec{\zeta}}$ $\quad (j\ 0)$

II $\bar{D}^{(j)}(R) = e^{-i\vec{J}^{(j)}\cdot(\vec{\theta})}$ $\quad \bar{D}^{(j)}(L) = e^{\vec{J}^{(j)}\cdot\vec{\zeta}}$ $\quad (0\ j)$

The full L.T. comes from doing the equivalent of the L.R.; for spin ½:

$$D^{(j)}(\Lambda) = D^{(1/2)}(R)\, D^{(1/2)}(L)$$
$$\bar{D}^{(j)}(\Lambda) = \bar{D}^{(1/2)}(R)\, \bar{D}^{(1/2)}(L)$$

call the basis vectors for Type I: ξ

Type II: η

So, $\xi \to \xi' = D^{(j)}(\Lambda)\,\xi$

$$\eta \to \eta' = \bar{D}^{(j)}(\Lambda)\,\eta$$

There is no Π such that $D = \Pi \bar{D} \Pi^{-1}$ so there are inequivalent subspaces.

Parity connects them, so something larger than D and $\bar{D}$ is required to characterize a state. So combine them into a $2(2j+1)$ IRR with matrix rep:

$$\mathcal{D}^{(j)}(\Lambda) = \begin{pmatrix} D^{(j)}(\Lambda) & 0 \\ 0 & \bar{D}^{(j)}(\Lambda) \end{pmatrix}$$

now

$\mathcal{D}^{(j)}$ and $\bar{\mathcal{D}}^{(j)}$ are equivalent since

$\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Pi^{-1}$ is the operation that does connect $\mathcal{D}$ and $\bar{\mathcal{D}}$ (related to parity in Dirac algebra)

by $\bar{\mathcal{D}} = \Pi \mathcal{D} \Pi^{-1}$

For spin 1/2 this is a $2(1+1) = 4$d space, spanned by the spinors, generally

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \xrightarrow{\Lambda} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} D^{(j)}(\Lambda) & 0 \\ 0 & \bar{D}^{(j)}(\Lambda) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Notice that

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \xrightarrow{P} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \eta \\ \xi \end{pmatrix}$$

So,

ψ is the basis of the IRR of the Lorentz group "extended by parity".

Consider a boost on ξ, $(1/2, 0)$

$$\xi \to \xi' = e^{-\vec{\sigma}/2 \cdot \vec{\xi}}\, \xi$$
$$= \left(\cosh \xi/2 - \vec{\sigma}\cdot\hat{p} \sinh \xi/2\right) \xi$$

remember $\cosh \xi/2 = \sqrt{\frac{\gamma+1}{2}}$ $\sinh \xi/2 = \sqrt{\frac{\gamma-1}{2}}$

(half angle formulae for $\cosh \xi = \gamma$)

So, $$\xi(\vec{p}) = \left(\sqrt{\frac{\gamma+1}{2}} - \vec{\sigma}\cdot\hat{p}\sqrt{\frac{\gamma-1}{2}}\right)\xi(0)$$

Interlude: remember the relativistic energy relation:

$$E^2 = p^2c^2 + m^2c^4$$

Suppose we transform a particle of mass m to rest from $p = p_3$

$$\begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ 0 \\ 0 \\ p_3 \end{pmatrix}$$

$$mc = E/c\,\gamma - \beta\gamma p_3 \qquad \Rightarrow \qquad \frac{\beta E}{c} = p$$

$$0 = -\beta\gamma E/c + \gamma p_3$$

$$\beta = \frac{pc}{E}$$

since $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

$$= \frac{1}{\sqrt{1 - \frac{p^2c^2}{E^2}}} \qquad \Rightarrow \qquad \gamma = E/mc^2$$

so, $$\rho(\vec{p}) = \left(\sqrt{\frac{\frac{E}{mc^2}+1}{2}} - \vec{\sigma}\cdot\hat{p}\sqrt{\frac{\frac{E}{mc^2}-1}{2}} \right) \rho(0)$$

with $\hat{p} = \vec{p}/|\vec{p}|$ $\qquad |p| = \sqrt{E^2/c^2 - m^2c^2}$

$\vdots$

$$\rho(\vec{p}) = \frac{E + mc^2 - \vec{\sigma}\cdot\vec{p}}{\sqrt{2mc^2(E+mc^2)}}\,\rho(0)$$

This is within the $D^{(1/2)}(L)$ space. Likewise,

$$\eta(\vec{p}) = \frac{E + mc^2 + \vec{\sigma}\cdot\vec{p}}{\sqrt{2mc^2(E+mc^2)}}\,\eta(0) \quad \text{in } \overline{D}^{(1/2)}(L)$$

The only thing different is $\vec{p}$, so $\xi(0) = \eta(0)$ and they can be combined.

$$\eta(\vec{p}) = \frac{E - \vec{\sigma}\cdot\vec{p}}{mc^2}\,\xi(\vec{p}) \quad \text{or.}$$

$$\left(E - \vec{\sigma}\cdot\vec{p}\right)\xi(\vec{p}) - mc^2\,\eta(\vec{p}) = 0$$

which can be written

$$\begin{pmatrix} -mc^2 & E + \vec{\sigma}\cdot\vec{p}\,c \\ E - \vec{\sigma}\cdot\vec{p}\,c & -mc^2 \end{pmatrix} \underbrace{\begin{pmatrix} \xi \\ \eta \end{pmatrix}}_{\psi} = 0$$

Define $\quad \gamma^{0\prime} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^{i\prime} = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$

then

$$\left(\gamma^{0\prime} p_0 + \gamma^{i\prime} p_i c - mc^2\right)\psi = 0$$

which is the Dirac Equation... * $\quad$ ($\gamma' \gamma^{0\prime}$ nonstandard basis)

If the particles are massless, then ξ and η decouple

$$\begin{pmatrix} 0 & E + \vec{\sigma}\cdot\vec{p}\,c \\ E - \vec{\sigma}\cdot\vec{p}\,c & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0$$

* ~~Standard~~ γ basis: $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$
(no "prime") $\rightarrow$ γ' and γ related by similarity

$$(E + \vec{\sigma}\cdot\vec{p}\,c)\,\eta = 0 \qquad\qquad \vec{\sigma}\cdot\hat{p}\,\eta = -\eta$$
$$\Rightarrow$$
$$(E - \vec{\sigma}\cdot\vec{p}\,c)\,\xi = 0 \qquad\qquad \vec{\sigma}\cdot\hat{p}\,\xi = \xi$$

The $\vec{\sigma}\cdot\hat{p}$ operation is HELICITY

There is a more general L.T. which includes shifts in space time

$$x^\mu \rightarrow x^{\mu'} = L(a,\Lambda)^\mu{}_\nu\, x^\nu = \Lambda^\mu{}_\nu\, x^\nu + a^\mu$$

$\Rightarrow$ a group with 10 parameters:

6 from Λ

4 from spacetime

The Poincare Group $\qquad \mathbb{P} \ni L\,;\, S$ $\leftarrow$ 4d translations

or $\qquad \mathbb{P}^\uparrow_+ \ni L^\uparrow_+\,;\, S$

Inhomogeneous Lorentz Group.

The overall group is generated by unitary operators

$$\mathcal{U}(a,\Lambda) = \mathcal{U}(a,1)\,\mathcal{U}(0,\Lambda) = \mathcal{U}_a\,\mathcal{U}_\Lambda$$

Now the generator of space translations is $\vec{P}$ and time translations is $\vec{E}$, so generator of

$\mathcal{U}(a,1)$ is p^μ

Infinitesimal: $L(\delta a, 1) = 1 - i\,\delta a^\mu P_\mu$

and a finite shift is from exponentiating:

$$U(a,1) = e^{-i a^\mu P_\mu} \quad \text{plus} \quad U(0,\Lambda) = e^{-i\frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

a trick: choose basis $\begin{pmatrix} x \\ 1 \end{pmatrix} \updownarrow 5$

Then,

$$U(a,\Lambda) = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}$$

so,

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda x + a \\ 1 \end{pmatrix}$$

and from $U(\delta a, 1) = 1 - \delta a^\mu P_\mu$, the infinitesimal generators can be written

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ & & & & 0 \\ & & 0 & & 0 \\ & & & & 0 \\ & & & & 0 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & & & & 1 \\ & & 0 & & 0 \\ & & & & 0 \\ & & & & 0 \end{pmatrix} \quad \text{etc.}$$

and the whole Lie Algebra structure can be constructed

$$[J_i, p_0] = 0$$

$$[J_i, p_i] = 0$$

$$[J_i, p_j] = i\,\varepsilon_{ijk}\, p^k$$

$$[K_i, p_0] = -i p_i$$

$$[K_i, p_i] = -i p_0$$

$$[K_i, p_j] = 0$$

$$\left.\begin{array}{l} [M_{\mu\nu}, P_\rho] = -i(g_{\mu\rho} P_\nu - g_{\nu\rho} P_\mu) \\ [P_\mu, P_\nu] = 0 \\ [M, M] = \text{before} \end{array}\right\}$$ Lie Algebra of $\mathbb{P}_+^\uparrow$

Also, $[p^2, p_\mu] = 0$, $[p^2, M_{\mu\nu}] = 0$ $\Rightarrow$ p^2 is a Casimir Operator.

Another construction

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\sigma\tau} M^{\nu\sigma} p^\tau$$

Pauli-Lubansky Tensor

.. a covariant angular momentum

Then, $P^\mu W_\mu = \frac{1}{2}\,\underline{\varepsilon_{\mu\nu\sigma\tau}}\,\underline{P^\mu}\,M^{\nu\sigma}\,\underline{P^\tau} = 0$

and $\quad W^2 = W_\mu W^\mu \; : \quad [W^2, P_\mu] = 0$

$$[W^2, M_{\mu\nu}] = 0$$

the <u>other</u> casimir operator for the rank 2 Poincare Group.

So, can label states with eigenvalues of P^2 and W^2.

Since $P^\mu P_\mu$ and $W^\mu W_\mu$ are scalars — evaluate eigenvalues in any frame

$$W_0 = \tfrac{1}{2}\,\varepsilon_{0\nu\sigma\tau}\,M^{\nu\sigma}p^\tau = \tfrac{1}{2}\,\varepsilon_{ijk}\,M^{ij}p^k$$

remember $\quad J_k = \frac{1}{2}\,\varepsilon_{ijk}\,M^{ij}\quad$ so

$$W_0 = J_k p^k = \vec{J}\cdot\vec{p}$$

$$\frac{W_0}{|p|} = \vec{J}\cdot\hat{p} \;\rightarrow\; \text{HELICITY again}$$

Look at $\quad W_i = \frac{1}{2}\,\varepsilon_{i\nu\sigma\tau}\,M^{\nu\sigma}p^\tau$

$$\vdots \quad W_i = J_i p^0 - (\vec{K}\times\vec{p})_i$$

$$\vec{W} = \vec{J}E - \vec{K}\times\vec{p}$$

In the rest frame $W^\mu = [0, \vec{J} m^2 c^4]$

so $W^\mu W_\mu = 0 - \vec{W}^2 = -J^2 m^2 c^4$

and in rest frame $p^\mu = [mc^2, \vec{0}]$

$p^\mu p_\nu = mc^2$

True in any frame, so, the bases of the IRR of the Poincare Group can be labeled thusly:

$|m, s; \vec{p}, \lambda\rangle$ ← helicity component

$$W^2 |m, s; \vec{p}, \lambda\rangle = -m^2 c^4 J^2 |m, s; \vec{p} \lambda\rangle$$

$$= -m^2 c^4 \underbrace{s(s+1)}_{\text{intrinsic angular momentum}} |m\, s; \vec{p}\, \lambda\rangle$$

$$p^2 |m, s; \vec{p}, \lambda\rangle = m^2 c^4 |m, s; \vec{p}, \lambda\rangle$$

Transformations within the "multiplet"?

$u(a, \Lambda)|m s; p \lambda\rangle$ give another.

Just like for $O(3)$, j and m uniquely label a state.
For $\mathbb{P}$ states are labeled by p and λ

⇒ Definition of a quantum particle: systems with definite mass and spin whose states are IRR of P.G.