GROUP THEORY

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Text. Group Theory in Physics Tung

Group Theory is an incredibly important part of physics! It's also a branch of mathematics with a fascinating history – even a dramatic history.

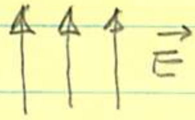
Its origins were the theory of equations (Galois and Abel), number theory and the "Invariant Theory" (Cayley and Sylvester. and Gordon and Noether – father and daughter) and geometry (Klein and Hilbert). 20th century developments then centered around physicists with Weyl and Wigner as the chief architects

Closely related and essential to Group Theory's application in physics is the notion of a vector space, whether Euclidean or in a Hilbert space. Here mathematics and physics collaborated (Grassman and Hamilton).

What good is Group Theory?

- can short-circuit calculations .. to even explore molecular and atomic spectra without doing any quantum mechanical calculations.

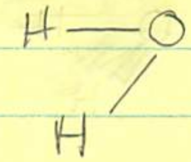
eg. imposition of an electric field picks a particular direction and can affect atomic degeneracies



Raman IR absorption → molecular vibrational normal modes

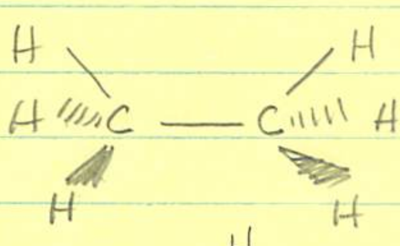
many molecules have specific symmetries

water

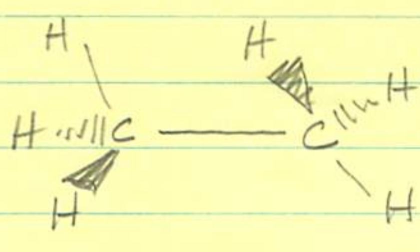


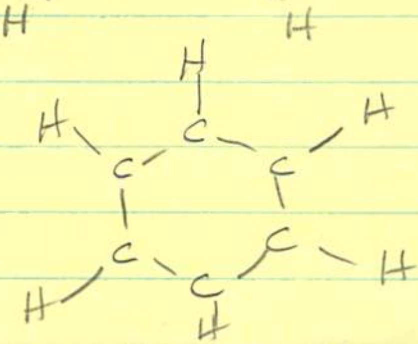
ammonia

methan - eclipsed.



methane





benzene

• Can explore conservation laws w/ Noether's Theorem.

demand a symmetry. (eg, translation invariance)

↓

identify the group and group generator. (eg P_μ.)

↓

find invariance of Lagrangian (eg. momentum conservation)

⇒ the importance of this is PROFOUND

demand of particular symmetries (eg. local phase invariance, $\psi \to \psi e^{i\theta(x)} = \psi'$

↓

need to postulate

1. spin 1 fields (eg. photon)
2. coupling (eg. α)

In the "construction" of the Universe.. the SYMMETRIES may have "come first"!

A Field. Remember in elementary school you were taught the postulates for algebra and arithmatic. Two operations were defined

$\oplus$ and $\otimes$

that had a set of properties among elements...

DEF: scalar; any abstract entity satisfying primary combinatoric relations

I. $\oplus$

A. To every pair, a and b, of scalars, there corresponds another scalar $a \oplus b$ called the sum or addition of a and b. CLOSURE property

B. $\forall_{a,b}$, $a \oplus b = b \oplus a$

COMMUTATIVE property

C. $\forall_{a,b,c}$ $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

ASSOCIATIVE property

D. $\exists! \phi \ni a \oplus \phi = a$

ZERO exists.

E. $\exists\ d \ni \forall_a$ $a \oplus d = \phi$ ... "$d = -a$"

NEGATIVE exists

II. $\otimes$

A. To every pair, a and b, of scalars, there exists another scalar $a \otimes b$ which is the <u>product</u> or <u>multiplication</u> of a & b
CLOSURE property.

B. $a \otimes b = b \otimes a$ COMMUTIVITY

C. $a \otimes (b \otimes c) = (a \otimes b) \otimes c$ ASSOCIATIVITY

D. $\forall_a \ \exists! \ \mathbb{1} \ \ni \ a \otimes \mathbb{1} = a$ IDENTITY

E. $\forall_a \ \exists \ c \ \ni \ a \otimes c = \mathbb{1}$ INVERSE

F. $a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c$ $\otimes$ DISTRIBUTIVE wrt $\oplus$

DEF. <u>field</u>; any set of entities which satisfies I A–E and II A–F is called a general, scalar field, $\mathcal{F}$.

examples: the set of real numbers, $\mathbb{R}$
" complex numbers, $\mathbb{C}$

the set of integers, $\mathbb{Z}$? NO why?

Hamilton actually found an algebra that <u>lacked</u> II B ... Cayley found one that <u>lacked</u> II B and II C.

This is all <u>abstract</u> .. not necessary numbers. Oddly, richness results from <u>removing</u> postulates from $\mathcal{F}$.

For example, the algebra called a GROUP.

DEF. A <u>group</u> is a set of abstract elements $g \in \{a, b, \ldots\}$ for which there is a <u>single</u> composition law,

"$\circ$"

(normally, and confusingly, called multiplication) which satisfies the following 4 properties:

1. If a and $b \in \mathcal{G}$ and $c = a \circ b$, then $c \in \mathcal{G}$.
   CLOSURE

2. $a \circ (b \circ c) = (a \circ b) \circ c$
   ASSOCIATIVITY

3. $\exists!\ g \equiv e \ni e \circ g_i = g_i \quad \forall g_i$
   IDENTITY

4. $\forall g_i \in \mathcal{G} \quad \exists \quad g' \ni g_i \circ g' = g' \circ g_i = e$

   INVERSE

THAT'S IT.

DEF. <u>order</u>; of a group is the number of elements
$\rightarrow$ can be finite or infinite

DEF. Abelian Group ; is one with a 5^{th} postulate:

5. $\forall_{g_i g_j \in \mathcal{G}} \quad g_i \circ g_j = g_j \circ g_i$

COMMUTIVITY holds

DEF. Discrete Group ; is one which has a countable, finite order.

There are very few Abelian Groups:

① $\{e\}$ a group of order one.

② $\{e, a\}$ " two

$$e \circ a = a$$
$$e \circ e = e$$
$$a \circ a = e \quad \longrightarrow \text{must be so.}$$

Cayley invented a nice way to show the operations and explicitly expose CLOSURE. .. the "multipication table"

For ②:

	e	a
e	e	a
a	a	e

Groups have names.. pretty standard.

② is called the Cyclic Group of order 2. , C_2

① " 1 , C_1

There is a 3-element group C_3:

	e	d	f
e	e	d	f
d	d	f	e
f	f	e	d

also.. Abelian

and C_4

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

GROUPS ARE ABSTRACT.

my word

But, there are many ways to "realize" a group .. and more conventional ways to "represent" a group.

standard word — the meat of practical G.T.

A. An arithmetic "realization" of C_2 ...

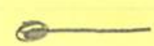
$a = -1 \qquad e = +1$

$\circ \equiv$ regular multiplication

B. A geometric "realization" of C_2 ...

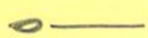
e and a are "operations"

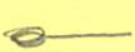
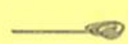
must be "on" something

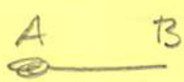
 a matchstick.

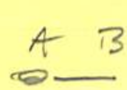
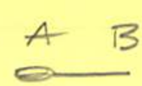
e operation: "let it be" (John Lennon, 1969 Gold)

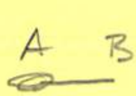
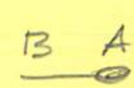
a operation: "flip it"

e () = ()

a () = ()

let me label the ends 

e () = ()

a () = ()

The group operations hold -- for moving matches:

$$e \circ a \circ \left(\overset{A}{\bullet}\!-\!\!-\overset{B}{\ }\right) = e \circ \left(\overset{B}{\ }\!-\!\!-\overset{A}{\bullet}\right) = \left(\overset{B}{\ }\!-\!\!-\overset{A}{\bullet}\right)$$
$$= a \circ \left(\overset{A}{\bullet}\!-\!\!-\overset{B}{\ }\right)$$

so $\quad e \circ a = a$

$$e \circ e \circ \left(\overset{A}{\bullet}\!-\!\!-\overset{B}{\ }\right) = e \circ \left(\overset{A}{\bullet}\!-\!\!-\overset{B}{\ }\right) = \left(\overset{A}{\bullet}\!-\!\!-\overset{B}{\ }\right)$$
$$= e \left(\overset{A}{\bullet}\!-\!\!-\overset{B}{\ }\right)$$

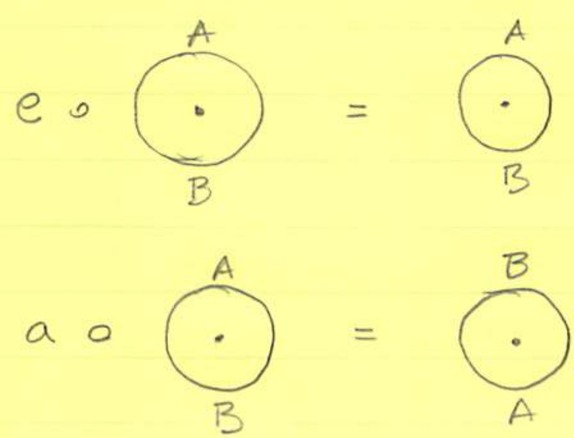
$e \circ e = e$

$$a \circ a \circ \left(\overset{A}{\bullet}\!-\!\!-\overset{B}{\ }\right) = a \circ \left(\overset{B}{\ }\!-\!\!-\overset{A}{\bullet}\right) = \left(\overset{A}{\bullet}\!-\!\!-\overset{B}{\ }\right)$$
$$= e \left(\overset{A}{\bullet}\!-\!\!-\overset{B}{\ }\right)$$

$a \circ a = e$

c. Another geometrical realization of C_2

$\circ \equiv$ rotate by $\pi, 2\pi$, ↻



So, in the last 2 realizations.. forget the geometry... C_2 swaps letters.

$$\left.\begin{array}{lll} e & \text{takes} & \begin{array}{l} A \to A \\ B \to B \end{array} \\ \\ a & \text{takes} & \begin{array}{l} A \to B \\ B \to A \end{array} \end{array}\right\} \text{a permutation.}$$

So,

$$a \circ a \circ \begin{pmatrix} A \\ B \end{pmatrix} = a \circ \begin{pmatrix} A \to B \\ B \to A \end{pmatrix} = a \circ \begin{pmatrix} B \\ A \end{pmatrix}$$
$$= \begin{pmatrix} A \\ B \end{pmatrix}$$

This is a big deal... and becomes a whole new group.. the Symmetric Group or Permutation Group

→ S_n is the Symmetric Group on n-objects. w/ $n!$ elements

Cayley invented a notation... for S_2

element in LH slot (1st)

$$\begin{pmatrix} L & R \\ i & j \end{pmatrix}$$

element in RH slot (2nd)

element in 2nd goes to j^{th}

element in 1st goes to the i^{th}

So, $$e = \begin{pmatrix} L & R \\ L & R \end{pmatrix}$$

$$a = \begin{pmatrix} L & R \\ R & L \end{pmatrix}$$

note. the notation implies: $$\begin{pmatrix} L & R \\ L & R \end{pmatrix} \equiv \begin{pmatrix} R & L \\ R & L \end{pmatrix}$$

$$\begin{pmatrix} L & R \\ R & L \end{pmatrix} \equiv \begin{pmatrix} R & L \\ L & R \end{pmatrix}$$

or $$\begin{pmatrix} a & b & c \dots \\ \alpha & \beta & \gamma \dots \end{pmatrix} \equiv \begin{pmatrix} b & a & c \dots \\ \beta & \alpha & \gamma \dots \end{pmatrix}$$

So, $$a \circ e = \begin{pmatrix} L & R \\ R & L \end{pmatrix} \circ \begin{pmatrix} L & R \\ L & R \end{pmatrix} = \begin{pmatrix} L & R \\ R & L \end{pmatrix} = a$$

$$a \circ a = \begin{pmatrix} L & R \\ R & L \end{pmatrix} \circ \begin{pmatrix} L & R \\ R & L \end{pmatrix} = \begin{pmatrix} L & R \\ L & R \end{pmatrix} = e$$

$$\downarrow$$

$$a \circ a = \begin{pmatrix} R & L \\ L & R \end{pmatrix} \circ \begin{pmatrix} L & R \\ R & L \end{pmatrix} = \begin{pmatrix} R & L \\ R & L \end{pmatrix} \rightarrow \begin{pmatrix} L & R \\ L & R \end{pmatrix}$$

and $$a \circ e = \begin{pmatrix} R & L \\ L & R \end{pmatrix} \circ \begin{pmatrix} L & R \\ L & R \end{pmatrix} = \begin{pmatrix} R & L \\ L & R \end{pmatrix} \rightarrow \begin{pmatrix} L & R \\ R & L \end{pmatrix} = a$$

notice that the algebra of $C_2 \equiv$ algebra of S_2

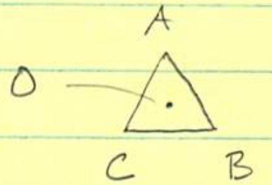
DEF. isomorphism; Two groups having the same multiplication table are isomorpic to one another.

"same algebra"

a 1:1 mapping between the elements.

These operations rearrange things... Step it up a notch:

C_3 has a nice geometrical realization



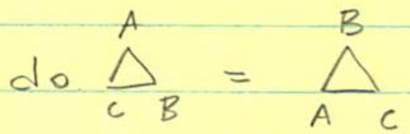
equilateral triangle

e: do nothing

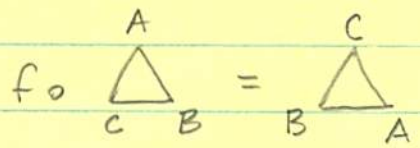
d: $\frac{2\pi}{3}$ ↺ about O

f: $\frac{4\pi}{3}$ ↺ $\equiv$ $\frac{2\pi}{3}$ ↻ about O

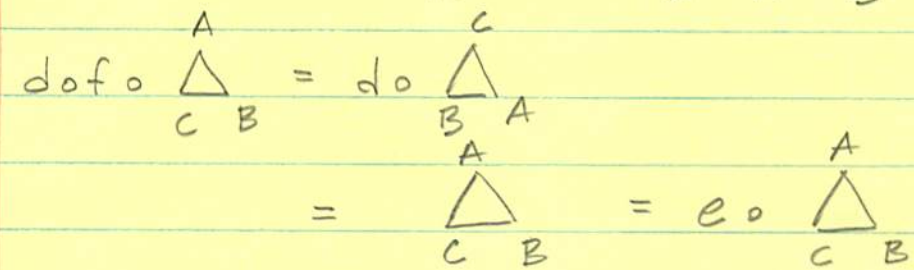
again, the replacement of adjacent letters... sort of...



$$\begin{array}{ccc} A & B & C \\ \downarrow & \downarrow & \downarrow \\ B & C & A \end{array}$$



$$\begin{array}{ccc} A & B & C \\ \downarrow & \downarrow & \downarrow \\ C & A & B \end{array}$$



$\Rightarrow$ $d \circ f = e$ a <u>cyclic</u> replacement

& a permutation... sort of..

$$d = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} \qquad f = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$$

$$d \circ f = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} \circ \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$$

$$\downarrow$$

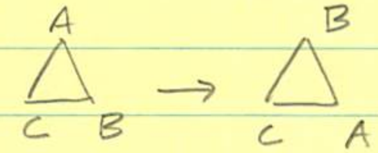
$$= \begin{pmatrix} C & A & B \\ A & B & C \end{pmatrix} \circ \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} = \begin{pmatrix} C & A & B \\ C & A & B \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix} = e$$

Okay. So is C_3 the same as S_3?

S_3 would have $3! = 6$ elements, C_3 has 3.

S_3 would do things like 

or $\begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$

The full S_3 .. the complete set of permutations on 3 things:

$$e = \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix} \quad a = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix} \quad b = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$$

$$c = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$$

$$d = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} \quad f = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$$

DEF. <u>subgroup</u>; If from a group $\mathcal{G}$, a subset of elements $\mathcal{H}$ can be selected which itself forms a group having the same combination law as $\mathcal{G}$ (same multiplication table subset), then $\mathcal{H}$ is a subgroup of $\mathcal{G}$.

Look back at S_3 and C_2

$\{e, a\}$, $\{e, b\}$, $\{e, c\}$ — 3 C_2 subgroups (or S_2 subgroups) of S_3

and C_3?

$\{e, d, f\}$ — a single C_3 subgroup of S_3

Also, inside of S_3:

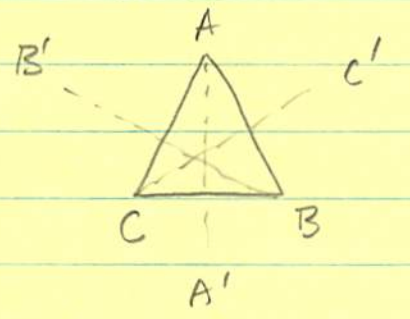
$$a \circ b = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix} \circ \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$$

$$= \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix} \circ \begin{pmatrix} A & C & B \\ C & A & B \end{pmatrix} = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} = f$$

$$\left.\begin{array}{l} a \circ b = f \\ \text{and} \\ b \circ a = d \end{array}\right\} \underline{\text{so}} \quad a \circ b \neq b \circ a$$

$\Rightarrow$ S_3 is non-Abelian... the smallest non-Abelian group.

A familiar geometrical realization of S_3



the above C_3 rotations

PLUS

3 rotations of π about axes $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$

This set "covers" all of the symmetry operations on an equilateral triangle ("geometric motions" was the 19th century phrase).

The smallest of The "dihedral groups of n-fold axes"

D_n

$\underline{1}$ n-fold axis set $\perp$ to the plane (principle axes)

$\underline{n}$ 2-fold axes $\perp$ to the principle and symmetrically arranged around it.

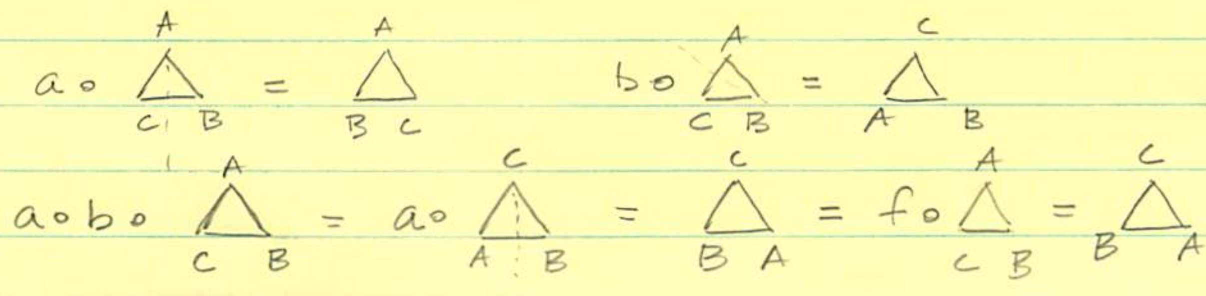
The covering group of the equilateral triangle

$$D_3 \xrightarrow{\text{isomorphic}} S_3$$

The multiplication table is (column × row)

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	f	d	c	b
b	b	d	e	f	a	c
c	c	f	d	e	b	a
d	d	b	c	a	f	e
f	f	c	a	b	e	d

So, check:



So, we establish the algebra: $a \circ b = f$

And.

$$b \circ a \; \underset{C\quad B}{\overset{A}{\triangle}} = b \circ \underset{B\quad C}{\overset{A}{\triangle}} = \underset{A\quad C}{\overset{B}{\triangle}} = d \circ \underset{C\quad B}{\overset{A}{\triangle}}$$

so $b \circ a = d \neq a \circ b \Rightarrow$ non-Abelian

So, easy to think of geometrically... but a weird realization:

Define 2 operators which take and symbol x and transform it to either

① $1-x$

or

② $\dfrac{1}{x}$

SO:

$$x \xrightarrow{①} 1-x$$

$$1-x \xrightarrow{①} 1-(1-x) = x \quad \text{repeats}$$

$$1-x \xrightarrow{②} \frac{1}{1-x}$$

$$\frac{1}{1-x} \xrightarrow{②} 1-x \quad \text{repeats}$$

$$\frac{1}{1-x} \xrightarrow{①} 1-\left(\frac{1}{1-x}\right) = \frac{1-x-1}{1-x} = \frac{x}{x-1}$$

$$\frac{x}{x-1} \xrightarrow{①} 1-\left(\frac{x}{x-1}\right) = \frac{x-1-x}{x-1} = \frac{1}{1-x} \quad$$

$$\frac{x}{x-1} \xrightarrow{②} \frac{x-1}{x}$$

$$\frac{x-1}{x} \xrightarrow{①} 1-\frac{x-1}{x} = \frac{x-x-1}{x} = \frac{1}{x}$$

$$\frac{x-1}{x} \xrightarrow{②} \frac{x}{x-1}$$

$$\frac{1}{x}$$

① ②

$$1-\frac{1}{x}=\frac{x-1}{x} \qquad x$$

assign

	e	x
	a	$1/x$
	b	$1-x$
	c	$\frac{x}{x-1}$
	d	$\frac{1}{1-x}$
	f	$\frac{x-1}{x}$

$$a \circ b = a \circ (1-x) = \frac{1}{1-x} = d$$

$$b \circ a = b \circ \frac{1}{x} = 1-\frac{1}{x} = \frac{x-1}{x} = f$$

hmmm... not just geometry

etc

same D_3 or S_3 algebra. No geometry here! Just algebra.

Algebra was the beginning of Group Theory.

SOME HISTORY —